1826

Solutio trium problematum difficiliorum ad methodum tangentium inversam pertinentium

Leonhard Euler

SOLUTIO
TRIUM PROBLEMATUM DIFFICILIORU
AD METHODUM TANGENTIUM INVERSAM PERTINENTIUM.

AUCTORE

L. EULERO.

Conventui exhibuit die 12. Nov. 1781.

Cum Ellipsis ea gaudeat proprietate, ut, ductis ex ejus fo ad punctum quodcunque in curva duobus rectis, eae aequaliter a curvam inclinentur, earumque summa simul ubique ejusdem sit qua titatis: hinc formari poterunt duae quaestiones reciprocae haud cilis indaginis, quae ob artificia calculi in solvendo adhibita atte tionem merere videntur. Eas igitur breviter hic exhibere a mus est.

Problema 1.

Tab. I. Fig. 4. *Datis duobus punctis A et B invenire lineam curvam FM ita comparatam ut, ductis ex singulis ejus punctis rectis MA et MB, eae utrinque aequaliter ad curv inclinentur.*

Solutio:

Sint rectae $AM = z$ et $BM = v$, vocenturque anguli $MAB = \Phi$ $MBA = \psi$ et anguli inclinationis $AMF = BMG = \omega$. Tum si co sideretur aliud punctum curvae proximum m, ducta recta Am d missoque ex m in AM perpendiculo mu, erit angulus $MAm = \partial$ $Mu = -\partial z$, $mu = z\partial\Phi$, ideoque $\cot. mMu = \cot.\omega = \frac{Mu}{mu} = -\frac{\partial}{z\partial}$
Simili modo ex altera parte reperietur $\cot. BMG = \cot.\omega = -\frac{\partial}{v\partial}$

ita ut haec prodeat aequatio: $\frac{\partial z}{z\partial\phi} = \frac{\partial v}{v\partial\psi}$, sive $\frac{\partial z}{z}\partial\psi = \frac{\partial v}{v}.\partial\phi$. Porro ex triangulo A M B, posita recta A B $= c$; ob angulum AMB $= 180^\circ - (\phi + \psi)$, erit $z = \frac{c \sin\psi}{\sin.(\phi+\psi)}$ et $v = \frac{c \sin.\phi}{\sin.(\phi+\psi)}$. Hinc fiet sumtis differentialibus logarithmicis

$$\frac{\partial z}{z} = \frac{\partial\psi}{\text{tag.}\psi} - \frac{(\partial\phi + \partial\psi)}{\text{tag.}(\phi+\psi)},$$

$$\frac{\partial v}{v} = \frac{\partial\phi}{\text{tag.}\phi} - \frac{(\partial\phi + \partial\psi)}{\text{tag.}(\phi+\psi)},$$

quibus substitutis aequatio illa hanc induet formam:

$$\frac{\partial\psi^2}{\text{tag.}\psi} - \frac{\partial\psi(\partial\phi+\partial\psi)}{\text{tag.}(\phi+\psi)} = \frac{\partial\phi^2}{\text{tag.}\phi} - \frac{\partial\phi(\partial\phi+\partial\psi)}{\text{tag.}(\phi+\psi)},$$

sive $\frac{\partial\psi^2}{\text{tag.}\psi} - \frac{\partial\phi^2}{\text{tag.}\phi} = \frac{\partial\psi^2 - \partial\phi^2}{\text{tag.}(\phi+\psi)}$, quae transmutatur in hanc:

$$\partial\psi^2\left(\frac{\cos.\psi}{\sin.\psi} - \frac{\cos.(\phi+\psi)}{\sin.(\phi+\psi)}\right) = \partial\phi^2\left(\frac{\cos.\phi}{\sin.\phi} - \frac{\cos.(\phi+\psi)}{\sin.(\phi+\psi)}\right),\ \text{unde}$$

$$\frac{\partial\psi^2 \sin.(\phi+\psi-\psi)}{\sin.\psi \sin.(\phi+\psi)} = \frac{\partial\phi^2 \sin.(\phi+\psi-\phi)}{\sin\phi \sin.(\phi+\psi)}$$

sive denique $\partial\psi^2 \sin.\phi^2 = \partial\phi^2 \sin.\psi^2$, ideoque $\frac{\partial\psi}{\sin.\psi} = \pm \frac{\partial\phi}{\sin\phi}$ unde integrando erit $l\,\text{tag.}\tfrac{1}{2}\psi = \pm\, l\,\text{tag.}\tfrac{1}{2}\phi + lC$, ita ut duae nascantur solutiones, quarum prima ex aequatione $\text{tag.}\tfrac{1}{2}\psi = C\,\text{tag.}\tfrac{1}{2}\phi$ est deducenda.

