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Methodus succincta summas serierum infinitarum per formulas differentiales investigandi

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METHODUS SUCCINCTA SUMMAS SERIERUM INFINITARUM PER FORMULAS DIFFERENTIALES INVESTIGANDI.

AUCTORE

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§. 1. Etsi hoc argumentum jam saepius pertractavi, tamen pleraque, quae ad summas commode exprimendas spectant, per varios libros sunt dispersa, atque etiam per ambages eruta; quamobrem hic succinctam methodum sum traditurus, cujus ope seriei cujuscunque summa facili calculo, sine ambagibus, per formam simplicissimam indagari poterit.

§. 2. Sit igitur X functio quaecunque ipsius x, et X', X'', X''', etc. inde oriantur, si loco x successive scribatur $x+1$, $x+2$, $x+3$, etc. Hinc ergo literae illae X, X', X'', X''', etc. mihi designabunt terminos cujusque seriei indicibus x, $x+1$, $x+2$, $x+3$, eic. respondentes. His positis duos casus serierum infinitarum sum contemplaturus, quorum priore termini omnes eodem signo $+$ affecti progrediuntur, ita ut series summanda sit:

$$X + X' + X'' + X''' + \text{etc.}$$

Altero vero casu iidem termini signis alternantibus procedant, ita ut series summanda sit $X - X' + X'' - X''' + \text{etc.}$ Hos igitur duos casus seorsim evolvam.

Casus I.

Summatio seriei infinitae

$$S = X + X' + X'' + X''' + \text{etc.}$$

§. 3. Denotet S' summam ejusdem seriei primo termino truncatae, ita ut sit $S' = X' + X'' + X''' + \text{etc.}$ et cum S sit certa functio ipsius x, quam hic potissimum investigamus, erit S' similis functio ipsius $x + 1$. Evidens ergo est, fore $S - S' = X$. Quare cum sit

$$S' = S + \partial S + \tfrac{1}{2}\partial\partial S + \text{etc.}$$

ubi denominatores, potestates elementi ∂x continentes, ut brevitati consulam, praetermitto, siquidem quasi sponte subintelliguntur, hinc nostra aequatio induet hanc formam:

$$0 = X + \partial S + \tfrac{1}{2}\partial\partial S + \tfrac{1}{6}\partial^3 S + \tfrac{1}{24}\partial^4 S + \text{etc.}$$

§. 4. Quodsi ergo ista series valde convergat, propemodum erit $\partial S = -X$, ideoque $S = -\int X \partial x$; quod integrale per constantem ita est determinandum, ut sumto x infinite magno evanescat, propterea quod termini infinitesimi pro nihilo haberi possunt, quia aliâs series ipsa nullam haberet summam finitam. Cognita propemodum summa, pro vera summa statuamus

$$S = -\int X\partial x - \alpha X - \beta\partial X - \gamma\partial\partial X - \text{etc.}$$

eritque hinc

$$\partial S = - X - \alpha\partial X - \beta\partial\partial X - \gamma\partial^3 X - \text{etc.}$$

Quod si jam pro singulis differentialibus ipsius S valores inde oriundi substituantur, pervenietur ad sequentem aequationem:

$$\left.\begin{array}{llllll}
+X - \alpha\partial X & - \beta\partial\partial X & - \gamma\partial^3 X & - \delta\partial^4 X & - \text{etc.} \\
-X - \frac{1}{2} & - \frac{1}{2}\alpha & - \frac{1}{2}\beta & - \frac{1}{2}\gamma & - \text{etc.} \\
 & - \frac{1}{6} & - \frac{1}{6}\alpha & - \frac{1}{6}\beta & - \text{etc.} \\
 & & - \frac{1}{24} & - \frac{1}{24}\alpha & - \text{etc.} \\
 & & & - \frac{1}{120} & - \text{etc.}
\end{array}\right\} = 0$$

et jam coëfficientes incogniti α, β, γ, etc. ex sequentibus aequalitatibus definiri debent:

$$\alpha + \tfrac{1}{2} = 0;\ \beta + \tfrac{1}{2}\alpha + \tfrac{1}{6} = 0;\ \gamma + \tfrac{1}{2}\beta + \tfrac{1}{6}\alpha + \tfrac{1}{24} = 0;\ \text{etc.}$$

unde fit $\alpha = -\frac{1}{2}$; $\beta = \frac{1}{12}$; $\gamma = 0$; etc.

