



1822

# De brachystochrona in medio resistente, dum corpus ad centrum virium utunque attrahitur

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Record Created:

2018-09-25

## Recommended Citation

Euler, Leonhard, "De brachystochrona in medio resistente, dum corpus ad centrum virium utunque attrahitur" (1822). *Euler Archive - All Works*. 761.

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DE BRACHISTOCHRONA  
IN MEDIO RESISTENTE  
DUM CORPUS AD CENTRUM VIRIUM  
UTCUNQUE ATTRAHITUR

AUCTORE  
L. EULERO.

Conventui exhibuit die 20. Nov. 1780.

§. 1. Sit  $O$  centrum virium, cujus attractio ad distantiam  $\equiv x$  sit  $X$ , functio quaecunque ipsius  $x$ ; tum vero si corporis celeritas fuerit  $\equiv v$ , sit vis resistens motui contraria  $\equiv V$ , functio quaecunque ipsius  $v$ . Sit jam curva  $AXC$  Brachystochrona quaesita, super qua corpus descendens tempore brevissimo ab  $A$  ad  $C$  perveniat, siquidem descensus in  $A$  ex quiete inceperit, Verum nihil obstat, quominus ipsi in  $A$  jam certa celeritas tribuatur. Pro initio hujus curvae  $A$  ponatur distantia  $OA \equiv a$  et pro fine  $C$  distantia  $OC \equiv c$ , angulusque  $AOC \equiv b$ . At vero pro ejus puncto quocunque  $X$  ponatur ejus distantia  $OX \equiv x$  et angulus  $AOX \equiv y$ ; atque manifestum est per relationem inter  $x$  et  $y$  curvam aeque determinari ac per aequationem inter coordinatas orthogonales. Ponatur autem arcus  $AX \equiv s$  ejusque elementum  $Xx \equiv ds$ , et ducta recta  $Ox$  ductoque ex  $x$  ad  $X$  perpendicularo  $Xy$ , erit  $Ky \equiv -dx$ , et ob angulum  $XOx \equiv dy$  erit  $xy \equiv xdy$ , unde fit elementum

$$Xx \equiv ds \equiv \sqrt{\partial x^2 + x x \partial y^2};$$

hinc si ponamus  $dy \equiv p \partial x$ , erit  $ds \equiv -\partial x \sqrt{1 + pp x x}$ .

Tab. I.  
Fig. 5.

§. 2. Cum nunc corpus in  $X$  sollicitetur in directione  $XO$  vi  $\equiv X$ , hinc pro directione motus  $Xx$  oriatur vis  $X: \frac{Xy}{Xx} \equiv -\frac{X \partial x}{\partial s}$ ; vis autem resistentiae, posita celeritate corporis in  $X \equiv u$ , est  $\equiv V$ ,

unde corpus accelerabitur a vi  $= -\frac{x\partial x}{\partial s} - V$ , quae ducta in elementum spatii  $\partial s$  dabit incrementum quadrati celeritatis, unde erit  $v\partial v = -X\partial x - V\partial s$ , hincque ob  $\partial s = \frac{\partial x\sqrt{1+ppxx}}{v}$  fiet

$$v\partial v = \partial x (V\sqrt{1+ppxx} - X),$$

quae aequatio exprimit relationem inter celeritatem  $v$  et quantitates proprie ad curvam pertinentes. Cum igitur tempusculum per  $Xx = \partial s$  sit  $\frac{\partial s}{v} = \frac{\partial x\sqrt{1+ppxx}}{v}$ , inter omnes curvas, ab A ad C ducendas, ea quaeritur, pro qua fiat valor hujus formulae integralis  $\int \frac{\partial x\sqrt{1+ppxx}}{v}$  omnium minimus.

§. 3. Hic ante omnia observasse juvabit, si terminus C in ipsa recta AO accipiatur, Brachystochoram in hanc ipsam rectam incidere debere, pro cujus ergo motu, ob  $y' = 0$  ideoque etiam  $p = 0$ , enascitur ista aequatio:  $v\partial v = \partial x(V - X)$ , quae quia in genere nequam resolvi potest, multo minus postulari poterit ut in genere pro Brachystochora AC motus determinatio penitus evolvatur, sed praecclare nobiscum agi censendum erit, si modo aequationem differentialem inter ternas variables  $x, y, v$  eruere valuerimus, quippe qua, cum formula:  $v\partial v = \partial x(V\sqrt{1+ppxx} - X)$  conjuncta, in se possibile esse intelligitur celeritatem  $v$  eliminari ideoque aequationem inter binas variables  $x$  et  $y$  obtineri posse.