I. Ponatur $\text{tag.}\tfrac{1}{2}\phi = \frac{t}{a}$ et $\text{tag.}\tfrac{1}{2}\psi = \frac{t}{b}$, fietque

$$\sin.\phi = \frac{2at}{aa+tt},\quad \cos.\phi = \frac{aa-tt}{aa+tt}$$

$$\sin.\psi = \frac{2bt}{bb+tt},\quad \cos.\psi = \frac{bb-tt}{bb+tt},\ \text{unde colligitur}$$

$$\sin.(\phi+\psi) = \frac{2t(a+b)(ab-tt)}{(aa+tt)(bb+tt)}.\quad \text{Hinc fit}$$

$$z = \frac{c\sin.\psi}{\sin.(\phi+\psi)} = \frac{bc(aa+tt)}{(a+b)(ab-tt)},$$

quo valore invento coordinatae pro curva quaesita facile determinantur, quae si vocentur $AP = x$, $PM = y$, erit

$$x = z \cos.\phi = \frac{bc\,(aa - tt)}{(a+b)\,(ab - tt)},$$

$$y = z \sin.\phi = \frac{2\,a\,b\,c\,t}{(a+b)\,(ab-tt)}.$$

Sit brevitatis gratia $\frac{bc}{a+b} = f$, eritque $x = \frac{f\,(aa-tt)}{ab-tt}$, unde $tt = \frac{a\,(af-bx)}{f-x}$, et $ab - tt = \frac{af\,(b-a)}{f-x}$, hincque colligitur $y = \frac{2}{b-a}\sqrt{a\,(f-x)(af-bx)}$, sive $yy = \frac{4a}{(b-a)^2}(f-x)\,(af-b$ aequatio pro Hyperbola.

II. Pro altero signo, iisdem denominationibus adhibitis, perietur:

$\sin.(\phi+\psi) = \frac{2t\,(a-b)\,(ab+tt)}{(aa+tt)\,(bb+tt)}$, ex quo fit $z = \frac{bc\,(aa+tt)}{(a-b)\,(ab+}$ sicque habebimus coordinatas

$$AP = x = z\cos.\phi = \frac{bc\,(aa-tt)}{(a-b)\,(ab+tt)}$$

$$PM = y = z\sin.\phi = \frac{2\,a\,b\,c\,t}{(a-b)\,(ab+tt)}.$$

unde, posito ut supra, $\frac{bc}{a-b} = f$, erit

$$x = \frac{f\,(aa-tt)}{ab+tt} \quad \text{et} \quad y = \frac{2\,a\,f\,t}{ab+tt},$$

atque ob $tt = \frac{a\,(af-bx)}{f+x}$ et $ab + tt = \frac{af\,(a+b)}{f+x}$, aequatio inter co dinatas prodit haec: $yy = \frac{4a}{(a+b)^2}(f+x)\,(af-bx)$, pro Ellip

Problema 2.

Tab. I. Fig. 5. *Invenire lineam curvam, ad axem AO et punctum fixum A ferendam, ejusmodi ut sumto radio incidente AM, cui pondeat radius reflexus MO, summa amborum AM + sit ubique constans* = a.

