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De seriebus memorabilibus, quibus sinus et cosinus angulorum multiplorum exprimere licet

Leonhard Euler

DE SERIEBUS MEMORABILIBUS QUIBUS SINUS ET COSINUS ANGULORAM MULTIPLORUM EXPRIMERE LICET.

AUCTORE

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§. 1. Series, quas hic sum expositurus, non tam ob usum in multiplicatione angulorum, quam ob eximia calculi artificia, quae me ad eas perduxerunt, imprimis autem propter egregiam simplicitatem legis, qua earum termini progrediuntur, omni attentione dignae videntur. Ad eas autem commodius investigandas utor characteribus, quibus coëfficientes potestatum binomialium designare soleo. Ita si x fuerit exponens potestatis, hi characteres sequentes habeant significationes:

$$\left(\frac{x}{1}\right) = x; \ \left(\frac{x}{2}\right) = \frac{x(x-1)}{1 \cdot 2}; \ \left(\frac{x}{3}\right) = \frac{x(x-1)(x-2)}{1 \cdot 2 \cdot 3}; \text{ etc.}$$

sicque in genere erit:

$$\left(\frac{x}{n}\right) = \frac{x(x-1)(x-2)(x-3)(x-4) \ldots (x-(n-1))}{1 \cdot 2 \cdot 3 \cdot 4 \ldots n}.$$

§. 2. Proposito nunc angulo quocunque ϕ, pro ejus multiplo quocunque $x\phi$ tales series, secundum memoratos characteres procedentes, indagabo, quae tam cosinum quam sinum hujus anguli multipli exprimant. Ac primo quidem pro cosinu istam fingo seriem:

$$\cos. x\phi = 1 + \left(\tfrac{x}{1}\right) A + \left(\tfrac{x}{2}\right) B + \left(\tfrac{x}{3}\right) C + \left(\tfrac{x}{4}\right) D \text{ etc.}$$

quae semper abrumpitur, quoties x denotat numerum integrum positivum; reliquis autem casibus in infinitum excurrit. Ad has autem literas A, B, C, D, etc. investigandas loco x successive assumo valores 1, 2, 3, 4, etc., ubi quidem valores $\cos.\phi$, $\cos. 2\phi$, $\cos. 3\phi$, $\cos. 4\phi$, etc. tanquam cognitos specto.

§. 3. Facta igitur hac evolutione sequentes valores pro literis A, B, C, D, etc. reperientur:

Si	erit
$x = 1$	$\cos.\phi = 1 + A$, ergo $A = \cos.\phi - 1$,
$x = 2$	$\cos.2\phi = 1 + 2A + B$, ergo $B = \cos.2\phi - 2\cos.\phi + 1$,
$x = 3$	$\cos.3\phi = 1 + 3A + 3B + C$, ergo $C = \cos.3\phi - 3\cos.2\phi + 3\cos.\phi - 1$,
$x = 4$	$\cos.4\phi = 1 + 4A + 6B + 4C + C + D$, ergo $D = \cos.4\phi - 4\cos.3\phi + 6\cos.2\phi - 4\cos.\phi + 1$,
$x = 5$	$\cos.5\phi = 1 + 5A + 10B + 10C + 5D + E$, ergo $E = \cos.5\phi - 5\cos.4\phi + 10\cos.8\phi - 10\cos.2\phi + 5\cos.\phi - 1$
etc.	etc.

§. 4. Hinc ergo in genere, pro casu $x = n$, si litera coëfficienti $\left(\tfrac{x}{n}\right)$ jungenda fuerit N, sequitur fore:

$N = \cos. N\phi - (\frac{n}{1})\cos.(n-1)\phi + (\frac{n}{2})\cos.(n-2)\phi - (\frac{n}{3})\cos.(n-3)\phi + \text{etc.}$

Nunc igitur praecipuum negotium huc redit, ut istius expressionis indefinitae valor ad formulam finitam reducatur, id quod fit, si illius seriei summam, quae est N, elicuerimus. Quanquam autem plures jam hujusmodi series, secundum cosinus procedentes, sunt summatae, tamen methodi, quibus auctores, ad eas investigandas, sunt usi, vix, ac ne vix quidem, ad hunc casum accommodari posse videntur. Singularem igitur methodum hic proponam, quae me ad hunc scopum perduxit.