§. 4. Cum igitur inter omnes curvas AG ea quaeri debeat, pro qua valor hujus formulae integralis  $\int \frac{\partial x\sqrt{1+ppxx}}{v}$  sit minimus, recurrendum erit ad problema generale isoperimetricum in praecedente dissertatione solutum. At quia hic circumstantiae non nihil sunt variatae, consultum erit solutionem ibi inventam sub forma theorematis huc transferre, quod ita si habebit:

**Theorema isoperimetricum generale.**

§. 5. Si inter omnes curvas, quae a puncto A ad C duci possunt, ea quaeratur, in qua valor formulae integralis  $\int W\partial x$

sit maximus vel minimus, ubi  $W$  praeter binas variables  $x$  et  $y$  earumque differentialia  $\frac{\partial y}{\partial x} = p$ ;  $\frac{\partial p}{\partial x} = q$ ;  $\frac{\partial q}{\partial x} = r$ ; etc. insuper involvat variabilem  $v$ , ita ut sit

$$\partial W = L \partial v + M \partial x + N \partial y + P \partial p + \text{etc.}$$

tum vero quantitas  $v$  ita, per aequationem differentialem detur, ut posito  $\partial v = \mathbb{B} \partial x$  sit

$$\partial \mathbb{B} = \mathfrak{L} \partial v + \mathfrak{M} \partial x + \mathfrak{N} \partial y + \mathfrak{P} \partial p + \mathfrak{Q} \partial q + \text{etc.},$$

his positis quaeratur  $\Lambda = e^{\int \mathfrak{L} \partial v}$ , hincque porro quantitas  $\Pi = \int L \Lambda \partial x$ , quod integrale ita capiatur, ut pro termino  $C$  evanescat, seu, quod eodem redit, terminus iste  $C$  ibi statuatur, ubi fit  $\Pi = 0$ , quibus inventis sumatur  $N' = N - \frac{\Pi \mathfrak{N}}{\Lambda}$ ,  $P' = P - \frac{\Pi \mathfrak{P}}{\Lambda}$ ;  $Q' = Q - \frac{\Pi \mathfrak{Q}}{\Lambda}$ ; etc. ex his pro natura curvae quaesitae ista deducitur aequatio:

$$0 = N' - \frac{\partial P'}{\partial x} + \frac{\partial \partial Q'}{\partial x^2} + \frac{\partial^2 R'}{\partial x^3} + \text{etc.}$$

ubi elementum  $\partial x$  sumtum est constans.

§. 6. Pro nostro igitur casu est  $W = \frac{\sqrt{1 + p p x x}}{v}$  et  $\mathbb{B} = \frac{v \sqrt{1 + p p x x} - x}{v}$ , quae formulae tantum tres variables involvunt, scilicet  $v$ ,  $x$  et  $p$ ; et quoniam litterae  $M$  et  $\mathfrak{M}$  in aequationem finalem non ingrediuntur, eas etiam evolvere non est opus. Hinc ex priore formula erit  $L = -\frac{\sqrt{1 + p p x x}}{v v}$ ;  $N = 0$ ,  $P = \frac{p x x}{v \sqrt{1 + p p x x}}$ . Ex altera vero formula fit:

$$\mathfrak{L} = -\frac{\sqrt{1 + p p x x} + x}{v v} + \frac{v' \sqrt{1 + p p x x}}{v^2},$$

posito scilicet  $\partial v = V' \partial v$ ; tum vero erit  $\mathfrak{N} = 0$  et  $\mathfrak{P} = \frac{V' p x x}{v \sqrt{1 + p p x x}}$ , quibus inventis nostra aequatio finalis erit  $\frac{\partial P'}{\partial x} = 0$ , ideoque  $P' = C$ , hoc est  $C = P - \frac{\Pi \mathfrak{P}}{\Lambda}$ . Unde patet, quantitatem  $\Pi$  evanescere, ubi fit  $P = C$ . Quare terminus Brachystochronae  $C$  ibi constitui debet, ubi fit  $\frac{p x x}{v \sqrt{1 + p p x x}} = C$ .