Solutio:

Ducta ad curvam normali MN anguli AMN et OMN er inter se aequales. Hinc si, ut in problemate praecedente, voc tum anguli MAN $= \phi$, MCN $= \psi$, tum vero AMN $=$ OMN $= \omega$, erit $\psi = 180^\circ - \phi - 2\omega$. Sit AM $= z$, OM $= v$, eritque $z + v = a$, ideoque $v = a - z$, unde ex triangulo AMO erit $z : a - z = \sin.\psi : \sin.\phi$, consequenter $z = \frac{a \sin.\psi}{\sin.\phi + \sin.\psi}$. Porro, ob $z : \sin.\psi =$ AO $: \sin.2\omega$, erit AO $= \frac{z \sin.2\omega}{\sin.\psi} = \frac{a \sin.2\omega}{\sin.\phi + \sin.\psi}$, ubi notetur esse $\sin.\psi = \sin.(\phi + 2\omega) = \sin.\phi \cos.2\omega + \cos.\phi \sin.2\omega$. Ex distantia z, cum angulo ϕ, prodit

$$\text{tag. AMF} = \text{cot. AMN} = -\frac{z\partial\phi}{\partial z},$$

ideoque $\text{tag.}\,\omega = -\frac{\partial z}{z\partial\phi}$ et $\frac{\partial z}{z} = -\partial\phi\,\text{tag.}\,\omega$, quae est aequatio problema determinans.

Pro ea evolvenda statuatur $\text{tag.}\,\phi = t$ et $\text{tag.}\,\omega = u$, eritque $\sin.\phi = \frac{t}{\sqrt{1+tt}}$, $\cos.\phi = \frac{1}{\sqrt{1+tt}}$, ut et $\sin.\omega = \frac{u}{\sqrt{1+uu}}$, $\cos.\omega = \frac{1}{\sqrt{1+uu}}$, unde fit $\sin.2\omega = \frac{2u}{1+uu}$ et $\cos.2\omega = \frac{1-uu}{1+uu}$; praeterea vero $\partial\phi = \frac{\partial t}{1+tt}$ et $\partial\omega = \frac{\partial u}{1+uu}$. Ex his valoribus colligitur $\sin.\psi = \frac{t(1-uu)+2u}{(1+uu)\sqrt{1+tt}}$, ideoque $\sin.\phi + \sin.\psi = \frac{2(t+u)}{(1+uu)\sqrt{1+tt}}$, unde porro fit $z = \frac{at(1-uu)+2au}{2(t+u)}$ et AO $= \frac{au\sqrt{1+tt}}{t+u}$, hinc

$$\frac{\partial z}{z} = -\partial\phi\,\text{tag.}\,\omega = -\frac{u\partial t}{1+tt} = \frac{t\partial u(1-2tu-uu) - u\partial t(1+uu)}{(t+u)(2u+t)(1-uu)}.$$

Si haec aequatio inter t et u evolvatur, prodit:

$$t\partial u(1 - 2tu - uu) = \frac{u\partial t(1-tu)(1-2tu-uu)}{1+tt},$$

quae, cum habeat divisorem $1 - 2tu - uu$, duplicem subministrat solutionem, quarum altera in aequatione $1 - 2tu - uu = 0$, altera in aequatione $t\partial u = \frac{u\partial t(1-tu)}{1+tt}$ continetur.

Ex priore aequatione prodit $t = \frac{1-uu}{2u}$, hoc est

$$\text{tag.}\,\phi = \frac{1 - \text{tag.}\,\omega^2}{2\,\text{tag.}\,\omega} = \text{cot.}\,2\omega,$$

unde concluditur fore $2\omega = 90^\circ - \phi$, ideoque $\psi = 90^\circ$. Erit igitur

$$z = \frac{a \sin \psi}{\sin.\phi + \sin.\psi} = \frac{a}{1 + \sin.\phi}, \text{ sive } z = a - z \sin.\phi.$$

Positis jam $AO = x$, $OM = z \sin.\phi = y$, erit

$$z = \sqrt{xx + yy} = a - z \sin.\phi = a - y,$$

sive $aa - 2ay = xx$; et posito $\frac{1}{2}a - y = v$, erit $xx = 2av$, qu

Tab. I. Fig. 6. aequatio est pro Parabola, cujus parameter $= 2a$. Sit $CA =$ erit A focus Parabolae CMB et CA axis: Constat autem si AM Am sint radii incidentes, radios reflexos MO, mo fore axi paral los atque angulos $AMC = BMO$, ut et $AmC = Bmo$.

