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De fractionibus continuis Wallisii

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DE FRACTIONIBUS CONTINUIS WALLISII.

AUCTORE

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§. 1. Postquam *Brounkerus* memorabilem suam frac-
tionem continuam pro quadratura circuli invenisset, eam-
que sine demonstratione cum *Wallisio* communicasset, hic
plurimum studii in eo collocavit, ut fontem, ex quo
Brounkerus hanc insignem formulam hausisset, detegeret.
Arbitratus autem est, eum usum fuisse egregiis illis for-
mulis, quas ipse in opere suo: *Arithmetica infinitorum*, ab-
eruerat. Quin etiam inde, per calculos non parum in-
strusos, non solum *Brounkeri* fractionem continuam, sed per-
super innumerabiles alias similes elicuit, quae utique, ut
inde ac *Brounkeri* expressio, dignae sunt judicandae,
oblivioni eripiantur.

§. 2. Quae autem ex *Wallisii* Arithmetica infini-
torum, diu ante inventam Analysin infinitorum in lu-
cem edita, huc pertinent, ea more nunc quidem recepto
ita repraesentari possunt, ut, formulis integralibus a ter-

mino $x = 0$ usque ad $x = 1$ extensis, sequentes quadraturae exhibeantur:

$$\int \frac{x\,\partial x}{\sqrt{1-xx}} = 1 = 1,$$

$$\int \frac{x^3\,\partial x}{\sqrt{1-xx}} = \frac{2}{3} = \frac{2.2}{2.3},$$

$$\int \frac{x^5\,\partial x}{\sqrt{1-xx}} = \frac{2.4}{3.5} = \frac{2.2.4.4}{2.3.4.5},$$

$$\int \frac{x^7\,\partial x}{\sqrt{1-xx}} = \frac{2.4.6}{3.5.7} = \frac{2.2.4.4.6.6}{2.3.4.5.6.7},$$

$$\int \frac{x^9\,\partial x}{\sqrt{1-xx}} = \frac{2.4.6.8}{3.5.7.9} = \frac{2.2.4.4.6.6.8.8}{2.3.4.5.6.7.8.9},$$

etc.

§. 3. Istas formulas in tertia columna ita adornavi, ut denominatores interpolationem manifesto admittant; sicque tantum superest, ut etiam numeratores ita transformentur, ut pariter interpolationem patiantur, id quod fiet, si talis series, secundum legem uniformem progrediens, scilicet A, B, C, D, E, F, etc. investigetur, ut sit:

$AB = 1.1$; $BC = 2.2$; $CD = 3.3$; $DE = 4.4$; etc.

id quod est id ipsum, in quo *Wallisius* summam ingenii sagacitatem manifestavit, quam autem investigationem deinceps multo generalius, et calculo longe faciliori, sum expediturus.

§. 4. Hac autem serie literarum A, B, C, D, etc. inventa totum negotium penitus erit confectum. Cum enim sit, uti sequens tabula declarat:

$$\int \frac{x\,\partial x}{\sqrt{1-xx}} = 1 = \frac{1}{A} \cdot \frac{A}{1},$$
$$\int \frac{x^3\,\partial x}{\sqrt{1-xx}} = \frac{B.C}{2.3} = \frac{1}{A} \cdot \frac{ABC}{1.2.3},$$
$$\int \frac{x^5\,\partial x}{\sqrt{1-xx}} = \frac{BCDE}{2.3.4.5} = \frac{1}{A} \cdot \frac{ABCDE}{1.2.3.4.5},$$
$$\int \frac{x^7\,\partial x}{\sqrt{1-xx}} = \frac{BCDEFG}{2.3.4.5.6.7} = \frac{1}{A} \cdot \frac{ABCDEFG}{1.2.3.4.5.6.7},$$
etc.

interpolatio nobis suppeditat sequentes quadraturas:

$$\int \frac{\partial x}{\sqrt{1-xx}} = \frac{1}{A} \cdot 1,$$
$$\int \frac{xx\,\partial x}{\sqrt{1-xx}} = \frac{1}{A} \cdot \frac{AB}{1.2},$$
$$\int \frac{x^4\,\partial x}{\sqrt{1-xx}} = \frac{1}{A} \cdot \frac{ABCD}{1.2.3.4},$$
$$\int \frac{x^6\,\partial x}{\sqrt{1-xx}} = \frac{1}{A} \cdot \frac{ABCDEF}{1.2.3.4.5.6},$$
etc.

§. 5. Cum nunc sit $\int \frac{\partial x}{\sqrt{1-xx}} = \frac{\pi}{2}$, denotante π peripheriam circuli, cujus diameter $= 1$, cujus loco brevitatis gratia scribamus $q = \frac{\pi}{2}$, omnes literarum A, B, C, D, etc. valores per hanc quantitatem q sequenti modo exprimentur:

	Differentiae
$A = \frac{1}{q} = 0{,}636620$	
	0,934176
$B = q = 1{,}570796$	
	0,975683
$C = \frac{4}{q} = 2{,}546479$	
	0,987813
$D = \frac{9q}{4} = 3{,}534292$	
	0,992782
$E = \frac{4 \cdot 16}{9q} = 4{,}527074$	
	0,995257
$F = \frac{9 \cdot 25}{4 \cdot 16} q = 5{,}522331$	

§. 6. Hic tertiam adjunxi columnam, quae valores numericos harum litterarum exhibet, quo clarius appareat, quemadmodum isti numeri secundum legem uniformem increscant, quod non evenisset, si loco q valorem falsum accepissem. His expositis methodum multo faciliorem tradam, qua pro singulis his literis fractiones continuae reperiri possunt, atque eadem opera hanc investigationem multo generaliorem instituam, dum sequens problema sum resoluturus:

Problema.

