1830

De infinitis curvis algebraicis, quarum longitudo arcui parabolico aequatur

Leonhard Euler

ON INFINITE ALGEBRAIC CURVES, WHOSE LONGITUDE IS EQUAL TO A PARABOLIC ARC.*

Leonhard Euler†

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Problem.

Fig. 1. *The proposed parabola AYC, with the related axis AB, whose parameter is $AB = BC$, will generate innumerable algebraic curves AZ, where the arc AZ is equal to the parabolic arc AY.*

Construction.

For the axis AB, produced back through the point F to describe the same *Parabola* AG. On this axis, we can take a point F as we please, in such a way that, the line FG drawn, this line FG and the parameter AB maintain a rational relationship, that is $\frac{AB}{FG} = n$. Thus for example from such a point F, a satisfying answer to the question can be constructed by one of the curves AZ.

For the point Y on the *Parabola*, the abscissa AX determines the ordinate XY, and we draw the normal line FG such that $GV = XY$, to obtain the angle $GFV = \theta$; and the angle found $AFZ = n\theta$, and we take $FZ = FX$, and Z is the point on the curve, whose *arc* AZ is equal to the *arc* AY. In this way, the point F can be assumed in an infinite number of ways, and innumerable curves AZ can be produced of the same character and the same properties.

Demonstration.

Supposing $AB = BC = 2a$ with $AX = x$ and $XY = y$, therefore $yy = 2ax$, and so $\partial x = \frac{y\partial y}{a}$, an element of a *Parabola*

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$$\partial s = \partial y \sqrt{1 + \tfrac{yy}{aa}}.$$

If I put $AF = f$ and $FG = g$ then $gg = 2af$. Then I set $FZ = FX = f + x = z$ and the angle $AFZ = \phi$, and then the curve in question will be $= \sqrt{\partial z^2 + zz\partial\phi^2}$. Therefore I will have:

$$\partial z^2 + zz\partial\phi^2 = \partial y^2 + \tfrac{yy\partial y^2}{aa}.$$

So from this I will have:

$\partial z = \partial x = \frac{y\partial y}{a}$ which becomes $zz\partial\phi^2 = \partial y^2$ therefore $\partial\phi = \frac{\partial y}{z}$.

Then it is true that $f = \frac{gg}{2a}$ and $x = \frac{yy}{2a}$, therefore $\partial\phi = \frac{2a\partial y}{gg+yy}$ consequently $\phi = \frac{2a}{g}\arctan\frac{y}{g}$. But then if $\arctan\frac{y}{g} = \theta$, and $\frac{2a}{g} = n$, it follows that $\phi = n\theta$. With the angle $AFZ = n\theta$ and the line $FZ = FX$, the point Z is on such a curve, which is equal to the element of the *Parabola*.

Note: The original diagram from the published copy of E781 has been amended below with the variables and constants used in the Demonstration shown in red. Also in the text the use of ArcTang has been replaced with the arctan function.

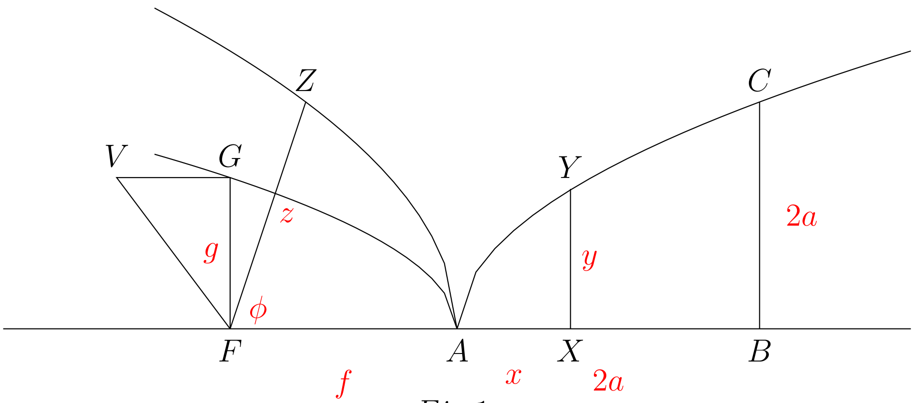

Fig.1.