§. 5. Hoc autem modo inventio literarum α, β, γ, etc. nimis foret operosa, neque tamen ulla lex perspiceretur, qua ulterius progrediantur; quamobrem modo prorsus singulari in valores istarum literarum inquiram. Considerabo scilicet seriem ordinariam, secundum eosdem coëfficientes procedentem, quae sit $V = 1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \text{etc.}$ atque evidens est, si hujus seriei summa V ad formam finitam perduci queat; tum, si eadem secundum potestates ipsius z evolvatur, eandem seriem necessario provenire debere, quo pacto valores litterarum α, β, γ, δ, etc. sponte innotescent.

§. 6. Ex relationibus igitur, quae inter litteras α, β, γ, δ, etc. intercedunt, supra §. 4. allatis, sequentes operationes instituantur:

$$\begin{array}{rcccccccc}
V = & 1 & +\alpha z & +\beta z^2 & +\gamma z^3 & +\delta z^4 & +\varepsilon z^5 & + \text{etc.} \\
\frac{1}{2} z\, V = & & +\frac{1}{2} & +\frac{1}{2}\alpha & +\frac{1}{2}\beta & +\frac{1}{2}\gamma & +\frac{1}{2}\delta & + \text{etc.} \\
\frac{1}{6} zz\, V = & & - & +\frac{1}{6} & +\frac{1}{6}\alpha & +\frac{1}{6}\beta & +\frac{1}{6}\gamma & + \text{etc.} \\
\frac{1}{24} z^3 V = & & - & - & +\frac{1}{24} & +\frac{1}{24}\alpha & +\frac{1}{24}\beta & + \text{etc.} \\
\frac{1}{120} z^4 V = & & - & - & - & +\frac{1}{120} & +\frac{1}{120}\alpha & + \text{etc.} \\
\frac{1}{720} z^5 V = & & - & - & - & - & +\frac{1}{720} & + \text{etc.} \\
\text{etc.} & & & & & & &
\end{array}$$

Hoc scilicet modo omnes termini, praeter primum, ad nihilum sunt redacti; eritque ergo

$$V\left(1 + \tfrac{1}{2}z + \tfrac{1}{6}z^2 + \tfrac{1}{24}z^3 + \tfrac{1}{120}z^4 + \tfrac{1}{720}z^5 + \text{etc.}\right) = 1.$$

§. 7. Cum igitur sit $e^z = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \text{etc.}$ erit $\frac{V(e^z - 1)}{z} = 1$, ideoque $V = \frac{z}{e^z - 1}$, quae expressio quo facilius iterum in seriem converti queat, ponamus $z = 2t$, ut sit $V = \frac{2t}{e^{2t} - 1}$, ideoque $V + t = t \cdot \frac{e^{2t} + 1}{e^{2t} - 1}$. Nunc statuatur $\frac{e^{2t} + 1}{e^{2t} - 1} = u$, fietque $V = tu - t$. Cum igitur sit $u = \frac{e^t + e^{-t}}{e^t - e^{-t}}$, hinc exponentialibus evolutis erit

$$u = \frac{1 + \frac{1}{2}t^2 + \frac{1}{24}t^4 + \frac{1}{720}t^6 + \text{etc.}}{t + \frac{1}{6}t^3 + \frac{1}{120}t^5 + \frac{1}{5040}t^7 + \text{etc.}},$$

ubi in numeratore solae potestates pares, in denominatore vero solae potestates impares occurrunt. Patet autem, sumto t quam minimo, fieri $u = \frac{1}{t}$, sequentes vero terminos per potestates t, t^3, t^5, etc. esse progressuros.