§. 5. Considero scilicet has binas formulas imaginarias: $p = \cos.\phi + \sqrt{-1}\sin.\phi$ et $q = \cos.\phi - \sqrt{-1}\sin.\phi$, ex quibus constat fore $p^n + q^n = 2\cos. n\phi$, ideoque $\cos. n\phi = \frac{1}{2}(p^n + q^n)$. Similique modo erit

$$\cos.(n-1)\phi = \tfrac{1}{2}(p^{n-1} + q^{n-1});$$

et ita porro, quibus valoribus substitutis, et potestatibus literarum p et q seorsim positis, facta multiplicatione per 2, habebimus:

$$\begin{aligned} 2N = &+ p^n - (\tfrac{n}{1})p^{n-1} + (\tfrac{n}{2})p^{n-2} - (\tfrac{n}{3})p^{n-3} + \text{etc.} \\ &+ q^n - (\tfrac{n}{1})q^{n-1} + (\tfrac{n}{2})q^{n-2} - (\tfrac{n}{3})q^{n-3} + \text{etc.} \end{aligned}$$

Hic autem evidens est superioris seriei summam esse $(p-1)^n$, inferioris vero $(q-1)^n$, ita ut jam futurum sit

$$2N = (p-1)^n + (q-1)^n,$$

quas formulas ergo ulterius prosequi oportet.

§. 6. Cum igitur sit $p = \cos.\phi + \sqrt{-1}\,\sin.\phi$, erit $p - 1 = \cos.\phi - 1 + \sqrt{-1}\,\sin.\phi$. Jam statuamus $\phi = 2\omega$, et cum sit $\cos.\phi = 1 - 2\sin.\omega^2$ et $\sin.\phi = 2\sin.\omega\cos.\omega$, habebimus $p - 1 = 2\sin.\omega(\sqrt{-1}\cos.\omega - \sin.\omega)$ quae expressio reducitur ad hanc:

$$p - 1 = 2\sqrt{-1}\,\sin.\omega\,(\cos.\omega + \sqrt{-1}\,\sin.\omega).$$

Simili autem modo reperietur

$$q - 1 = -2\sqrt{-1}\,\sin.\omega\,(\cos.\omega - \sqrt{-1}\,\sin.\omega).$$

Ex his igitur formulis conficietur

$$(p-1)^n = 2^n(\sqrt{-1})^n\,\sin.\omega^n(\cos.n\omega + \sqrt{-1}\,\sin.n\omega),$$
$$(q-1)^n = 2^n(-\sqrt{-1})^n\,\sin.\omega^n(\cos.n\omega - \sqrt{-1}\,\sin.n\omega),$$

quarum ergo formularum summa praebet valorem ipsius 2 N, quem quaerimus.

§. 7. Potestates autem imaginariorum $\sqrt{-1}$ et $-\sqrt{-1}$ modo fiunt $+1$, modo -1, modo imaginariae $\pm\sqrt{-1}$, prout exponens n fuerit numerus vel formae $4i$, vel $4i+1$, vel $4i+2$, vel $4i+3$, quandoquidem constat esse:

$$(\sqrt{-1})^{4i} = +1;\quad (-\sqrt{-1})^{4i} = +1,$$
$$(\sqrt{-1})^{4i+1} = \sqrt{-1};\quad (-\sqrt{-1})^{4i+1} = -\sqrt{-1},$$
$$(\sqrt{-1})^{4i+2} = -1;\quad (-\sqrt{-1})^{4i+2} = -1,$$
$$(\sqrt{-1})^{4i+3} = -\sqrt{-1};\quad (-\sqrt{-1})^{4i+3} = +\sqrt{-1}.$$

§. 8. Hac observatione praemissa tribuamus nunc successive exponenti n valores 1, 2, 3, 4, etc. quo pacto N denotabit successive literas A, B, C, D, etc. quarum ergo valores sequenti modo per angulum $\omega = \frac{1}{2}\Phi$ expressos reperiemus. Sit igitur primo $n = 1$, erit:

$$2A = 2\sqrt{-1}\sin.\omega(\cos.\omega + \sqrt{-1}\sin.\omega)$$
$$-2\sqrt{-1}\sin.\omega(\cos.\omega - \sqrt{-1}\sin.\omega),$$

qui ergo valor reducitur ad hanc formam:

$$2A = -4\sin.\omega\sin.\omega,\ \text{ideoque}\ A = -2\sin.\omega\sin.\omega.$$

§. 9. Sumto autem $n = 2$ fiet

$$2B = -4\sin.\omega^2(\cos.2\omega + \sqrt{-1}\sin.2\omega)$$
$$-4\sin.\omega^2(\cos.2\omega - \sqrt{-1}\sin.2\omega),$$

unde colligitur $B = -4\sin.\omega^2\cos.2\omega$.

§. 10. Sit $n = 3$, eritque

$$2C = -8\sqrt{-1}\sin.\omega^3(\cos.3\omega + \sqrt{-1}\sin.3\omega)$$
$$+8\sqrt{-1}\sin.\omega^3(\cos.3\omega - \sqrt{-1}\sin.3\omega),$$

ex quo fit $C = 8\sin.\omega^3\sin.3\omega$.

§. 11. Sumatur $n = 4$, atque nanciscemur

$$2D = 16\sin.\omega^4(\cos.4\omega + \sqrt{-1}\sin.4\omega)$$
$$+16\sin.\omega^4(\cos.4\omega - \sqrt{-1}\sin.4\omega),$$

hincque oritur $D = 16\sin.\omega^4\cos.4\omega$.

