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Observationes circa fractiones continuas in hac
forma contentas $S =$
 $n / (1 + (n+1) / (2 + (n+2) / (3 + (n+3) / (4 + \text{etc.}))))$

Leonhard Euler

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OBSERVATIONES CIRCA FRACTIONES CONTINUAS.
IN HAC FORMA CONTENTAS:

$$S = \frac{n}{1 + \frac{n+1}{2 + \frac{n+2}{3 + \frac{n+3}{4 + \text{etc.}}}}}$$

AUCTORE

L. EULERO.

Conventui exhib. die 18. Novembris 1779.

I. Cum plures adhuc inventae sint methodi ad fractiones continuas deveniendi, earumque visissim valores assignandi; nulla tamen earum ita est comparata, cujus ope valores earum fractionum continuarum, quae in hac forma sunt contentae, investigari queant, unico casu excepto, quo $n = 1$. Memini enim a me jam olim istius formae:

$$\frac{1}{1 + \frac{2}{2 + \frac{3}{3 + \frac{4}{4 + \text{etc.}}}}}$$

valorem inventum esse $= \frac{1}{e-1}$, denotante e numerum, cujus logarithmus hyperbolicus est unitas, id quod mihi quidem eo magis mirum videtur, quod reliquis casibus omnibus, quibus n est numerus integer positivus, summa

adeo per numeros rationales exprimi queat; quamobrem mea investigatio super hoc argumento, qua istas summas inveni, plurimum lucis allatura videtur ad hanc doctrinam summi momenti uberius excolendam.

II. Quo indolem hujus formae accuratius perscrutari liceat, eam sequentibus formulis sum complexus:

$$\begin{aligned} S &= \frac{n}{1+A}, \\ A &= \frac{n+1}{2+B}, \\ B &= \frac{n+2}{3+C}, \\ C &= \frac{n+3}{4+D}, \\ &\text{etc.} \end{aligned}$$

ex quibus vicissim sequentes derivantur:

$$\begin{aligned} A &= \frac{n}{S-1}, \\ B &= \frac{n+1}{A} - 2, \\ C &= \frac{n+2}{B} - 3, \\ D &= \frac{n+3}{C} - 4, \\ &\text{etc.} \end{aligned}$$

III. Hinc jam facile patet semper esse:

$$S < \frac{n}{1}; \quad A < \frac{n+1}{2}; \quad B < \frac{n+2}{3}; \quad C < \frac{n+3}{4} \text{ etc.}$$

Quodsi jam in prima formula loco A scribatur valor $\frac{n+1}{2}$, qui est valde magnus, tum fractio $\frac{n}{1+A}$ erit nimis parva, ideoque erit $S > \frac{n}{1+\frac{n+1}{2}}$, sive erit $S > \frac{2n}{n+3}$. Simili modo circa sequentes formulas ratiocinando fiet:

$$A > \frac{n+1}{2+n+2}, \text{ sive } A > \frac{3(n+1)}{n+8};$$

$$B > \frac{n+2}{3+n+3}, \text{ sive } B > \frac{4(n+2)}{n+15};$$

$$C > \frac{n+3}{4+n+4}, \text{ sive } C > \frac{5(n+3)}{n+24};$$

$$D > \frac{n+4}{5+n+5}, \text{ sive } D > \frac{6(n+4)}{n+35};$$

Conveniet autem has formulas conspectui clarius exponi.

Tabula I.

$S = \frac{n}{1+A}$	$A = \frac{n}{S} - 1$
$A = \frac{n+1}{2+B}$	$B = \frac{n+1}{A} - 2$
$B = \frac{n+2}{3+C}$	$C = \frac{n+2}{B} - 3$
$C = \frac{n+3}{4+D}$	$D = \frac{n+3}{C} - 4$
$D = \frac{n+4}{5+E}$	$E = \frac{n+4}{D} - 5$
$E = \frac{n+5}{6+F}$	$F = \frac{n+5}{E} - 6$
$F = \frac{n+6}{7+G}$	

Tabula II.