§. 8. Cum igitur posuerimus $u = \frac{e^{2t}+1}{e^{2t}-1}$, erit $e^{2t} = \frac{u+1}{u-1}$, ideoque $2t = l\frac{u+1}{u-1}$. Hinc ergo differentiando erit $\partial t = -\frac{\partial u}{uu-1}$, unde concluditur fore $\frac{\partial u}{\partial t} + uu - 1 = 0$. Quia autem novimus, primum terminum seriei, qua u exprimitur, esse $\frac{1}{t}$ et sequentium potestatum exponentes binario crescere, statuatur:

$$u = \frac{1}{t} + 2At - 2Bt^3 + 2Ct^5 - 2Dt^7 + \text{etc.}$$

fiatque substitutio sequenti modo:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= -\frac{1}{tt} + 2A - 6Btt + 10Ct^4 - 14Dt^6 + 18Et^8 - \text{etc.}\\
uu &= +\frac{1}{tt} + 4A - 4B + 4C - 4D + 4E - \text{etc.}\\
&\qquad + 4AA - 8AB + 8AC - 8AD + \text{etc.}\\
&\qquad\qquad + 4BB - 8BC + \text{etc.}\\
-1 &= -1
\end{aligned}$$

ubi termini primi se sponte destruunt, reliqui vero sequentes praebent determinationes:

$$\begin{aligned}
6A &= 1 & &\text{ergo } A = \tfrac{2}{3}\cdot\tfrac{1}{4} = \tfrac{1}{6},\\
10B &= 4AA & &\ldots\ B = \tfrac{2}{5}AA = \tfrac{1}{90},\\
14C &= 8AB & &\ldots\ C = \tfrac{2}{7}\cdot 2AB = \tfrac{1}{945},\\
18D &= 8AC + 4BB & &\ldots\ D = \tfrac{2}{9}(2AC + BB) = \tfrac{1}{9540},
\end{aligned}$$

$$22E = 8(AD + BC), \text{ ergo } E = \tfrac{2}{11}(2AD + 2BC) = \tfrac{1}{93555},$$
etc.

§. 9. Hae ergo litterae A, B, C, D, etc. prorsus eaedem sunt, quibus olim ad summas potestatum reciprocarum exprimendas sum usus, siquidem inveni esse:

$$1 + \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16} + \tfrac{1}{25} + \text{etc.} = A\pi^2,$$
$$1 + \tfrac{1}{4^2} + \tfrac{1}{9^2} + \tfrac{1}{16^2} + \tfrac{1}{25^2} + \text{etc.} = B\pi^4,$$
$$1 + \tfrac{1}{4^3} + \tfrac{1}{9^3} + \tfrac{1}{16^3} + \tfrac{1}{25^3} + \text{etc.} = C\pi^6,$$
etc.

quos valores usque ad potestatem trigesimam quartam per calculos valde operosos sum exsecutus.

§. 10. Cum igitur sumserimus:

$$u = \tfrac{1}{t} + 2At - 2Bt^3 + 2Ct^5 - \text{etc.}$$

ob $V = tu - t$ erit

$$V = 1 - t + 2At^2 - 2Bt^4 + 2Ct^6 - 2Dt^8 + \text{etc.}$$

ubi nil aliud superest, nisi ut loco t scribatur $\frac{1}{2}z$, unde prodit

$$V = 1 - \tfrac{z}{2} + \tfrac{Azz}{2} - \tfrac{Bz^4}{8} + \tfrac{Cz^6}{32} - \tfrac{Dz^8}{128} + \text{etc.}$$

Cum igitur habuerimus

$$V = 1 + \alpha z + \beta z^2 + \gamma z^3 + \text{etc.}$$

collatione instituta reperiemus $\alpha = \frac{1}{2}$; $\beta = \frac{1}{2}A$; $\gamma = 0$; $\delta = -\frac{1}{8}B$; $\varepsilon = 0$; $\zeta = \frac{1}{32}C$; $\eta = 0$; etc.

§. 11. Inventis jam valoribus harum litterarum summa seriei propositae

$$S = X + X' + X'' + X''' + \text{etc.}$$

sequenti modo exprimetur:

$$S = -\int X \partial x + \tfrac{1}{2} X - \tfrac{1}{2} A \partial X + \tfrac{1}{8} B \partial^3 X - \tfrac{1}{32} C \partial^5 X + \tfrac{1}{128} D \partial^7 X - \tfrac{1}{512} E \partial^9 X + \text{etc.}$$

ubi integrale $\int X \partial x$ ita capi debet, ut posito $x = \infty$ evanescat; unde patet, si constans adjicienda debeat esse infinita, etiam ipsam seriei summam fore infinitam.