§. 12. Sumto porro $n = 5$, fit

$$2E = 32\sqrt{-1}\sin.\omega^5(\cos.5\omega + \sqrt{-1}\sin.5\omega)$$
$$-32\sqrt{-1}\sin.\omega^5(\cos.5\omega - \sqrt{-1}\sin.5\omega),$$

ergo colligendo prodit $E = -32\sin.\omega^5\sin.5\omega$.

§. 13. Pro casu $n = 6$ invenitur

$$2F = -64 \sin.\omega^6 (\cos.6\omega + \sqrt{-1}\sin.6\omega)$$
$$- 64 \sin.\omega^6 (\cos.6\omega - \sqrt{-1}\sin.6\omega),$$

sive $F = -64 \sin.\omega^6 \cos.6\omega$.

§. 14. Statuatur porro $n = 7$, eritque

$$2G = -128\sqrt{-1}\sin.\omega^7 (\cos.7\omega + \sqrt{-1}\sin.7\omega)$$
$$+ 128\sqrt{-1}\sin.\omega^7 (\cos.7\omega - \sqrt{-1}\sin.7\omega);$$

ideoque $G = +128 \sin.\omega^7 \sin.7\omega$.

§. 15. Denique posito $n = 8$ prodit

$$2H = +256 \sin.\omega^8 (\cos.8\omega + \sqrt{-1}\sin.8\omega)$$
$$+ 256 \sin.\omega^8 (\cos.8\omega - \sqrt{-1}\sin.8\omega),$$

hincque $G = +256 \sin.\omega^8 \cos.8\omega$.

§. 16. Istos igitur valores, per periodos quadripartitas progredientes, in sequentibus duabus columnis junctim repraesentemus:

$A = -2 \sin.\omega \sin.\omega$	$F = -2^6 \sin.\omega^6 \cos.6\omega$
$B = -2^2 \sin.\omega^2 \cos.2\omega$	$G = +2^7 \sin.\omega^7 \sin.7\omega$
$C = +2^3 \sin.\omega^3 \sin.3\omega$	$H = +2^8 \sin.\omega^8 \cos.8\omega$
$D = +2^4 \sin.\omega^4 \cos.4\omega$	$I = -2^9 \sin.\omega^9 \sin.9\omega$
$E = -2^5 \sin.\omega^5 \sin.5\omega$	$K = -2^{10} \sin.\omega^{10} \cos.10\omega$
	etc.

consequenter valor formulae propositae, scilicet $\cos.x\Phi$, sive $\cos.2x\omega$, per sequentem seriem satis concinnam exprimetur:

$$\cos. 2x\omega = \left\{\begin{array}{l} 1 - 2\left(\frac{x}{1}\right)\sin.\omega\ \sin.\ \omega - \ 4\ \left(\frac{x}{2}\right)\sin.\omega^2\cos.2\omega \\ + \ 8\left(\frac{x}{3}\right)\sin.\omega^3\sin.3\omega + \ 16\ \left(\frac{x}{4}\right)\sin.\omega^4\cos.4\omega \\ - 32\left(\frac{x}{5}\right)\sin.\omega^5\sin.5\omega - \ 64\ \left(\frac{x}{6}\right)\sin.\omega^6\cos.6\omega \\ + 128\left(\frac{x}{7}\right)\sin.\omega^7\sin.7\omega + 256\left(\frac{x}{8}\right)\sin.\omega^8\cos.8\omega \\ - \text{etc.} \qquad\qquad\qquad - \text{etc.} \end{array}\right\}$$

§. 17. Antequam hanc formulam maxime generalem ad casus particulares aceommodemus, observationem prorsus singularem, eamque maximi momenti, in medium attulisse operae pretium est, inde petitam, quod per evolutionem communem sit

$$\cos. x\Phi = 1 - \tfrac{1}{2}x^2\Phi^2 + \tfrac{1}{24}x^4\Phi^4 - \tfrac{1}{720}x^6\Phi^6 + \text{etc.}$$

ubi tantum potestates pares ipsius x occurrunt; quam ob rem necesse est, ut in nostra serie inventa, facta evolutione characterum $\frac{x}{n}$, omnes termini, potestatibus imparibus ipsius x affecti, seorsim se mutuo destruant; quare etiam omnes termini inde resultantes sola litera x affecti junctimque sumti nihilo aequari debebunt, unde istos terminos ex singulis characteribus oriundos hic exponamus:

$\left(\frac{x}{1}\right)$ dat $+ x$	$\left(\frac{x}{2}\right)$ dat $- \frac{1}{2}x$	$\left(\frac{x}{3}\right)$ dat $+ \frac{1}{3}x$	$\left(\frac{x}{4}\right)$ dat $- \frac{1}{4}x$
$\left(\frac{x}{5}\right)$ . . $+ \frac{1}{5}x$	$\left(\frac{x}{6}\right)$ . . $- \frac{1}{6}x$	$\left(\frac{x}{7}\right)$ . . $+ \frac{1}{7}x$	$\left(\frac{x}{8}\right)$ . . $- \frac{1}{8}x$
$\left(\frac{x}{9}\right)$ . . $+ \frac{1}{9}x$	$\left(\frac{x}{10}\right)$ . . $- \frac{1}{10}x$	$\left(\frac{x}{11}\right)$ . . $+ \frac{1}{11}x$	$\left(\frac{x}{12}\right)$ . . $- \frac{1}{12}x$
etc.	etc.	etc.	etc.