Evolvamus alteram aequationem $t\partial u = \frac{u(1 - tu)\partial t}{1 + tt}$, quae separabilitatem reducetur ponendo $t = \frac{p - u}{1 + pu}$, unde differenti do fit elementum $\partial t = \frac{\partial p(1 + uu) - \partial u(1 + pp)}{(1 + pu)^2}$, tum vero

$$1 + tt = \frac{(1 + uu)(1 + pp)}{(1 + pu)^2},$$

hincque colligitur $\frac{\partial t}{1 + tt} = \frac{\partial p}{1 + pp} - \frac{\partial u}{1 + uu}$. Porro est $1 - tu = \frac{1+}{1+}$ unde facta substitutione obtinetur haec aequatio:

$$\frac{(p - u)\partial u}{1 + pu} = \frac{u(1 + uu)}{1 + pu}\left(\frac{\partial p}{1 + pp} - \frac{\partial u}{1 + uu}\right),$$

sive $p\partial u = \frac{u(1 + uu)\partial p}{1 + pp}$, seu $\frac{\partial u}{u(1 + uu)} = \frac{\partial p}{p(1 + pp)}$, cujus aequa nis, penitus separatae, integrale est $l\frac{u}{\sqrt{1 + uu}} = lC + l\frac{p}{\sqrt{1 +}}$ ejusque evolutio, nisi ad angulos recurrere liceret, non parum ret molesta. Cum autem posuerimus $t = \frac{p - u}{1 + pu}$, erit

$$p = \frac{t + u}{1 - tu} = \frac{\text{tag.}\,\phi + \text{tag.}\,\omega}{1 - \text{tag.}\,\phi\ \text{tag.}\,\omega} = \text{tag.}\,(\phi + \omega)$$

ideoque $\frac{p}{\sqrt{1 + pp}} = \sin.(\phi + \omega)$, unde ob $\frac{u}{\sqrt{1 + uu}} = \sin.\omega$

Fig. 5. $\sin.\omega = C \sin.(\phi + \omega)$. Cum igitur in figura sit ang $MNO = \phi + \omega$, erit $C = \frac{\sin.\omega}{\sin.(\phi + \omega)} = \frac{AN}{AM}$, nec minus erit $C =$ et componendo $C = \frac{AN + ON}{AM + OM} = \frac{AO}{a}$, ideoque $AO = aC$, hoc constans. Punctum igitur O erit fixum, ex qua conditione sta manifesto sequitur curvam esse sectionem coni, ita ut praeter

rabolam, Hyperbolam et Ellipsin nullae aliae curvae dentur problemati satisfacientes.

Posterior aequatio $t\partial u = \frac{u(1 - tu)\partial t}{1 + tt}$ etiam sequenti modo resolvi potest. Reducatur ea primo ad hanc formam:

$$t\partial u - u\partial t + t^3\partial u + tuu\partial t = 0.$$

Ponatur $u = pt$ atque ob $\partial u = p\partial t + t\partial p$ prodibit haec aequatio:

$$tt(1 + tt)\partial p + pt^3(1 + p)\partial t = 0, \text{ sive}$$

$$\frac{\partial p}{p(1+p)} = -\frac{t\partial t}{1+tt} \text{ sive } \frac{\partial p}{p} - \frac{\partial p}{1+p} + \frac{t\partial t}{1+tt} = 0,$$

unde fit integrando $lp - l(1 + p) + l\sqrt{1 + tt} = lC$ et ad numeros descendendo $\frac{p\sqrt{1+tt}}{1+p} = C$, unde colligitur $p = \frac{C}{\sqrt{1+tt} - C}$ et $u = \frac{Ct}{\sqrt{1+tt} - C}$, hincque $t + u = \frac{t\sqrt{1+tt}}{\sqrt{1+tt} - C}$. Supra autem invenimus $AO = \frac{au\sqrt{1+tt}}{t+u}$, unde concluditur fore $AO = aC$, ideoque constantem ut supra, ita ut inde iterum sectio conica oriatur.