§. 12. Consideremus exemplum, quo $X = \frac{1}{x^n}$, ita ut hujus seriei summa sit quaerenda:

$$S = \frac{1}{x^n} + \frac{1}{(x+1)^n} + \frac{1}{(x+2)^n} + \frac{1}{(x+3)^n} + \text{etc.}$$

Hic igitur erit $\int X \partial x = -\frac{1}{(n-1)x^{n-1}}$, quae forma ut evanescat posito $x = \infty$, necesse est ut exponens n sit unitate major. Alioquin enim, si esset $n = 1$ vel $n < 1$, summa seriei certe foret infinite magna. Porro vero erit $\partial X = -\frac{n}{x^{n+1}}$, hinc $\partial^3 X = -\frac{n(n+1)(n+2)}{x^{n+3}}$; $\partial^5 X = -\frac{n \ldots (n+4)}{x^{n+5}}$; etc. quibus valoribus substitutis summa quaesita erit:

$$S = \frac{1}{(n-1)x^{n-1}} + \frac{1}{2x^n} + \frac{A}{2} \cdot \frac{n}{x^{n+1}} - \frac{B}{8} \cdot \frac{n(n+1)(n+2)}{x^{n+3}} + \frac{C}{32} \cdot \frac{n \ldots (n+4)}{x^{n+5}} - \text{etc.}$$

quae series eo magis converget, quo major accipietur numerus x, praeterquam quod literae A, B, C, etc. progressionem valde convergentem constituunt.

§. 13. Quod si ergo ab unitate incipiendo hi termini $1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} \ldots \ldots \frac{1}{(x-1)^n}$ actu colligantur, eorumque summa vocetur Δ, ejusdem seriei in infinitum continuatae summa erit $\Delta + S$. Hoc modo olim summas talium serierum infinitarum pro singulis exponentis n valoribus 2, 3, 4, 5, etc. ad plures figuras decimales computavi, sumto scilicet $x = 10$, quo pacto calculus satis expedite absolvi poterat.

Casus 2.

Summatio seriei infinitae

$$S = X - X' + X'' - X''' + X^{IV} - \text{etc.}$$

§. 14. Quod si igitur index x unitate augeatur, habebimus $S' = X' - X'' + X''' - X^{IV} + \text{etc.}$ Addatur haec aequatio ad praecedentem, prodibitque aequatio finita $S + S' = X$. Quare per formulas differentiales habebimus:

$$X = 2S + \partial S + \tfrac{1}{2}\partial\partial S + \tfrac{1}{6}\partial^3 S + \tfrac{1}{24}\partial^4 S + \text{etc.}$$

unde neglectis differentialibus erit $S = \frac{1}{2}X$, qui ergo erit primus terminus seriei quam quaerimus. Statuamus igitur

$$S = \tfrac{1}{2}X + \alpha\partial X + \beta\partial\partial X + \gamma\partial^3 X + \text{etc.}$$

et facta substitutione fiet:

$$\begin{array}{rcccccc}
2S = & X & + 2\alpha\partial X & + 2\beta\partial\partial X & + 2\gamma\partial^3 X & + 2\delta\partial^4 X & + \text{etc.} \\
\partial S = & \tfrac{1}{2} & + \alpha & + \beta & + \gamma & & + \text{etc.} \\
\tfrac{1}{2}\partial\partial S = & - & + \tfrac{1}{4} & + \tfrac{1}{2}\alpha & + \tfrac{1}{2}\beta & & + \text{etc.} \\
\tfrac{1}{6}\partial^3 S = & - & - & + \tfrac{1}{12} & + \tfrac{1}{6}\alpha & & + \text{etc.} \\
\tfrac{1}{24}\partial^4 S = & - & - & - & + \tfrac{1}{48} & & + \text{etc.} \\
\text{etc.} & & & & \text{etc.} & &
\end{array}$$

quae expressio tota soli X est aequanda.

§. 15. Singulis igitur columnis verticalibus ad nihilum redactis orientur sequentes aequalitates:

$$2\alpha + \tfrac{1}{2} = 0;\ 2\beta + \alpha + \tfrac{1}{4} = 0;\ 2\gamma + \beta + \tfrac{1}{2}\alpha + \tfrac{1}{12} = 0;$$
$$2\delta + \gamma + \tfrac{1}{2}\beta + \tfrac{1}{6}\alpha + \tfrac{1}{24} = 0; \text{ etc.}$$

unde priores saltem literae has recipiunt determinationes:

$$\alpha = -\tfrac{1}{4};\ \beta = 0;\ \gamma = \tfrac{1}{48};\ \delta = 0; \text{ etc.}$$