§. 18. Colligamus igitur omnes istos terminos, ac dividendo per x perveniemus ad sequentem seriem maxime memorabilem:

$$0 = -2\sin.\omega\sin.\omega + \tfrac{1}{2}.\,2^2\sin.\omega^2\cos.2\omega + \tfrac{1}{3}.\,2^3\sin.\omega^3\sin.3\omega$$
$$-\tfrac{1}{4}.\,2^4\sin\omega^4\cos.4\omega - \tfrac{1}{5}.\,2^5\sin.\omega^5\sin.5\omega + \tfrac{1}{6}.\,2^6\sin.\omega^6\cos.6\omega + \text{etc.}$$

unde duas series inter se aequales deducimus, quae sunt

$$2\sin.\omega\sin.\omega - \tfrac{1}{3}.\,2^3\sin.\omega^3\sin.3\omega + \tfrac{1}{5}.\,2^5\sin.\omega^5\sin.5\omega - \text{etc.}$$
$$= \tfrac{1}{2}.\,2^2\sin.\omega^2\cos.2\omega - \tfrac{1}{4}.\,2^4\sin.\omega^4\cos.4\omega + \tfrac{1}{6}.\,2^6\sin.\omega^6\cos.6\omega - \text{etc.}$$

Hinc ergo pulcherrimum theorema condi potest:

Theorema.

Denotante ω *angulum quemcunque duae sequentes series:*

$$s = \tfrac{2}{1}\sin.\omega\ \sin.\omega - \tfrac{2^3}{3}\sin.\omega^3\sin.3\omega + \tfrac{2^5}{5}\sin.\omega^5\sin.5\omega - \text{etc.}$$
$$t = \tfrac{2^2}{2}\sin.\omega^2\cos.2\omega - \tfrac{2^4}{4}\sin.\omega^4\cos.4\omega + \tfrac{2^6}{6}\sin.\omega^6\cos.6\omega - \text{etc.}$$

semper erunt inter se aequales, sive erit $s = t$.

Demonstratio.

§. 19. Hic ubique loco $2\sin.\omega$ scribamus litteram b, ut sit:

$$s = \frac{b\sin.\omega}{1} - \frac{b^3\sin.3\omega}{3} + \frac{b^5\sin.5\omega}{5} - \frac{b^7\sin.7\omega}{7} + \text{etc.}$$
$$t = \frac{b^2\cos.2\omega}{2} - \frac{b^4\cos.4\omega}{4} + \frac{b^6\cos.6\omega}{6} - \frac{b^8\cos.8\omega}{8} + \text{etc.}$$

quarum serierum summas investigemus, nullo habito respectu ad relationem, quae inter litteras b et ω intercedit, quam ob rem nihil impediet, quo minus littera b tanquam constans spectetur; utriusque autem summa inventa loco b restituemus valorem assumtum $2\sin.\omega$, atque videbimus hoc casu revera futurum esse $t = s$.

§. 20. Incipiamus igitur a serie priore, de qua observemus, sumto angulo $\omega = 0$ fore etiam $s = 0$, atque differentiata hac serie reperiemus fore:

$$\frac{\partial s}{\partial \omega} = b \cos. \omega - b^3 \cos. 3\omega + b^5 \cos. 5\omega - b^7 \cos. 7\omega + \text{etc.}$$

quae multiplicetur per $1 + 2bb \cos. 2\omega + b^4$, atque ob

$$2 \cos. 2\omega \cos. n\omega = \cos. (n+2)\omega + \cos. (n-2)\omega,$$

obtinebimus sequentem aequationem:

$$\frac{\partial s}{\partial \omega}(1 + 2bb \cos. 2\omega + b^4)$$

$$\begin{aligned} = b \cos \omega - b^3 \cos. 3\omega + b^5 \cos. 5\omega - b^7 \cos. 7\omega + b^9 \cos. 9\omega - \text{etc.} \\ + b^3 \cos. 3\omega - b^5 \cos. 5\omega + b^7 \cos. 7\omega - b^9 \cos. 9\omega + \text{etc.} \\ + b^3 \cos. \omega - b^5 \cos. \omega + b^7 \cos. 3\omega - b^9 \cos. 5\omega - \text{etc.} \\ + b^5 \cos. \omega - b^7 \cos. 3\omega + b^9 \cos. 5\omega - \text{etc.} \end{aligned}$$

quibus terminis collectis nanciscemur

$$\frac{\partial s}{\partial \omega}(1 + 2bb \cos. 2\omega + b^4) = b \cos. \omega + b^3 \cos. \omega = b(1 + bb) \cos. \omega,$$

sicque erit $\partial s = \frac{b(1+bb)\partial\omega \cos. \omega}{1 + 2bb \cos. 2\omega + b^4}$.