Invenire seriem literarum A, B, C, D, *etc. uniformi lege procedentem, ita ut sit* $AB = ff$; $BC = (f + a)^2$; $CD = (f + 2a)^2$; *etc.*

Solutio:

§. 7. Hinc statim patet, qualis functio fuerit A ipsius f, talem esse debere B functionem ipsius $f + a$, tum vero C ipsius $f + 2a$, D ipsius $f + 3a$ et ita porro. Hac lege observata, si statuamus $A = f - \frac{1}{2}a + \frac{\frac{1}{2}s}{A'}$, poni debebit $B = f + \frac{1}{2}a + \frac{\frac{1}{2}s}{B'}$; ubi literae A′ et B′ eandem inter se rationem tenere debent, ita ut ex A′ oriatur B′, si loco f scribatur $f + a$. Cum igitur fractionibus sublatis, sit $2A = 2f - a + \frac{s}{A'}$ et $2B = 2f + a + \frac{s}{B'}$, harum formularum productum ipsi $4ff$ est aequandum, unde oritur haec aequatio a fractionibus liberata:

$$aaA'B' - A's(2f-a) - B's(2f+a) - ss = 0.$$

Sumamus igitur $s = aa$, ut aequatio, per aa divisa, sit

$$A'B' - A'(2f-a) - B'(2f+a) = aa,$$

quae commode per factores repraesentari poterit ita:

$$(A' - 2f - a)(B' - 2f + a) = 4ff.$$

§. 8. Quia nnnc, si ambae literae A' et B' essent aequales, ex parte sinistra foret $A' = B' = 4f$, legem supra allatam sequentes, statuamus $A' = 4f - 2a + \frac{s'}{A''}$ et $B' = 4f + 2a + \frac{s'}{B''}$, quibus substitutis ultima aequatio induet hanc formam:

$$\left(2f - 3a + \frac{s'}{A''}\right)\left(2f + 3a + \frac{s'}{B''}\right) = 4ff.$$

Facta igitur evolutione et sublatis fractionibus orietur sequens aequatio:

$$9aaA''B'' - A''s'(2f-3a) - B''s'(2f+3a) - s's' = 0.$$

Sumatur ergo hic $s' = 9aa$, ut habeatur ista:

$$A''B'' - A''(2f-3a) - B''(2f+3a) = 9aa,$$

quae iterum per factores hoc modo repraesentari potest:

$$(A'' - 2f - 3a)(B'' - 2f + 3a) = 4ff.$$

§. 9. Cum nunc iterum medius valor inter A'' et B'' sit $4f$, statuamus porro

$$A'' = 4f - 2a + \frac{s''}{A'''} \text{ et } B'' = 4f + 2a + \frac{s''}{B'''},$$

et facta substitutione emerget ista aequatio:

$$\left(2f - 5a + \frac{s''}{A'''}\right)\left(2f + 5a + \frac{s''}{B'''}\right) = 4ff.$$

Facta igitur evolutione, sublatisque fractionibus, erit

$$25\,aa\,A'''B''' - A'''s''(2f-5a) - B'''s''(2f+5a) - s''s'' = 0.$$

Statuatur $s''=25\,aa$, et ista aequatio hanc induet formam:

$$A'''B''' - A'''(2f-5a) - B'''(2f+5a) = 25\,aa,$$

quae per factores hoc modo repraesentari potest:

$$(A'''-2f-5a)(B'''-2f+5a) = 4ff.$$

§. 10. Statuatur denuo ut ante $A''' = 4f - 2a + \frac{s'''}{A^{IV}}$ et $B''' = 4f + 2a + \frac{s'''}{B^{IV}}$, fietque facta substitutione

$$\left(2f - 7a + \frac{s'''}{A^{IV}}\right)\left(2f + 7a + \frac{s'''}{B^{IV}}\right) = 4ff,$$

qua aequatione evoluta et in ordinem redacta obtinetur

$$A^{IV}B^{IV} = A^{IV}(2f-7a) - B^{IV}(2f+7a) = 49\,aa,$$

ubi scilicet posuimus $s'''=49\,aa$; tum vero per factores erit

$$(A^{IV}-2f-7a)(B^{IV}-2f+7a) = 4ff.$$

Unde perspicuum est quomodo hae operationes sint ulterius continuandae.