§. 16. Quo autem hos valores facilius investigemus, consideremus hanc seriem:

$$V = \tfrac{1}{2} + \alpha z + \beta z^2 + \gamma z^3 + \text{etc.}$$

cujus scilicet summam V quaeri oporteat. Inde ergo sequentes derivemus series:

$$\begin{array}{lcccccccc}
2V = & 1 & + 2\alpha z & + 2\beta zz & + 2\gamma z^3 & + 2\delta z^4 & + 2\varepsilon z^5 & + \text{etc.} \\
Vz = & & + \tfrac{1}{2}z & + \alpha zz & + \beta z^3 & + \gamma z^4 & + \delta z^5 & + \text{etc.} \\
\tfrac{1}{2}Vzz = & & - & + \tfrac{1}{4} & + \tfrac{1}{2}\alpha & + \tfrac{1}{2}\beta & + \tfrac{1}{2}\gamma & + \text{etc.} \\
\tfrac{1}{6}Vz^3 = & & - & - & + \tfrac{1}{12} & + \tfrac{1}{6}\alpha & + \tfrac{1}{6}\beta & + \text{etc.} \\
\tfrac{1}{24}Vz^4 = & & - & - & - & + \tfrac{1}{48} & + \tfrac{1}{24}\alpha & + \text{etc.} \\
\text{etc.} & & & & & & &
\end{array}$$

Harum igitur serierum summa, ob aequalitates ante allatas, fiet $= 1$, sicque habebimus istam aequationem:

$$V\left(2 + z + \tfrac{1}{2}z^2 + \tfrac{1}{6}z^3 + \tfrac{1}{24}z^4 + \text{etc.}\right) = 1.$$

Quare cum sit

$$e^z = 1 + z + \tfrac{1}{2}z^2 + \tfrac{1}{6}z^3 + \text{etc.}$$

erit manifesto $V(1 + e^z) = 1$, sive $V = \frac{1}{1 + e^z}$, unde fit $2V - 1 = \frac{1 - e^z}{1 + e^z}$.

§. 17. Ponatur igitur ut ante $\frac{e^z - 1}{e^z + 1} = u$, ut sit $2V = 1 - u$, sitque iterum $z = 2t$, ita ut $u = \frac{e^t - e^{-t}}{e^t + e^{-t}}$, et facta evolutione erit $u = \frac{t + \frac{1}{6}t^3 + \frac{1}{120}t^5 + \frac{1}{5040}t^7 + etc.}{1 + \frac{1}{2}t^2 + \frac{1}{24}t^4 + \frac{1}{720}t^6 + etc.}$. Unde patet seriei, valorem ipsius u exprimentis, primum terminum fore t, sequentes vero per potestates impares ipsius t progredi.

§. 18. Cum igitur sit $u = \frac{e^{2t} - 1}{e^{2t} + 1}$, erit $e^{2t} = \frac{1+u}{1-u}$, ideoque $2t = l\frac{1+u}{1-u}$, unde differentiando fit $\partial t = \frac{\partial u}{1-uu}$, ita ut $\frac{\partial u}{\partial t} + uu - 1 = 0$, quae est ipsa aequatio pro casu priore inventa. Neque tamen propterea pro u eadem series provenit. Quoniam enim hic primus seriei terminus debet esse $= t$, fingenda est hujusmodi series:

$$u = t - \mathfrak{A}t^3 + \mathfrak{B}t^5 - \mathfrak{C}t^7 + \mathfrak{D}t^9 - \mathfrak{E}t^{11} + \text{etc.}$$

fierique debebit facta substitutione:

$$\begin{array}{rccccccl}
\frac{\partial u}{\partial t} = & 1 & - 3\mathfrak{A}tt & + 5\mathfrak{B}t^4 & - 7\mathfrak{C}t^6 & + 9\mathfrak{D}t^8 & - 11\mathfrak{E}t^{10} & + \text{etc.} \\
uu = & & + 1 & - 2\mathfrak{A} & + 2\mathfrak{B} & - 2\mathfrak{C} & + 2\mathfrak{D} & - \text{etc.} \\
 & & & + \mathfrak{A}^2 & - 2\mathfrak{A}\mathfrak{B} & + 2\mathfrak{A}\mathfrak{C} & & - \text{etc.} \\
 & & & & & + \mathfrak{B}^2 & & - \text{etc.} \\
-1 = & -1 & & & & & &
\end{array}$$