§. 21. Simili modo tractemus alteram seriem, de qua notasse juvabit, sumto $\omega = 0$ fore $t = \frac{1}{2} l(1 + bb)$, cum sit

$$t = \frac{b^2}{2} - \frac{b^4}{4} + \frac{b^6}{6} - \frac{b^8}{8} + \text{etc.}$$

Facta jam differentiatione prodibit

$$\frac{\partial t}{\partial \omega} = - bb \sin. 2\omega + b^4 \sin. 4\omega - b^6 \sin. 6\omega + \text{etc.}$$

Hic jam iterum utrinque multiplicetur per $1 + 2bb \cos. \omega + b^4$ et calculus ita adornetur:

$$1 \cdot \frac{\partial t}{\partial \omega} = - bb \sin. 2\omega + b^4 \sin. 4\omega - b^6 \sin. 6\omega + b^8 \sin. 8\omega - \text{etc.}$$

$$2b^2 \cos. 2\omega \cdot \frac{\partial t}{\partial \omega} = \quad - b^4 \sin. 4\omega + b^6 \sin. 6\omega - b^8 \sin. 8\omega + \text{etc.}$$

$$+ b^6 \sin. 2\omega - b^8 \sin. 4\omega + \text{etc.}$$

$$+ b^4 \cdot \frac{\partial t}{\partial \omega} = \quad - b^6 \sin. 2\omega + b^8 \sin. 4\omega - \text{etc.}$$

unde collectis membris nascitur haec aequatio:

$$\frac{\partial t}{\partial \omega}(1 + 2bb \cos. 2\omega + b^4) = - bb \sin. 2\omega,$$

consequenter erit $\partial t = - \frac{bb \partial \omega \sin. 2\omega}{1 + 2bb \cos. 2\omega + b^4}$.

§. 22. Inventis his duabus formulis differentialibus, utriusque integrationem investigemus, ac pro priore quidem, ob $\partial \omega \cos. \omega = \partial . \sin. \omega$, habebimus:

$$\partial s = \frac{b(1 + bb) \partial . \sin. \omega}{1 + 2bb \cos. 2\omega + b^4},$$

quae expressio, ob $\cos. 2\omega = 1 - 2 \sin. \omega^2$, transformatur in hanc:

$$\partial s = \frac{b(1 + bb) \partial . \sin. \omega}{(1 + bb)^2 - 4bb \sin. \omega^2}.$$

Quia vero constat esse $\int \frac{\partial z}{ff - ggzz} = \frac{1}{2fg} l \cdot \frac{f + gz}{f - gz}$, nostro autem casu sit $f = 1 + bb$ et $g = 2b$ et $z = \sin. \omega$, invenitur hoc integrale:

$$s = \frac{1}{4} l \frac{1 + bb + 2b \sin. \omega}{1 + bb - 2b \sin. \omega},$$

quae formula casu $\omega = 0$ evanescit, ideoque constantis additione non indiget.

§. 23. Pro altera formula, ob $-\partial \omega \sin. 2\omega = {}^1\partial . \cos. 2\omega$ habebimus $\partial t = + \frac{\frac{1}{2} bb \partial . \cos. 2\omega}{1 + 2bb \cos. 2\omega + b^4}$, ubi numerator aequatur quartae parti differentialis denominatoris, unde integrale erit $t = \frac{1}{4} l (1 + 2bb \cos. 2\omega + b^4)$. Necesse autem est ut posito $\omega = 0$ fiat $t = \frac{1}{2} l (1 + bb)$, atque commode hic evenit ut isto casu idem valor prodeat, sicque adjectione constantis non est opus. Notasse autem hic juvabit esse etiam:

$$t = \tfrac{1}{4} l (1 + bb + 2b \sin. \omega) + \tfrac{1}{4} l (1 + bb - 2b \sin. \omega).$$

§. 24. His jam integralibus inventis,

ob $s = \frac{1}{4} l (1 + bb - 2b \sin. \omega) - \frac{1}{4} l (1 + bb - 2b \sin. \omega)$, erit eorum differentia:

$$t - s = \tfrac{1}{2} l (1 + bb - 2b \sin. \omega).$$

At vero pro casu nostri theorematis est $b = 2 \sin. \omega$, quo valore substituto prodit $t - s = \frac{1}{2} l\, 1 = 0$, quae est demonstratio nostri theorematis.

Exemplum 1.