§. 11. His igitur colligendis, ob $s = aa$, $s' = 9\,aa$, $s'' = 25\,aa$, $s''' = 49\,aa$, etc. pro $2\,A$ adipiscemur sequentem fractionem continuam:

$$2\,A = 2f - a + \cfrac{aa}{4f - 2a + \cfrac{9\,aa}{4f - 2a + \cfrac{25\,aa}{4f - 2a + \cfrac{49\,aa}{4f - 2a + \text{etc.}}}}}$$

ubi si loco f ordine scribamus $f+a$, $f+2a$, $f+3a$, etc. similes fractiones continuae prodibunt, pro $2\,B$, $2\,C$, $2\,D$, etc. quae ita se habebunt:

$$2B = 2f + a + \cfrac{aa}{4f + 2a + \cfrac{9aa}{4f + 2a + \cfrac{25aa}{4f + 2a + \cfrac{49aa}{4f + 2a + \text{etc.}}}}}$$

$$2C = 2f + 3a + \cfrac{aa}{4f + 6a + \cfrac{9aa}{4f + 6a + \cfrac{25aa}{4f + 6a + \cfrac{49aa}{4f + 6a + \text{etc.}}}}}$$

$$2D = 2f + 5a + \cfrac{aa}{4f + 10a + \cfrac{9aa}{4f + 10a + \cfrac{25aa}{4f + 10a + \cfrac{49aa}{4f + 10a + \text{etc.}}}}}$$

etc.

§. 12. Quod si jam hic ponamus $f = 1$ et $a = 1$ prodibit ipse casus a *Wallisio* tractatus, unde fractiones continuae a *Wallisio* inventae, cum suis valoribus per quadraturam circuli expressis, erunt sequentes:

FRACTIONES CONTINUAE WALLISIANAE.

$$2A = 1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{1 + \text{etc.}}}}} = \frac{2}{q} = \frac{4}{\pi},$$

$$2B = 3 + \cfrac{1}{6 + \cfrac{9}{6 + \cfrac{25}{6 + \cfrac{49}{6 + \text{etc.}}}}} = 2q = \pi,$$

$$2C = 5 + \cfrac{1}{10 + \cfrac{9}{10 + \cfrac{25}{10 + \cfrac{49}{10 + \text{etc.}}}}} = \frac{8}{q} = \frac{16}{\pi},$$

$$2D = 7 + \cfrac{1}{14 + \cfrac{9}{14 + \cfrac{25}{14 + \cfrac{49}{14 + \text{etc.}}}}} = \frac{9q}{2} = \frac{9\pi}{4},$$

$$2E = 9 + \cfrac{1}{18 + \cfrac{9}{18 + \cfrac{25}{18 + \cfrac{49}{18 + \text{etc.}}}}} = \frac{128}{9q} = \frac{256}{9\pi}.$$

quarum prima est ipsa fractio continua a *Brounkero* inventa.

§. 13. Neutiquam autem vero simile est, *Brounkerum* per tantas ambages ad suam formulam pervenisse; equidem credo potius, illam ex consideratione hujus seriei notissimae: $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \text{etc.} = \frac{\pi}{4}$, quae vulgo *Leibnitio* tribui solet, multo autem ante a *Jacobo Gregorio* erat eruta, a quo *Brounkerus* eam nosse poterat, derivasse, quippe quod per operationes satis faciles et obvias fieri potuit sequentem in modum:

Posito	erit
$\frac{\pi}{4} = 1 - \alpha$	$\frac{4}{\pi} = \frac{1}{1-\alpha} = 1 + \frac{\alpha}{1-\alpha} = 1 + \cfrac{1}{-1 + \frac{1}{\alpha}}$
$\alpha = \frac{1}{3} - \beta$	$\frac{1}{\alpha} = \frac{3}{1-3\beta} = 3 + \frac{9\beta}{1-3\beta} = 3 + \cfrac{9}{-3 + \frac{1}{\beta}}$
$\beta = \frac{1}{5} - \gamma$	$\frac{1}{\beta} = \frac{5}{1-5\gamma} = 5 + \frac{25\gamma}{1-5\gamma} = 5 + \cfrac{25}{-5 + \frac{1}{\gamma}}$
$\gamma = \frac{1}{7} - \delta$	$\frac{1}{\gamma} = \frac{7}{1-7\delta} = 7 + \frac{49\delta}{1-7\delta} = 7 + \cfrac{49}{-7 + \frac{1}{\delta}}$
etc.	etc.

Quodsi jam hic loco $\frac{1}{\alpha}$, $\frac{1}{\beta}$, $\frac{1}{\gamma}$, etc. valores modo inventi substituantur, ultro se offert ipsa fractio continua *Brounkeri*, siquidem hinc sequitur fore

$$\frac{4}{\pi} = 1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \text{etc.}}}}}$$

§. 14. Quod autem ad nostram problematis solutionem generalem attinet, etiam singularum fractionum continuarum valores per certas quadraturas exprimere licet, id quod in sequente problemate ostendamus:

Problema.

Proposita serie A, B, C, D, *etc. secundum legem uniformem procedente, ita ut sit* $AB = ff$; $BC = (f + a)^2$; $CD = (f + 2a)^2$; *etc. singularum harum litterarum valores, primo quidem per producta continua, tum vero per formulas integrales expressas investigare.*

Solutio.