Sin autem aequationem inter coordinatas eruere atque inde naturam curvae concludere velimus, ex valore modo ante invento $u = \frac{Ct}{\sqrt{1+tt} - C}$ quaeratur $1 - uu = \frac{1 + tt + CC(1 - tt) - 2C\sqrt{1+tt}}{(\sqrt{1+tt} - C)^2}$, atque ob $t + u = \frac{t\sqrt{1+tt}}{\sqrt{1+tt} - C}$, substitutione facta colligitur

$$z = \frac{at(1 - uu) + 2au}{2(t + u)} = \frac{a(1 - CC)\sqrt{1+tt}}{2(\sqrt{1+tt} - C)},$$

sive posito brevitatis gratia $\frac{a(1 - CC)}{2} = b$, erit $z = \frac{b\sqrt{1+tt}}{\sqrt{1+tt} - C}$. Quod si jam introducantur coordinatae orthogonales $AN = x = z\cos.\Phi$ et $MN = y = z\sin.\Phi$, ob $\tan.\Phi = \frac{y}{x} = t$ erit $\sqrt{1 + tt} = \frac{\sqrt{x^2 + y^2}}{x} = \frac{z}{x}$. Hinc prodit $z = \frac{b\sqrt{1+tt}}{\sqrt{1+tt} - C} = \frac{bz}{x - Cx}$, sive $z - Cx = b$ et $z = b + Cx$, quo valore substituto in aequatione $\sqrt{xx + yy} = z$, ea abibit in istam: $yy + (1 - CC)xx = 2bCx + bb$, quae est pro Ellipsi, si $C < 1$, at vero pro Hyperbola, si $C > 1$.

Alia solutio ejusdem problematis.

Maneant omnes denominationes, ut in praecedentibus sunt s bilitae, et cum tota solutio his duabus formulis innitatur: tag. $\omega =$ — et $\frac{z}{a-z} = \frac{\sin.\psi}{\sin.\phi}$, ponatur $\cot.\phi = v$, ut sit $v = \frac{1}{t}$ atque $\partial\phi = \frac{-\partial}{1+u}$ unde fit $\frac{\partial z}{z} = -\partial\phi \operatorname{tag}.\omega = -u\partial\phi$, hoc est $\frac{\partial z}{z} = \frac{u\partial v}{1+vv}$. Alte aequatio $\frac{z}{a-z} = \frac{\sin.\psi}{\sin.\phi}$, ob

$$\sin.\psi = \sin.(\phi + 2\omega) = \sin.\phi \cos.2\omega + \cos.\phi \sin.2\omega,$$

fit $\frac{z}{a-z} = \cos.2\omega + \cot.\phi \sin.2\omega = \frac{1-uu+2vu}{1+uu}$, unde collig $v = \frac{2z-a(1-uu)}{2u(a-z)}$, hincque

$$\partial v = \frac{2au(1+uu)\partial z + (a-z)(2a(1+uu)-4z)\partial u}{4uu(a-z)^2} \text{ et}$$

$$1 + vv = \frac{(1+uu)(aa(1+uu)-4z(a-z))}{4uu(a-z)^2}.$$

Habebimus igitur

$$\frac{\partial v}{1+vv} = \frac{2au(1+uu)\partial z + 2(a-z)(a(1+uu)-2z)\partial u}{(1+uu)(aa(1+uu)-4z(a-z))} = \frac{\partial z}{uz}.$$

Quod si jam differentialia ∂z et ∂u separentur, prodibit seque aequatio:

$$\partial z(1+uu)(a-2z)\left(2z-a(1+uu)\right) = 2zu(a-z)\partial u\left(2z-a(1+uu\right.$$

quae, cum habeat divisorem, scil. $2z - a(1+uu)$, duas pra bebit solutiones, quarum prior ex aequatione $2z = a(1+u$ altera ex aequatione $\frac{\partial z(a-2z)}{z(a-z)} = \frac{2u\partial u}{1+uu}$ erit petenda.