§. 25. Contemplemur nunc etiam nonnullos casus particulares, ac primo quidem, si sumeremus $\omega = 180$ omnes plane termini in nihilum abirent. Quamobrem incipiamus a casu $\omega = 90^\circ = \frac{\pi}{2}$; ubi ergo erit:

$\sin. \omega = 1$; $\cos. 2\omega = -1$; $\cos. 4\omega = +1$; $\cos. 6\omega = -1$; etc.
$\sin. 3\omega = -1$; $\sin. 5\omega = +1$; $\sin. 7\omega = -1$; $\sin. 9\omega = +1$; etc.
quamobrem series pro $\cos. x\pi$ inventa erit:

$$\cos. x\pi = 1 - 2\left(\tfrac{x}{1}\right) + 4\left(\tfrac{x}{2}\right) - 8\left(\tfrac{x}{3}\right) + 16\left(\tfrac{x}{4}\right) - 32\left(\tfrac{x}{5}\right) + \text{etc.}$$

quae series manifesto nascitur ex evolutione potestatis $(1-2)^x = -1^x$, cujus valores sunt alternatim $+1$ et -1 id quod egregie convenit cum formula $\cos. x\pi$, siquidem ipsi x tribuantur numeri integri.

§. 26. Hoc autem casu binae illae series, quas inter se aequales esse §. 18. invenimus, erunt:

$$2+\tfrac{1}{3}\cdot 2^3+\tfrac{1}{5}\cdot 2^5+\tfrac{1}{7}\cdot 2^7+\text{etc.}=-\tfrac{1}{2}\cdot 2^2-\tfrac{1}{4}\cdot 2^4-\tfrac{1}{6}\cdot 2^6-\text{etc.}$$

Cum autem haec series maxime sit divergens, nullum consensum apertum cum veritate expectare licet, quod quidem maxime paradoxon videtur, at vero novimus utique dari ejusmodi series divergentes omnes terminos positivos habentes, quarum summa tamen non solum sit nulla sed adeo negativa. Ceterum veritas in superiori theoremate jam solidissime est demonstrata.

Exemplum 2.

§. 27. Sumatur nunc $\omega=60^\circ=\frac{\pi}{3}$, erit $2\sin.\omega=b=\sqrt{3}$, ob $\sin.\omega=\frac{\sqrt{3}}{2}$. Tum vero erit:

$\sin.3\omega=0$; $\sin.5\omega=-\frac{\sqrt{3}}{2}$; $\sin.7\omega=+\frac{\sqrt{3}}{2}$; $\sin.9\omega=0$; etc.

$\cos.2\omega=-\frac{1}{2}$; $\cos.4\omega=-\frac{1}{2}$; $\cos.6\omega=1$; $\cos.8\omega=-\frac{1}{2}$; etc.

Hinc ergo sequentem nanciscimur seriem:

$$\cos.\frac{2\pi x}{3}=1-\tfrac{3}{2}\left(\tfrac{x}{1}\right)+\tfrac{3}{2}\left(\tfrac{x}{2}\right)+\tfrac{9}{2}\left(\tfrac{x}{4}\right)+\tfrac{27}{2}\left(\tfrac{x}{5}\right)-\tfrac{27}{2}\left(\tfrac{x}{6}\right)$$
$$+\tfrac{81}{2}\left(\tfrac{x}{7}\right)-\tfrac{81}{2}\left(\tfrac{x}{8}\right)+\text{etc.}$$

Illae autem binae series pro s et t inventae hoc casu erunt:

$$s=\tfrac{3}{2}-\tfrac{27}{2\cdot 5}-\tfrac{81}{2\cdot 7}+\tfrac{729}{2\cdot 11}+\tfrac{2187}{2\cdot 13}-\text{etc. sive}$$
$$2s=3^1-\tfrac{3^3}{5}-\tfrac{3^4}{7}+\tfrac{3^6}{11}+\tfrac{3^7}{13}-\tfrac{3^9}{17}+\tfrac{3^{10}}{19}+\text{etc.}$$

tum vero

$$t = -\frac{3}{2.2} + \frac{9}{2.4} + \frac{27}{1.6} + \frac{81}{2.8} - \frac{243}{2.10} - \frac{729}{1.12} - \text{etc. sive}$$

$$2t = -3^1 + \frac{3^2}{4} + \frac{3^3}{3} + \frac{3^4}{8} - \frac{3^5}{10} - \frac{3^6}{6} - \text{etc.}$$

quae ergo duae series certe sunt aequales, etiamsi hoc absurdum videri queat, cujus rei causa in eo est quaerenda, quod hae series sunt divergentes.

Exemplum 3.