§. 15. Cum igitur sit $A = \frac{ff}{B}$; $B = \frac{(f+a)^2}{C}$; $C = \frac{(f+2a)^2}{D}$; etc. his valoribus continuo substitutis reperietur

$$A = \frac{ff(f+2a)^2(f+4a)^2(f+6a)^2 \text{ (etc.}}{(f+a)^2(f+3a)^2(f+5a)^2 \text{ (etc.}},$$

in infinitum. Cum autem hoc modo nullus determinatus valor oriatur, quoniam, ubicunque abrumpitur, vel in numeratoribus vel in denominatoribus factor redundat, hoc incommodum tolletur, si factores simplices sequenti modo disponamus:

$$A = f \cdot \frac{f(f+2a)}{(f+a)(f+a)} \cdot \frac{(f+2a)(f+4a)}{(f+3a)(f+3a)} \cdot \frac{(f+4a)(f+6a)}{(f+5a)(f+5a)} \cdot \text{etc.}$$

Sic enim membra continuo propius ad unitatem accedent et in infinitum ipsi unitati aequabuntur, sicque ista expressio utique determinatum valorem habebit.

§. 16. Quo autem ostendamus quomodo ejus valorem ad formulas integrales reduci oporteat, in subsidium vocemus hoc lemma:

Integralibus ab $x = 0$ *ad* $x = 1$ *extensis erit:*

$$\int \frac{x^{m-1}\partial x}{\sqrt[n]{(1-x^n)^{n-k}}} = \frac{m+k}{m} \cdot \frac{m+k+n}{m+n} \cdot \frac{m+k+2n}{m+2n} \cdot \frac{m+k+3n}{m+3n} \cdot \frac{m+k+4n}{m+4n} \cdots \int \frac{x^{\infty}\partial x}{\sqrt[n]{(1-x^n)^{n-k}}}.$$

Quo jam hoc lemma ad nostrum casum accomodemus, quoniam in nostris membris singuli factores incrementum capiunt $= 2a$, statui debet $n = 2a$; tum vero sumto $m = f$ et $k = a$ habebimus:

$$\int \frac{x^{f-1}\partial x}{\sqrt{1-x^{2a}}} = \frac{f+a}{f} \cdot \frac{f+3a}{f+2a} \cdot \frac{f+5a}{f+3a} \cdots \int \frac{x^{\infty}\partial x}{\sqrt{1-x^{2a}}},$$

quae expressio, inversa, praebet priores singulorum membrorum factores. Pro posterioribus sumamus $m = f + a$, manente $k = a$, hocque facto erit:

$$\int \frac{x^{f+a-1}\partial x}{\sqrt{1-x^{2a}}} = \frac{f+2a}{f+a} \cdot \frac{f+4a}{f+3a} \cdot \frac{f+6a}{f+5a} \cdots \int \frac{x^{\infty}\partial x}{\sqrt{1-x^{2a}}}.$$

§. 17. Evidens nunc est, posteriorem formulam per priorem divisam ipsum nostrum productum continuum exhibere, quo pacto ambo integralia infinitesima se mutuo tollunt, consequenter habemus:

$$A = \int \frac{x^{f+a-1}\partial x}{\sqrt{1-x^{2a}}} : \int \frac{x^{f-1}\partial x}{\sqrt{1-x^{2a}}}.$$ Simili modo protinus

$$B = \int \frac{x^{f+2a-1}\partial x}{\sqrt{1-x^{2a}}} : \int \frac{x^{f+a-1}\partial x}{\sqrt{1-x^{2a}}},$$

$$C = \int \frac{x^{f+3a-1}\partial x}{\sqrt{1-x^{2a}}} : \int \frac{x^{f+2a-1}\partial x}{\sqrt{1-x^{2a}}},$$

etc.

At vero haec investigatio adhuc generalior reddi potest, quemadmodum sequens problema docebit.

Problema generalius.

Invenire seriem uniformi lege procedentem A, B, C, D, *etc. ita ut sit* $AB = ff + c$; $BC = (f+a)^2 + c$; $CD = (f+2a)^2 + c$; $DE = (f+3a)^2 + c$; *ubi in singulis productis litera* f *quantitate* a *augeatur.*

Solutio prior per fractiones continuas.

§. 18. Hic iterum evidens est, qualis A fuerit functio ipsius f, talem esse debere B functionem ipsius $f+a$; C ipsius $f+2a$; D ipsius $f+3a$ et ita porro. Cum igitur sit $AB = ff + c$, si A et B essent aequales, omisso c foret $A = B = f$. Quanto igitur A minor accipitur quam f, tanto B debet esse major; unde posito $A = f - x$ erit $B = f + x$. Quoniam autem B ex A nascitur, si loco f scribatur $f+a$, etiam esse debet $B=f+a-x$, unde concludimus fore $x=\frac{1}{2}a$; sicque partes principales pro A et B erunt $A=f-\frac{1}{2}a$ et $B=f+\frac{1}{2}a$, sive $2A=2f-a$ et $2B=2f+a$, ideoque pro sequentibus $2C=2f+3a$; $2D=2f+5a$; $2E=2f+7a$; etc.