$S < \frac{n}{1}$	$S > \frac{2n}{n+3}$
$A < \frac{n+1}{2}$	$A > \frac{3(n+1)}{n+8}$
$B < \frac{n+2}{3}$	$B > \frac{4(n+2)}{n+15}$
$C < \frac{n+3}{4}$	$C > \frac{5(n+3)}{n+24}$
$D < \frac{n+4}{5}$	$D > \frac{6(n+4)}{n+35}$
$E < \frac{n+5}{6}$	$E > \frac{7(n+5)}{n+48}$
$F < \frac{n+6}{7}$	$F > \frac{8(n+6)}{n+63}$

IV. Ope harum tabularum facile erit, assumpto pro S valore quocunque, dignoscere, utrum is sit veritati consentaneus, nec ne? Quodsi enim inde, ex primae tabulae secunda columna, quaerantur valores A, B, C etc. statim atque eorum aliquis extra limites in secunda tabula assignatos cadat, id certum erit signum valorem assumptum esse falsum, ideoque vel nimis magnum vel nimis parvum; hocque modo, plures pro S valores fingendo, continuo pro-

pius ad verum valorem accedere licebit. His igitur observationibus utamur ad quosdam casus simpliciores evolvendos.

Evolutio casus quo $n=2$, ideoque

$$S = \frac{2}{1 + \frac{3}{2 + \frac{4}{3 + \frac{5}{4 + \frac{6}{5 + \text{etc.}}}}}}$$

V. Pro hoc igitur casu limites erunt $S < \frac{2}{3}$ et $S > \frac{4}{5}$; $A < \frac{3}{2}$ et $A > \frac{2}{10}$; $B < \frac{4}{3}$ et $B > \frac{16}{17}$; $C < \frac{5}{4}$ et $C > \frac{25}{26}$; $D < \frac{6}{5}$ et $D > \frac{36}{37}$ etc. Sumamus igitur $S=1$, atque si inde sequentium litterarum valores deducantur, reperiemus $A=1$; $B=1$; $C=1$; $D=1$ etc. qui valores cum omnes intra assignatos limites cadant, id certum est signum valorem assumptum $S=1$ veritati esse consentaneum, quod quidem facile ex ipsa forma perspicere licuisset.

Evolutio casus quo $n=3$ et

$$S = \frac{3}{1 + \frac{4}{2 + \frac{5}{3 + \frac{6}{4 + \frac{7}{5 + \text{etc.}}}}}}$$

VI. Hoc ergo casu limites nostri erunt:

$$S < \frac{3}{4}; \quad S > 1,$$

$$A < 2; \quad A > \frac{16}{17},$$

$$B < \frac{5}{3}; \quad B > \frac{26}{18},$$

$$C < \frac{6}{4}; \quad C > \frac{36}{17},$$

$$D < \frac{7}{5}; \quad D > \frac{42}{23},$$

etc.

Hinc jam statim patet non esse $S=2$; foret enim $A=\frac{1}{2}$, quod excluditur. Sumto $S=\frac{3}{2}$, fit $A=1$, qui valor etiam extra limites cadit. Sumatur igitur $S=\frac{4}{3}$ et prodibit $A=\frac{5}{4}$, qui valor jam intra limites cadit; hinc ergo fiet $B=\frac{6}{5}$; $C=\frac{7}{6}$; $D=\frac{8}{7}$; $E=\frac{9}{8}$; $F=\frac{10}{9}$ etc., qui valores cum omnes intra limites praescriptos cadant, hoc certum est signum hujus fractionis continuae propositae verum valorem esse $S=\frac{4}{3}$.

VII. Commode hic usu venit, ut omnes valores litterarum S, A, B, C, D etc. manifesto ordine se insequantur, scilicet $\frac{4}{3}; \frac{5}{4}; \frac{6}{5}; \frac{7}{6}; \frac{8}{7}$ etc., quandoquidem termini harum fractionum progressionem arithmeticam constituunt. At vero ipsa rei natura postulat, ut haec litterae S, A, B, C, D etc. secundum legem quandam uniformem progrediantur, quemadmodum hoc casu prodit progressio arithmetica, cujus differentiae primae sunt constantes; unde concludere licet, etiam pro reliquis casibus ejusmodi valores pro litteris S, A, B, C, D , prodire debere, qui differentiis continuo sumendis tandem ad differentias evanescentes perducant. Hoc observato relinquamus ipsi S suum valorem indefinitum, indeque computemus valores sequentium

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litterarum: $A = \frac{3-S}{S}$; $B = \frac{6S-6}{3-S}$; $C = \frac{33-23S}{6S-6}$; $D = \frac{128S-168}{33-23S}$ etc.