§. 28. Sumatur $\omega = 45^\circ = \frac{\pi}{4}$, eritque sin. $\omega = \frac{1}{\sqrt{2}}$ ideoque $b = \sqrt{2}$. Porro vero notetur esse:

sin. $3\omega = \frac{1}{\sqrt{2}}$; sin. $5\omega = -\frac{1}{\sqrt{2}}$; sin. $7\omega = -\frac{1}{\sqrt{2}}$; sin. $9\omega = \frac{1}{\sqrt{2}}$; etc.

cos. $2\omega = 0$; cos. $4\omega = -1$; cos. $6\omega = 0$; cos. $8\omega = +1$; etc.

unde series nostra principalis erit:

$$\cos.\frac{\pi x}{2} = 1 - \left(\frac{x}{1}\right) + 2\left(\frac{x}{3}\right) - 4\left(\frac{x}{4}\right) + 4\left(\frac{x}{5}\right) - 8\left(\frac{x}{7}\right) + 16\left(\frac{x}{8}\right) - \text{etc.}$$

Haec autem seriem adhuc est divergens. Illae autem duae series s et t, quas aequales esse ostendimus, ita se habebunt:

$$s = 1 - \frac{2}{3} - \frac{4}{5} + \frac{8}{7} + \frac{16}{9} - \frac{32}{11} - \frac{64}{18} + \text{etc.}$$

$$t = \frac{4}{4} - \frac{16}{8} + \frac{64}{12} - \frac{256}{16} + \frac{1024}{20} - \frac{4096}{24} - \text{etc. sive}$$

$$s = 1 - \frac{2}{3} - \frac{2^2}{5} + \frac{2^3}{7} + \frac{2^4}{9} - \frac{2^5}{11} - \frac{2^6}{13} + \text{etc.}$$

$$t = \frac{4}{4} - \frac{4^2}{8} + \frac{4^3}{12} - \frac{4^4}{16} + \frac{4^5}{20} - \frac{4^6}{24} + \frac{4^7}{28} - \text{etc.}$$

ubi nihil absoni occurrit.

Exemplum 4.

§. 29. Sit denique $\omega = 30^\circ = \frac{\pi}{6}$, unde ob sin. $\omega = \frac{1}{2}$ erit $b = 1$, qui ergo casus ad series convergentes perducet. Est vero

$\sin. 3\omega = 1$; $\sin. 5\omega = \frac{1}{2}$; $\sin. 7\omega = -\frac{1}{2}$; $\sin. 9\omega = -1$; $\sin. 11\omega = -\frac{1}{2}$
$\cos. 2\omega = \frac{1}{2}$; $\cos. 4\omega = -\frac{1}{2}$; $\cos. 6\omega = -1$; $\cos. 8\omega = -\frac{1}{2}$; $\cos. 10\omega = +\frac{1}{2}$.
Hinc ergo nostra series erit:

$$\cos. \frac{\pi x}{3} = 1 - \frac{1}{2}\left(\frac{x}{1}\right) - \frac{1}{2}\left(\frac{x}{2}\right) + \left(\frac{x}{3}\right) - \frac{1}{2}\left(\frac{x}{4}\right) - \frac{1}{2}\left(\frac{x}{5}\right) + \left(\frac{x}{6}\right) - \frac{1}{2}\left(\frac{x}{7}\right)$$
$$- \frac{1}{2}\left(\frac{x}{8}\right) + \left(\frac{x}{9}\right) - \frac{1}{2}\left(\frac{x}{10}\right) - \frac{1}{2}\left(\frac{x}{11}\right) + \text{etc.}$$

quae expressio commode in ternas sequentes series decomponitur:

$$\cos. \frac{\pi x}{3} = \left\{ \begin{array}{l} 1 \left(1 + \left(\frac{x}{3}\right) + \left(\frac{x}{6}\right) + \left(\frac{x}{9}\right) + \left(\frac{x}{12}\right) + \text{etc.}\right) \\ - \frac{1}{2}\left(\left(\frac{x}{1}\right) + \left(\frac{x}{4}\right) + \left(\frac{x}{7}\right) + \left(\frac{x}{10}\right) + \left(\frac{x}{13}\right) + \text{etc.}\right) \\ - \frac{1}{2}\left(\left(\frac{x}{2}\right) + \left(\frac{x}{5}\right) + \left(\frac{x}{8}\right) + \left(\frac{x}{11}\right) + \left(\frac{x}{14}\right) + \text{etc.}\right) \end{array} \right\}$$

Binae autem series s et t hoc casu erunt:

$$s = \frac{1}{2 \cdot 1} - \frac{1}{3} + \frac{1}{2 \cdot 5} + \frac{1}{2 \cdot 7} - \frac{1}{9} + \frac{1}{2 \cdot 11} + \frac{1}{13} - \text{etc.}$$
$$t = \frac{1}{2 \cdot 2} + \frac{1}{2 \cdot 4} - \frac{1}{6} + \frac{1}{2 \cdot 8} + \frac{1}{2 \cdot 10} - \frac{1}{12} + \frac{1}{2 \cdot 14} + \text{etc.}$$