§. 19. His valoribus principalibus inventis ponamus revera esse $2A=2f-a+\frac{s}{A'}$; $2B=2f+a+\frac{s}{B'}$. At pro s mox idoneus valor emerget. Hinc igitur erit:

$$4AB=4ff-aa+\frac{s}{A'}(2f+a)+\frac{s}{B'}(2f-a)+\frac{ss}{A'B'}=4ff+4c,$$

quae aequatio, sublatis fractionibus, hanc induet formam:

$$A'B'(aa+4c)-A's(2f-a)-B's(2f+a)-ss=0.$$

Sumamus jam $s=aa+4c$, eritque facta divisione:

$$A'B'-A'(2f-a)-B(2f+a)=aa+4c,$$

quae aequatio ita per factores repraesentetur:

$$(A'-2f-a)(B'-2f+a)=4ff+ac.$$

§. 20. Nunc simili modo ut ante ratiocinando intelligitur, si A' et B' fuerint aequales, membrum sinistrum fore $A'A'-4fA'=0$, ideoque $A'=B'=4f$. Quia autem B' oriri debet ex A', si loco f scribatur $f+a$, evidens est partes principales fore $A'=4f-2a$ et $B'=4f+2a$. Revera igitur ponamus esse $A'=4f-2a+\frac{s'}{A''}$ et $B'=4f+2a+\frac{s'}{B''}$, unde, si hi valores substituantur, aequatio praecedens, per factores exhibita, hanc induet formam:

$$(2f - 3a + \tfrac{s'}{A''})(2f + 3a + \tfrac{s'}{B''}) = 4ff + 4c,$$

quae, facta evolutione, ad istam perducit aequationem:

$$(4ff - 9aa) + \tfrac{s'}{A''}(2f + 3a) + \tfrac{s'}{B''}(2f - 3a) + \tfrac{s's'}{A''B''} = 4ff + 4c,$$

haecque sublatis fractionibus abit in hanc:

$$A''B''(9aa + 4c) - A''s'(2f - 3a) - B''s'(2f + 3a) - s's' = 0.$$

Sumto igitur $s' = 9aa + 4c$, et facta divisione, oritur haec aequatio:

$$A''B'' - A''(2f - 3a) - B''(2f + 3a) = 9aa + 4c,$$

quae per factores repraesentari potest hoc modo:

$$(A'' - 2f - 3a)(B'' - 2f + 3a) = 4ff + 4c.$$

§. 21. Quia haec aequatio similis est praecedenti, iterumque pro casu $A'' = B''$ prodiret $4f$, statuatur ulterius $A'' = 4f - 2a + \frac{s''}{A'''}$ et $B'' = 4f + 2a + \frac{s''}{B'''}$, unde postrema aequatio per factores foret:

$$(2f - 5a + \tfrac{s''}{A'''})(2f + 5a + \tfrac{s''}{B'''}) = 4ff + 4c.$$

At facta evolutione sublatisque fractionibus prodit:

$$A'''B'''(25aa + 4c) - A'''s''(2f - 5a) - B'''s''(2f + 5a) - s''s'' = 0.$$

Sumendo igitur $s'' = 25aa + 4c$ et dividendo per s'' fiet:

$$A'''B''' - A'''(2f - 5a) - B'''(2f + 5a) = 25aa + 4c,$$

sive per productum:

$$(A''' - 2f - 5a)(B''' - 2f + 5a) = 4ff + 4c.$$

§. 22. Statuatur ulterius $A''' = 4f - 2a + \frac{s'''}{A^{IV}}$ et $B''' = 4f + 2a + \frac{s'''}{B^{IV}}$, et superior aequatio per productum, substitutis his valoribus, erit:

$$(2f - 7a + \tfrac{s'''}{A^{IV}})(2f + 7a + \tfrac{s'''}{B^{IV}}) = 4ff + 4c,$$

quae, iisdem operationibus repetitis, sumtoque $s''' = 49aa + 4c$ ad sequentem reducitur:

$$A^{IV}B^{IV} - A^{IV}(2f - 7a) - B^{IV}(2f + 7a) = 49aa + 4c,$$

sive in factoribus erit:

$$(A^{IV} - 2f - 7a)(B^{IV} - 2f + 7a) = 4ff + 4c.$$

Ex quibus jam abunde liquet, quomodo calculum ulterius prosequi oporteat.

§. 23. His igitur valoribus successive substitutis ob $s = aa + 4c$; $s' = 9aa + 4c$; $s'' = 25aa + 4c$; $s''' = 49aa + 4c$; etc. pro A obtinebimus sequentem fractionem continuam:

$$2A = 2f - a + \cfrac{aa + 4c}{4f - 2a + \cfrac{9aa + 4c}{4f - 2a + \cfrac{25aa + 4c}{4f - 2a + \cfrac{49aa + 4c}{4f - 2a + \text{etc.}}}}}$$

Simili modo hinc erit:

$$2B = 2f + a + \cfrac{aa + 4c}{4f + 2a + \cfrac{9aa + 4c}{4f + 2a + \cfrac{25aa + 4c}{4f + 2a + \cfrac{49aa + 4c}{4f + 2a + \text{etc.}}}}}$$

$$2C = 2f + 3a + \cfrac{aa + 4c}{4f + 6a + \cfrac{9aa + 4c}{4f + 6a + \cfrac{25aa + 4c}{4f + 6a + \cfrac{49aa + 4c}{4f + 6a + \text{etc.}}}}}$$

$$2D = 2f + 5a + \cfrac{aa + 4c}{4f + 10a + \cfrac{9aa + 4c}{4f + 10a + \cfrac{25aa + 4c}{4f + 10a + \cfrac{49aa + 4c}{4f + 10a + \text{etc.}}}}}$$

etc.