atque hinc sequentes oriuntur determinationes:

$$\begin{array}{lll}
3\mathfrak{A} = 1 & \text{ideoque} & \mathfrak{A} = \frac{1}{3}, \\
5\mathfrak{B} = 2\mathfrak{A} & \text{ideoque} & \mathfrak{B} = \frac{2}{5}\mathfrak{A} = \frac{2}{15}, \\
7\mathfrak{C} = 2\mathfrak{B} + \mathfrak{A}^2 & \text{hinc} & \mathfrak{C} = \frac{2}{7}\mathfrak{B} + \frac{1}{7}\mathfrak{A}^2 = \frac{17}{315}, \\
9\mathfrak{D} = 2\mathfrak{C} + 2\mathfrak{A}\mathfrak{B} & \text{ergo} & \mathfrak{D} = \frac{2}{9}\mathfrak{C} + \frac{2}{9}\mathfrak{A}\mathfrak{B} = \frac{62}{2835}, \\
\text{etc.} & & \text{etc.}
\end{array}$$

§. 19. Cum igitur sit $V = \frac{1}{2} - \frac{1}{2}u$, si loco t restituamus $\frac{z}{2}$, pro V hanc reperiemus seriem:

$$V = \frac{1}{2} - \frac{1}{4}z + \frac{1}{16}\mathfrak{A}z^3 - \frac{1}{64}\mathfrak{B}z^5 + \frac{1}{256}\mathfrak{C}z^7 - \frac{1}{1024}\mathfrak{D}z^9 + \text{etc.}$$

Quare cum posuerimus

$$V = \frac{1}{2} + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 \text{ etc.}$$

hinc colligimus valores literarum α, β, γ, δ, etc. qui ergo erunt $\alpha = -\frac{1}{4}$; $\beta = 0$; $\gamma = \frac{1}{16}\mathfrak{A}$; $\delta = 0$; $\varepsilon = -\frac{1}{64}\mathfrak{B}$; $\zeta = 0$; $\eta = \frac{1}{256}\mathfrak{C}$; $\theta = 0$; etc. consequenter summa quaesita erit:

$$S = \frac{1}{2}X - \frac{1}{4}\partial X + \frac{1}{16}\mathfrak{A}\partial^3 X + \frac{1}{64}\mathfrak{B}\partial^5 X + \frac{1}{256}\mathfrak{C}\partial^7 X - \text{etc.}$$

§. 20. Comparemus nunc istos coëfficientes cum iis quos in casu praecedente pro similibus differentialibus sumus adepti, qui erant $\frac{A}{2}$, $\frac{B}{8}$, $\frac{C}{32}$, etc. atque egregiam relationem inter utrosque deprehendemus, uti ex hoc schemate videre licet:

∂X	$\frac{1}{4}$: $\frac{A}{2}$	$=$	3	$=$	$2^2 - 1$,
$\partial^3 X$	$\frac{\mathfrak{A}}{16}$: $\frac{B}{8}$	$=$	15	$=$	$2^4 - 1$,
$\partial^5 X$	$\frac{\mathfrak{B}}{64}$: $\frac{C}{32}$	$=$	63	$=$	$2^6 - 1$,
$\partial^7 X$	$\frac{\mathfrak{C}}{256}$: $\frac{D}{128}$	$=$	255	$=$	$2^8 - 1$,
$\partial^9 X$	$\frac{\mathfrak{D}}{1024}$: $\frac{E}{512}$	$=$	1023	$=$	$2^{10} - 1$,
etc.	etc.				

§. 21. Per eosdem igitur numeros notissimos A, B, C, D, etc. etiam hoc casu summa quaesita sequenti modo commode exprimetur:

$$S = \tfrac{1}{2}X - (2^2-1)\tfrac{A}{2}\,.\,\partial X + (2^4-1)\tfrac{B}{8}\,.\,\partial^3 X - (2^6-1)\tfrac{C}{32}\,.\,\partial^5 X$$
$$+ (2^8-1)\tfrac{D}{128}\,.\,\partial^7 X - (2^{10}-1)\tfrac{E}{512}\,.\,\partial^9 X + \text{etc.}$$

quam seriem quousque lubuerit continuare licet.