Haec posterior aequatio integrata dat $lz(a-z) = lC + l(1+u$ sive in numeris $az - zz = C(1+uu)$, unde si in expressione s pra pro $1+vv$ data loco $az - zz$ hic valor $C(1+uu)$ subs tuatur, orietur sequens expressio: $1 + vv = \frac{(1+uu)^2(aa-4C)}{4uu(a-z)^2}$, ut, ob $\cot.\phi = v$ et $\sin.\phi = \frac{1}{\sqrt{1+vv}}$, fiat $\sin.\phi = \frac{2u(a-z)}{(1+uu)\sqrt{aa-4C}}$

Tab. I. Fig. 5. Hinc cum sit $AO : \sin.2\omega = MO : \sin.\phi$, erit

$$AO = \frac{(a-z)\sin.2\omega}{\sin.\phi} = \frac{2u(a-z)}{(1+uu)\sin.\phi} = \sqrt{aa-4C};$$

unde iterum patet, intervallum AO esse constans ideoque punctum O fixum, ex quo statim sequitur sectio conica.

Altera aequatio $2z = a(1+uu)$ dat $a - z = \frac{a(1-uu)}{2}$, unde [illegible] $\frac{1+uu}{1-uu}$ atque $v = \cot.\Phi = \frac{2z - a(1-uu)}{2u(a-z)} = \frac{(1+uu)-(1-uu)}{u(1-uu)}$, sive $\cot.\Phi = \frac{2u}{1-uu} = \text{tag.}\, 2\omega$, unde concluditur fore $90^\circ - \Phi = 2\omega$, sive $90^\circ = \Phi + 2\omega$, quo, ut ante, parabola indicatur.

Cum invenerimus $z(a-z) = C(1+uu) = \frac{C}{\cos.\omega^2}$, erit $z\cos.\omega \times (a-z)\cos.\omega = C$. Ducatur recta PQ, curvam in M tangens, et ex A et O in hanc tangentem demittantur perpendicula AP, OQ, eritque $AP = z\cos.\omega$ et $OQ = (a-z)\cos.\omega$, unde patet rectangulum ex his perpendiculis AP . OQ fore constans. Constat autem, in omnibus sectionibus conicis, quarum foci in A et O, rectangulum AP . OQ aequale esse quadrato semiaxis conjugati, unde semiaxis conjugatus sectionis conicae, quam hic eruimus, erit $= \sqrt{C}$. Tab. I. Fig. 5.

Tertia solutio sine calculo expedita.

Consideretur curvae punctum M, ejusque proximum m, ex quo radius reflexus mo cadat in axis punctum o, et cum requiratur ut sit tam $AM + MO = a$, quam $Am + mo = a$, erit $Am - AM = MO - mo$. Jam ex M in Am demittatur perpendiculum Mp, similique modo ex m in MO perpendiculum mq; et cum sit angulus $Mmp = mMq$, erunt triangula Mmp et mMq inter se aequalia, ob communem hypothenusam, ideoque $Mq = mp$. Atqui est $mp = Am - AM$ et $Mq = MO - mo$; tum vero $Mq = MO - Oq$, unde sequitur $Oq = mo$, id quod duplici modo fieri potest: 1°) quando omnes radii reflexi ad axem sunt perpendiculares, qui casus statim dat Parabolam. Praeterea vero fiet 2°) $qO = mo$, si punctum o cadet in O, sive quando O est punctum fixum, qui casus statim perducit ad Ellipsin vel Hyperbolam. Fig. 7.

Problema.

Invenire curvam LMN, in cujus tangentes MT si ex datis Fig. 8.

duobus punctis A et B demittantur perpendicula AF, BG, eorum rectangulum sit constans, hoc est AF . BG =

Solutio.

Bisecto intervallo AB in C sit $CA = CB = b$, ac pona… $CP = x$, $PM = y$, eritque tag. $MTP = -\frac{\partial y}{\partial x} = -p$, po… $\partial y = p\partial x$; tum vero habebimus $PT = -\frac{y}{p}$ et $CT = \frac{px - y}{p}$, un… colligitur $AT = \frac{px - y - bp}{p}$, hincque $BT = \frac{px - y + bp}{p}$. C… jam sit $AF = AT . \sin. T$ et $BG = BT . \sin. T$, ob $\sin. T = \frac{…}{\sqrt{1 + …}}$ habebimus $AF . BG = \frac{(px - y)^2 - bbpp}{pp} \times \frac{pp}{1 + pp} = cc$, sive