Solutio altera per producta continua.

§. 24. Cum sit $AB = ff + c$; $BC = (f+a)^2 + c$; $CD = (f+2a)^2 + c$; $DE = (f+3a)^2 + c$; etc. erit:

$$A = \frac{(ff+c)((f+2a)^2+c)((f+4a)^2+c)((f+6a)^2+c)(etc.}{((f+a)^2+c)((f+3a)^2+c)((f+5a)^2+c)(etc.}.$$

At vero in hac expressione, ubicunque sistas, vel in numeratoribus vel in denominatoribus factor redundabit. Quod quo clarius appareat, subsistamus primo in littera F, eritque:

$$A = \frac{ff+c}{(f+a)^2+c} \cdot \frac{(f+2a)^2+c}{(f+3a)^2+c} \cdot (f+4a)^2+c \cdot \frac{1}{F},$$

Quando autem in sequente littera G subsistimus fiet:

$$A = \frac{ff+c}{(f+a)^2+c} \cdot \frac{(f+2a)^2+c}{(f+3a)^2+c} \cdot \frac{(f+4a)^2+c}{(f+5a)^2+c} \cdot G.$$

§. 25. Quod si ergo istae binae expressiones in infinitum continuentur et in se invicem ducantur, ultimus factor literalis, qui hic est $\frac{G}{F}$, manifesto unitati aequabitur. Quia vero hoc casu numerus factorum in numeratore unitate redundat, ejus factorem primum in fronte seorsim scribamus, atque productum sequenti modo exprimetur:

$$A^2 = (ff+c) \cdot \frac{(ff+c)((f+2a)^2+c)}{((f+a)^2+c)((f+a)^2+c)} \cdot \frac{((f+2a)^2+c)((f+4a)^2+c)}{((f+3a)^2+c)((f+3a)^2+c)} \text{ etc.}$$

ubi jam infinitesimi factores unitati aequabuntur, sicque ista expressio uniformi lege procedit.

Hic autem duos casus distingui conveniet, prouti c fuerit numerus vel negativus vel positivus.

Casus 1, quo $c = -bb$.

§. 26. Priore casu quilibet factor in duos resolvi se patietur. Statuamus igitur primo $c = -bb$, quo casu fractio continua sequenti modo exhiberi potest:

$$2A = 2f - a + \cfrac{(a+2b)(a-2b)}{4f - 2a + \cfrac{(3a+2b)(3a-2b)}{4f - 2a + \cfrac{(5a+2b)(5a-2b)}{4f - 2a + \cfrac{(7a+2b)(7a-2b)}{4f - 2a + \text{etc.}}}}}$$

atque loco expressionis per factores continuos nunc habebimus sequentem pro simplici litera A, scilicet:

$$A = (f-b) \cdot \frac{(f+b)(f+2a-b)}{(f+a+b)(f+a-b)} \cdot \frac{(f+2a-b)(f+4a-b)}{(f+3a+b)(f+3a-b)} \cdot \text{etc.}$$

in cujus expressionis quolibet membro summa factorum numeratoris aequatur summae factorum denominatoris; ob quam proprietatem hi factores per formulam integralem exprimi poterunt.

§. 27. Constat enim, si haec formula integralis:

$$\int \frac{x^{m-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-k}}},$$

ab $x = 0$ usque ad $x = 1$ extendatur, valorem reduci ad sequens productum infinitum:

$$\frac{m+k}{m} \cdot \frac{m+k+n}{m+n} \cdot \frac{m+k+2n}{m+2n} \cdot \cdot \cdot \cdot \cdot \int \frac{x^{\infty}\,\partial x}{\sqrt[n]{(1-x^n)^{n-k}}}.$$

Quo igitur hanc formam ad nostram expressionem accomodemus, quia singuli factores in sequenti membro quantitate $2a$ augentur, sumi debet $n = 2a$; tum vero posito

$m = f + b$ et $k = a$ reperietur fore:

$$\frac{f+a+b}{f+b} \cdot \frac{f+3a+b}{f+2a+b} \cdot \frac{f+5a+b}{f+4a+b} \cdots \int \frac{x^{\infty}\,\partial x}{\sqrt{(1-x^{2a})}} = \int \frac{x^{f+b-1}\,\partial x}{\sqrt{1-x^{2a}}},$$

quae expressio inversa priores factores cujusque membri continet. Pro posterioribus autem, manente $n = 2a$, sumatur $m = f + a - b$ et $k = a$, quo facto prodibit haec aequatio:

$$\frac{f+2a-b}{f+a-b} \cdot \frac{f+4a-b}{f+3a-b} \cdot \frac{f+6a-b}{f+5a-b} \cdots \int \frac{x^{\infty}\,\partial x}{\sqrt{1-x^{2a}}} = \int \frac{x^{f+a-b-1}\,\partial x}{\sqrt{1-x^{2a}}}.$$