Hinc ergo erit:

$$2(s - t) = 1 - \frac{1}{2} - \frac{2}{3} - \frac{1}{4} + \frac{1}{5} + \frac{2}{6} + \frac{1}{7} - \frac{1}{8} - \frac{2}{9} - \frac{1}{10} + \frac{1}{11} + \frac{2}{12} + \text{etc.}$$

hancque seriem, cujus summa est $= 0$, hoc modo in tres series relolvere licet:

$$0 = \left\{ \begin{array}{l} \frac{1}{1} - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \frac{1}{13} - \text{etc.} \\ - 1\left(\frac{1}{2} - \frac{1}{5} + \frac{1}{8} - \frac{1}{11} + \frac{1}{14} - \text{etc.}\right) \\ - 2\left(\frac{1}{3} - \frac{1}{6} + \frac{1}{9} - \frac{1}{12} + \frac{1}{15} - \text{etc.}\right) \end{array} \right\}$$

§. 30. Eodem plane modo quo supra seriem pro $\cos 2x\omega$ investigavimus, etiam series pro sinu ejusdem anguli multipli eruitur sequenti modo. Fingatur, ut supra, haec series:

$$\sin. x\phi = (\tfrac{x}{1})A + (\tfrac{x}{2})B + (\tfrac{x}{3})C + \text{etc.}$$

quae semper abrumpitur, quoties x denotat numerum integrum positivum. Evolvendo autem, ut jam supra fecimus, literae A, B, C, etc. ita reperientur expressae, ut facile pateat characteri $(\tfrac{x}{n})$ respondere seriem:

$$N = \sin. n\phi - (\tfrac{n}{1})\sin.(n-1)\phi + (\tfrac{n}{2})\sin. n-2)\phi - \text{etc.}$$

postremo membro existente $\pm \sin. 0\,\phi$.

§. 31. Cum jam sit $\sin. \lambda\phi = \frac{p^\lambda - q^\lambda}{2\sqrt{-1}}$, erit

$$2N\sqrt{-1} = \left\{\begin{matrix} +p^n - (\tfrac{n}{1})p^{n-1} + (\tfrac{n}{2})p^{n-2} - \text{etc.} = (p-1)^n \\ -q^n + (\tfrac{n}{1})q^{n-1} - (\tfrac{n}{2})q^{n-2} + \text{etc.} = -(q-1)^n \end{matrix}\right\}.$$

At vero ex superioribus manifestum est fore

$$p - 1 = 2\sin. \omega\sqrt{-1}(\cos. \omega + \sqrt{-1}\sin. \omega)$$
$$q - 1 = -2\sin. \omega\sqrt{-1}(\cos. \omega - \sqrt{-1}\sin. \omega)$$

ideoque

$$2N\sqrt{-1} = (\ 2\sin. \omega\sqrt{-1})^n(\cos. n\omega + \sqrt{-1}\sin. n\omega)$$
$$-(-2\sin. \omega\sqrt{-1})^n(\cos. n\omega - \sqrt{-1}\sin. n\omega)$$

ubi notandum, pro quatuor formis, quas littera n habere potest, fore:

Si $n = 4i$, $\quad N = (2\sin. \omega)^n \sin. n\omega$;
.. $n = 4i + 1$, $\quad N = (2\sin. \omega)^n \cos. n\omega$;
.. $n = 4i + 2$, $\quad N = -(2\sin. \omega)^n \sin. n\omega$;
.. $n = 4i + 3$, $\quad N = -(2\sin. \omega)^n \cos. n\omega$.

§. 32. Quod si igitur successive litterae n tribuantur valores 1, 2, 3, 4, etc. erit

$$A = 2\sin.\omega\cos.\omega \qquad E = 2^5\sin.\omega^5\cos.5\omega$$
$$B = -2^2\sin.\omega^2\sin.2\omega \qquad F = -2^6\sin.\omega^6\sin.6\omega$$
$$C = -2^3\sin.\omega^3\cos.3\omega \qquad G = -2^7\sin.\omega^7\cos.7\omega$$
$$D = +2^4\sin.\omega^4\sin.4\omega \qquad H = +2^8\sin.\omega^8\sin.8\omega$$

etc.

consequenter series quaesita pro sinu, restituto loco ϕ valore 2ω, ita se habebit

$$\sin.2x\omega = \left\{\begin{array}{l} +\ 2\left(\frac{x}{1}\right)\sin.\omega\cos.\omega - 4\left(\frac{x}{2}\right)\sin.\omega^2\sin.2\omega \\ -\ 8\left(\frac{x}{3}\right)\sin.\omega^3\cos.3\omega + 16\left(\frac{x}{4}\right)\sin.\omega^4\sin.4\omega \\ +\ 32\left(\frac{x}{5}\right)\sin.\omega^5\cos.5\omega - 64\left(\frac{x}{6}\right)\sin.\omega^6\sin.6\omega \\ -\ 128\left(\frac{x}{7}\right)\sin.\omega^7\cos.7\omega + 256\left(\frac{x}{8}\right)\sin.\omega^8\sin.8\omega \\ \qquad\qquad \text{etc.} \qquad\qquad\qquad\qquad \text{etc.} \end{array}\right\}$$