$$(px - y)^2 - bbpp = cc\,(1 + pp),$$

unde, posito brevitatis gratia $bb + cc = aa$, haec oritur aequati…

$$(y - px)^2 = cc + aapp, \text{ sive } y - px = \sqrt{cc + aapp}.$$

Ista aequatio, ob $p = \frac{\partial y}{\partial x}$, est differentialis ideoque integra… debere videtur: interim tamen hic ope differentiationis integrale e… potest. Cum enim sit $\partial y = p\partial x$, differentiatione facta prodit

$$x\partial p = -\frac{aap\partial p}{\sqrt{cc + aapp}},$$

quae aequatio, cum divisorem habeat ∂p, subministrat statim so… tionem ex aequatione $\partial p = 0$ petendam, unde fit p constans, pu… $p = \alpha$, ex quo colligitur $\partial y = \alpha\partial x$, ideoque $y = \alpha x + \beta$, qu… aequatio est pro linea recta.

Altera solutio ex aequatione $x = \frac{-aap}{\sqrt{cc + aapp}}$ erit deducend… ex qua fit $y = px + \sqrt{cc + aapp} = \frac{cc}{\sqrt{cc + aapp}}$. Hinc patet fo… $\frac{xx}{aa} + \frac{yy}{cc} = 1$, quae aequatio est pro Ellipsi, quoties cc est quantit… positiva, sive quoties $a > b$; at pro Hyperbola quoties $a < b$.

Quodsi autem aequatio $(y - px)^2 = cc + aapp$ evolvatur e… loco p scribatur $\frac{\partial y}{\partial x}$, ita ut prodeat

$$yy\partial x^2 - 2xy\partial x\partial y + xx\partial y^2 = cc\partial x^2 + aa\partial y^2$$

Cum vero haec aequatio more solito tractetur, ob

$$\partial y^2 (xx - aa) = 2xy\partial x\partial y + (cc - yy)\,\partial x^2, \text{ erit}$$

$$(xx - aa)\,\partial y = xy\partial x \pm \partial x\sqrt{(xxyy + (cc - yy)(xx - aa))}, \text{ sive}$$

$$\partial y = \frac{xy\partial x \pm \partial x\sqrt{ccxx + aayy - aacc}}{xx - aa}$$

cujus resolutio non parum difficultatis habet.

[illegible] Ponatur $a = 1$, [illegible] $= 1$, $c = 1$, erit aequatio illa

$$(xx - 1)\,\partial y = xy\partial x \pm \partial x\sqrt{xx + yy - cc}.$$

Sit porro $\sqrt{xx + yy - cc} = z$ et $y = ux$, atque ob $\partial y = u\partial x + x\partial u$ emerget sequens aequatio:

$$u\partial x (xx - 1) + x\partial u (xx - 1) = uxx\partial x + z\partial x, \text{ sive}$$

$$x\partial u\,(xx - 1) - u\,\partial x = z\,\partial x.$$

Cum igitur sit $xx(1 + uu) - 1 = zz$, erit $xx = \frac{zz + 1}{uu + 1}$, unde nostra aequatio: $\partial u\,(xx - 1) - u\frac{\partial x}{x} = z\frac{\partial x}{x}$, ob $xx - 1 = \frac{zz - uu}{uu + 1}$ et $\frac{\partial x}{x} = \frac{z\partial z}{1 + zz} - \frac{u\partial u}{1 + uu}$, hanc induet formam:

$$\frac{\partial u\,(zz - uu)}{1 + uu} = \frac{z\,(u + z)\,\partial z}{1 + zz} - \frac{u\,(u + z)\,\partial u}{1 + uu}$$

quae manifesto reducitur ad hanc:

$$\frac{(zz + uz)\,\partial u}{1 + nu} = \frac{(zz + uz)\,\partial z}{1 + zz}.$$

Haec aequatio factores habet z et $u + z$, quorum uterque dat solutionem. Primo enim prodit aequatio $zz = xx + yy - 1 = 0$, sive $xx + yy = 1$, cujus natura neminem latet. Secundo fit