Si igitur haec aequatio per praecedentem dividatur, postremi factores integrales se mutuo destruent, prodibitque productum infinitum, in valore A occurrens, per duas formulas integrales expressum, ita ut sit:

$$A = (f-b) \cdot \int \frac{x^{f+a-b-1}\,\partial x}{\sqrt{1-x^{2a}}} : \int \frac{x^{f+b-1}\,\partial x}{\sqrt{1-x^{2a}}}.$$

§. 28. Quo haec exemplo illustremus, sumamus $f = 2$, $a = 1$, $b = 1$, ut habeamus hos valores: $AB = 3$, $BC = 8$, $CD = 15$, $DE = 24$, etc. hocque casu nostra fractio continua evadit:

$$2A = 3 - \cfrac{3}{6 + \cfrac{5}{6 + \cfrac{21}{6 + \cfrac{45}{6 + \cfrac{77}{6 + \text{etc.}}}}}}$$

At per productum continuum erit:

$$A = \frac{3 \cdot 3}{2 \cdot 4} \cdot \frac{5 \cdot 5}{4 \cdot 6} \cdot \frac{7 \cdot 7}{6 \cdot 8} \cdot \frac{9 \cdot 9}{8 \cdot 10} \cdot \text{etc.}$$

Tum vero per formulas integrales habebitur:

$$A = \int \frac{x\,\partial x}{\sqrt{1-xx}} : \int \frac{xx\,\partial x}{\sqrt{1-xx}}.$$

Constat autem pro nostris terminis integrationis, ab $x=0$ usque ad $x=1$, esse $\int \frac{x\,\partial x}{\sqrt{1-xx}}=1$ et $\int \frac{xx\,\partial x}{\sqrt{1-xx}}=\frac{\pi}{4}$, unde colligitur $A=\frac{4}{\pi}$, id quod cum ipso producto *Wallisiano*, quo $\frac{\pi}{2}=\frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{6\cdot 6}{5\cdot 7}\cdot$ etc. egregie convenit.

Casus 2, quo $c=+bb$.

§. 29. Evolvamus nunc quoque alterum casum $c=+bb$, pro quo fractio continua hanc formam induit:

$$2A=2f-a+\cfrac{aa+4bb}{4f-2a+\cfrac{9aa+4bb}{4f-2a+\cfrac{25aa+4bb}{4f-2a+\cfrac{49aa+4bb}{4f-2a+\text{etc.}}}}}$$

At vero productum continuum, ex praecedente forma, loco b scribendo $b\sqrt{-1}$, ita imaginarie expressum se prodit:

$$A=(f-b\sqrt{-1})\cdot\frac{(f+b\sqrt{-1})(f+2a-b\sqrt{-1})}{(f+a+b\sqrt{-1})(f+a-b\sqrt{-1})}\cdot\frac{(f+2a+b\sqrt{-1})(f+4a-b\sqrt{-1})}{(f+3a+b\sqrt{-1})(f+3a-b\sqrt{-1})}\ \text{etc.}$$

Evidens autem est in eadem expressione §. 26. allata etiam loco b scribi potuisse $-b\sqrt{-1}$, unde prodiisset:

$$A=(f+b\sqrt{-1})\frac{(f-b\sqrt{-1})(f+2a+b\sqrt{-1})}{(f+a-b\sqrt{-1})(f+a+b\sqrt{-1})}\cdot\frac{(f+2a-b\sqrt{-1})(f+4a+b\sqrt{-1})}{(f+3a-b\sqrt{-1})(f+3a+b\sqrt{-1})}.$$

Productum igitur harum duarum expressionum fit reale, erit enim

$$A^2=(ff+bb)\frac{(ff+bb)((f+2a)^2+bb)}{((f+a)^2+bb)((f+a)^2+bb)}\cdot\frac{((f+2a)^2+bb)((f+4a)^2+bb)}{((f+3a)^2+bb)((f+3a)^2+bb)},\ \text{etc.}$$

quae expressio congruit cum superiore, §. 25. inventa.

§. 30. At vero etiam expressio per formulas integrales evadit imaginaria. Si enim in formulis §. 27. loco b scribatur $b\sqrt{-1}$, orietur sequens expressio:

$$A=(f-b\sqrt{-1})\int\frac{x^{f+a-1-b\sqrt{-1}}\,\partial x}{\sqrt{1-x^{2a}}}:\int\frac{x^{f-1+b\sqrt{-1}}\,\partial x}{\sqrt{1-x^{2a}}}.$$

Verum mutato imaginariorum signo erit

$$A = (f + b\sqrt{-1}) \int \frac{x^{f+a-1+b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}} : \int \frac{x^{f-1-b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}};$$

ubi nullum est dubium, quin in utraque expressione imaginaria se mutuo destruant, etiamsi nulla pateat methodus hanc mutuam imaginariorum destructionem actu evolvere.