§. 7. Cum nunc sit  $\Lambda = e^{\int \xi dx}$  erit  $\frac{\partial \Lambda}{\Lambda} = \xi dx$ , ergo  $\partial \Lambda = \Lambda \xi dx$ .  
Hinc autem porro habebimus  $\Pi = \int L \Lambda dx$ . Quare cum ex aequatione finali fiat

$$\Pi = \frac{AP}{\eta} - \frac{CA}{\eta}, \text{ hoc est } \Pi = \frac{\Lambda}{V} - \frac{C \Lambda v \sqrt{1 + pp'xx}}{V p'xx},$$

statuamus brevitatis gratia  $\sqrt{1 + pp'xx} = \omega$  et  $\frac{\sqrt{1 + pp'xx}}{p'xx} = t$ ,  
ita ut sit  $t = \frac{\omega}{x \sqrt{\omega \omega - 1}}$ . Differentiemus nunc aequationem inventam, et cum sit  $\partial \Pi = L \Lambda dx$  et  $\partial \Lambda = \Lambda \xi dx$ , facta hac substitutione tota aequatio per  $\Lambda$  dividi poterit, ideoque non opus erit ejus valorem integralem determinare. Nunc ergo pro  $L$  et  $\xi$  valores inventos substituendo pervenietur ad hanc aequationem:

$$0 = \frac{\omega dx}{v \omega} + \frac{\xi dx}{V} - \frac{\partial v}{V v} - \frac{C \xi t v \partial x}{V} - C x \frac{(v \partial t + t \partial v)}{V} + \frac{C t v \partial v}{V v},$$

$$\text{ubi est } \xi = -\frac{v \omega + Y}{u v} + \frac{v' \omega}{u}.$$

§. 8. Nunc cum sit  $v dv = dx (V \omega - X)$ , erit  $dx = \frac{v dv}{V \omega - X}$ , quem valorem in nostra aequatione loco  $dx$  substituamus, scilicet pro  $dx$  ubique scribamus  $v dv$ , reliquos vero terminos multiplicemus per  $V \omega - X$  et loco  $V dv$  scribamus  $\partial V$ , quo facto aequatio sequentem induet formam:

$$0 = \frac{\omega \partial v}{v} + \frac{\omega \partial v}{V} - \frac{C v \omega \partial v}{V} + \frac{v \omega - X}{V v} (C v t \partial V - C v v \partial t - \partial V - \frac{v \partial v}{v}),$$

§. 9. Quia haec aequatio non parum est complexa, primo eos tantum terminos evolvamus, in quibus non inest constans  $C$ , sique reperientur

$$\frac{\omega \partial v}{v} + \frac{\omega \partial v}{V} - \frac{\omega \partial v}{V} + \frac{X \partial v}{V v} = \frac{\omega \partial v}{v} + \frac{X \partial v}{V v}, \text{ sive } \frac{X}{V} \left( \frac{\partial v}{V} + \frac{\partial v}{v} \right).$$

At vero termini constantem  $C$  continentis erunt

$$\frac{C v \omega t \partial v}{V} + C v \omega \partial t + \frac{C v \omega \partial v}{V} + \frac{C X v \partial t}{V} - \frac{C X t v \partial v}{V v}$$

sive deletis terminis se destruentibus

$$-\frac{C v t X \partial v}{V v} + \frac{C v X \partial t}{V} - v \omega \partial t,$$

quocirca tota aequatio ita se habebit:

$$\frac{x}{v} \left( \frac{\partial v}{\partial x} + \frac{\partial v}{\partial x} \right) - C v \omega \partial t + \frac{C v X \partial t}{v} - \frac{C v t X \partial v}{v v} = 0.$$

§. 10. Quod si jam haec aequatio dividatur per  $CvX$ , prodibit haec forma:

$$\frac{1}{Cv} \partial \cdot v v - \frac{\omega \partial t}{x} + \frac{\partial t}{v} - \frac{t \partial v}{v v} = 0,$$

cujus aequationis tam primum membrum quam duo postrema integrationem admittunt. Sumto igitur integrali erit  $-\frac{1}{Cv} + \frac{t}{v} - \int \frac{\omega \partial t}{x} = \Delta$ , ubi in signo summatorio tantum binae variables  $p$  et  $x$  involvuntur, quia est  $\omega = \sqrt{1 + p p x x}$  et  $t = \frac{\sqrt{1 + p p x x}}{p x x}$ , ac praeterea  $X$  functio ipsius  $x$ . Quamobrem per hanc aequationem tertia variabilis  $v$ , cum sua functione data  $V$ , determinari est censenda; Quodsi hi valores in aequatione  $v \partial v (V \sqrt{1 + p p x x} - X)$  substituerentur, oriretur aequatio binas tantum variables  $x$  et  $p$ , vel  $x$  et  $y$  involvens, qua ergo natura curvae Brachystochronae quaesitae exprimitur; neque quicquam ulterius pro solutione hujus problematis postulari potest. Curva autem hac inventa terminus descensus  $C$  ibi statui debet, ubi fit, uti jam observavimus,  $P = C$ , seu ubi fit

$$C = \frac{p x x}{v \sqrt{1 + p p x x}}.$$