$$z + u = \sqrt{xx + yy - 1} + \frac{y}{x} = 0,$$

hoc est $xx(xx + yy) = xx + yy$, unde fit $x = -1$ et $x = -y$, pro recta. Dividendo autem aequationem illam per factorem communem colligitur $\frac{\partial u}{1 + uu} = \frac{\partial z}{1 + zz}$, unde integrando

$$\text{A tag.}\,u = \text{A tag.}\,z + C, \text{ sive A tag.}\,z = \text{A tag.}\,u - \text{A tag.}\,n$$

hoc est $\text{A tag.}\,z = \text{A tag.}\,\frac{n + u}{1 - nu}$, hincque $z = \frac{n + u}{1 - nu}$, sive

$$\sqrt{xx + yy - 1} = \frac{nx + y}{x - ny},$$

ergo $xx + yy = \frac{(1 + nn)(xx + yy)}{(x - ny)^2}$, consequenter $(x - ny)^2 = 1$ vel $x - ny = \sqrt{1 + nn}$, iterum pro recta. Hac autem met[...] uti non licet simulac littera p ad altiores potestates ascendit.

Aequatio autem generalis, quae integrationem per differenti[...] nem admimittit, est, quando, posito $\frac{\partial y}{\partial x} = p$, formula $px - y$ cu[...] que functioni ipsius p aequatur. Posita enim hac functione Π, $\Pi = px - y$, quae aequatio differentiata dat $\partial \Pi = x \partial p = \Pi$[...] unde factor $\partial p = 0$ ostendit, semper lineam rectam satisfa[...] Praeterea vero habetur haec solutio: $x = \Pi'$ et $y = p\Pi' - \Pi$[...]

Fig. 1.

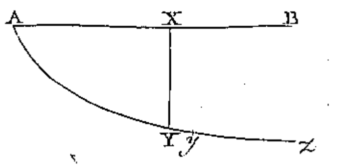

Fig. 2.

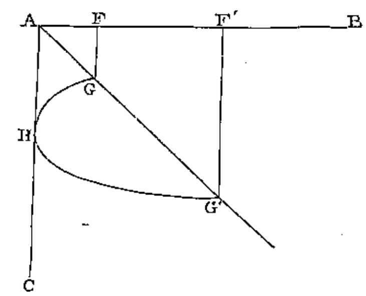

Fig. 3.

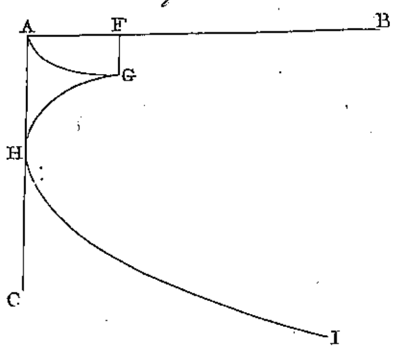

Fig. 4.

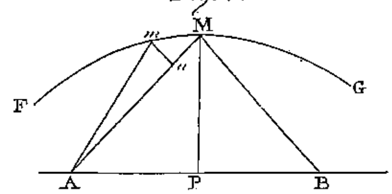

Fig. 5.

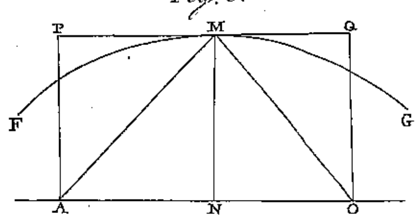

Fig. 6.

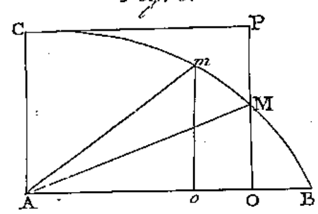

Fig. 7.

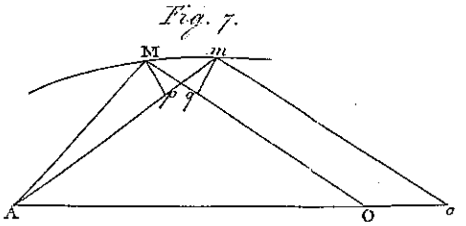

Fig. 8.