§. 31. Verum si hae ambae expressiones in se mutuo ducantur, tum ista destructio haud difficulter ostendi poterit. Cum enim productum sit

$$A^2 = (ff + bb) \frac{\int \frac{x^{f+a-1-b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}} \cdot \int \frac{x^{f+a-1+b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}}}{\int \frac{x^{f-1+b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}} \cdot \int \frac{x^{f-1-b\sqrt{-1}}\partial x}{\sqrt{1-x^{2a}}}},$$

demonstrari potest tam in numeratore quam in denominatore imaginaria seorsim se destruere, quod quidem pro denominatore ostendisse sufficiet, cum numerator inde oriatur, scribendo $f + a$ loco f.

§. 32. Quo demonstratio succinctior reddatur, ponamus brevitatis gratia $\frac{x^{f-1}\partial x}{\sqrt{1-x^{2a}}} = \partial V$, quo facto denominator nostrae expressionis, imaginariis affectae, erit

$$\int x^{+b\sqrt{-1}}\partial V \cdot \int x^{-b\sqrt{-1}}\partial V.$$

Jam statuatur factorum

summa $= \int (x^{b\sqrt{-1}} + x^{-b\sqrt{-1}})\partial V = p$,

differentia $= \int (x^{b\sqrt{-1}} - x^{-b\sqrt{-1}})\partial V = q$,

atque notum est productum propositum fore

$$\int x^{b\sqrt{-1}}\partial V \cdot \int x^{-b\sqrt{-1}}\partial V = \frac{pp-qq}{4}.$$

Monstrabo igitur tam pp quam qq ad quantitates reales reduci posse.

§. 33. Hunc in finem loco x in potestatibus imaginariis scribamus e^{lx}, ut fiat

$$p = \int (e^{blx\sqrt{-1}} + e^{-blx\sqrt{-1}})\partial V,$$
$$q = \int (e^{blx\sqrt{-1}} - e^{-blx\sqrt{-1}})\partial V.$$

Cum igitur noverimus esse

$$e^{\Phi\sqrt{-1}} + e^{-\Phi\sqrt{-1}} = 2\cos.\Phi \text{ et}$$
$$e^{\Phi\sqrt{-1}} - e^{-\Phi\sqrt{-1}} = 2\sqrt{-1}\sin.\Phi,$$

posito brevitatis gratia $blx = \Phi$ fiet

$$p = 2\int\partial V\cos.\Phi \text{ et } q = 2\sqrt{-1}\int\partial V\sin.\Phi,$$

unde sponte fluit denominator

$$\frac{pp-qq}{4} = (\int\partial V\cos.\Phi)^2 + (\int\partial V\sin.\Phi)^2,$$

expressio quae manifesto est realis.

§. 34. Hinc facile colligitur valor numeratoris, quippe qui erit

$$(\int x^a\partial V\cos.\Phi)^2 + (\int x^a\partial V\sin.\Phi)^2,$$

ita ut expressio nostra, imaginariis turbata, pro A^2 sequenti modo realiter repraesentetur:

$$A^2 = (ff+bb)\frac{(\int x^a\partial V\cos.\Phi)^2 + (\int x^a\partial V\sin.\Phi)^2}{(\int\partial V\cos.\Phi)^2 + \int\partial V\sin.\Phi)^2},$$

existente $\partial V = \frac{x^{f-1}\partial x}{\sqrt{1-x^{2a}}}$ et $\Phi = blx$.

§. 35. In analysi autem adhuc desideratur methodus per integrationem tractandi hujusmodi formulas:

$$\int \frac{x^{f-1}\partial x \cos. b l x}{\sqrt{-x^{2a}}} \text{ et } \int \frac{x^{f-1}\partial x \sin. b l x}{\sqrt{1-x^{2a}}}.$$

Interim tamen si denominator abesset, utraque formula revera integrari posset, id quod sequenti modo ostendisse operae pretium erit.

§. 36. Praestari enim hoc poterit ope reductionis notissimae $\int P\partial Q = PQ - \int Q\partial P$. Si scilicet pro formula priore sumatur $P = \cos. b l x$ et $\partial Q = x^{f-1}\partial x$, fiet

$$\int x^{f-1}\partial x \cos. b l x = \frac{x^f}{f}\cos. b l x + \frac{b}{f}\int x^{f-1}\partial x \sin. b l x.$$

Pro altera vero, sumto $P = \sin. b l x$ et $\partial Q = x^{f-1}\partial x$, erit

$$\int x^{f-1}\partial x \sin. b l x = \frac{x^f}{f}\sin. b l x - \frac{b}{f}\int x^{f-1}\partial x \cos. b l x.$$

Hinc porro colligitur substituendo

$$\int x^{f-1}\partial x \cos. b l x = \frac{x^f}{ff+bb}(f\cos. b l x + b \sin. b l x);$$

$$\int x^{f-1}\partial x \sin. b l x = \frac{x^f}{ff+bb}(f\sin. b l x - b \cos. b l x).$$

At vero, accedente denominatore, nihil aliud intelligitur, nisi integrale ad genus quantitatum maxime transcendentium, adhuc ignotum, revolvi.