Jam termini harum fractionum, scilicet denominatores, in hac serie progrediuntur: 1; S; 3-S; 6S-6; 33-23S; 128S-168. Hinc erunt:

Different. primae S-1; 3-2S; 7S-9; 39-29S; 151S-201.

Different. secundae: 4-3S; 9S-12; 48-36S; 180S-240.

Different. tertiae: 12S-16; 60-45S; 216S-288.

Hic statim patet differentias primas non evanescere, quia ex iis, nihilo aequatis, prodirent diversi valores pro S sequentes: 1; $\frac{3}{2}$; $\frac{9}{7}$; $\frac{39}{29}$ etc. At vero si differentiae secundae nihilo aequentur, omnes praebent $S = \frac{4}{3}$, quem eundem valorem differentiae tertiae, et sequentes, nihilo aequatae, producant, sicque certi esse possumus revera fore $S = \frac{4}{3}$.

Evolutio casus quoniam $n = 4$ et

$$S = \frac{4}{1} + \frac{5}{2} + \frac{6}{3} + \frac{7}{4} + \text{etc.}$$

VIII. Hic statim adhibeamus methodum modo ante expositam, et ex valore indefinito S colligimus valores litterarum A, B, C, D etc. qui erunt:

$$A = \frac{4-S}{S}; \quad B = \frac{7S-8}{4-S}; \quad C = \frac{48-27S}{7S-8}; \quad D = \frac{157S-248}{48-27S};$$

$$E = \frac{1624-1001S}{157S-248}.$$

Jam termini harum formularum in seriem disponantur, et continuo differentiae capiantur, ut sequitur:

$$\begin{array}{l}
 1; S; 4 - S; 7S - 8; 48 - 27S; 157S - 248; \\
 \text{D. I. } S - 1; 4 - 2S; 8S - 12; 56 - 34S; 184S - 296; \\
 \text{D. II. } 5 - 3S; 10S - 16; 68 - 42S; 218S - 352; \\
 \text{D. III. } \quad \quad 13S - 21; 84 - 52S; 260S - 420; \\
 \text{D. IV. } \quad \quad \quad \quad 105 - 65S; 312S - 504.
 \end{array}$$

Hic statim perspicuum est neque differentias primas, neque secundas, scopo satisfacere; quia ex iis, nihilo aequatis, diversi valores pro S essent prodituri; at vero differentiae tertiae omnes dant $S = \frac{21}{13}$, qui ergo pro vero valore fractionis continuae propositae est habendus.

IX. Quo autem de hoc certiores reddamur, exploremus istum valorem $\frac{21}{13}$ per methodum primo indicatam, ex eoque computemus sequentes valores ope secundae columnae primae tabulae, sumendo $n=4$, ut sequitur: $A = \frac{31}{21}$; $B = \frac{43}{31}$; $C = \frac{57}{43}$; $D = \frac{73}{57}$; $E = \frac{91}{73}$; qui valores omnes intra limites in tabula secunda datos cadere deprehenduntur. Praeterea vero egregium ordinem progressionis inter se servant, cum eorum termini crescant secundum differentias; 8, 10, 12, 14, 16 etc. quae scilicet binario continuo crescunt; cum contra quilibet alii valores pro S assumti ad valores ab-

surdos deducerent, qui mox extra limites praescriptos extravagarentur.

Evolutio casus, quo $n = 5$ et

$$S = \frac{5}{1 + \frac{6}{2 + \frac{7}{3 + \frac{8}{4 + \text{etc.}}}}}$$

X. Applicemus hic etiam methodum ante usitatam, ac reperientur sequentes valores:

$$A = \frac{5-S}{S}; \quad B = \frac{8S-10}{5-S}; \quad C = \frac{65-31S}{8S-10}; \quad D = \frac{188S-340}{65-31S};$$

$$E = \frac{2285-1219S}{188S-340}.$$

Jam termini harum fractionum in seriem disponantur, et differentiae continuae sumantur hoc modo:

$$1; S; 5-S; 8S-10; 65-31S; 188S-340; 2285-1219S;$$

$$D. I. \quad S-1; 5-2S; 9S-15; 75-39S; 219S-405; 2625-1407S;$$

$$D. II. \quad 6-3S; 11S-20; 90-48S; 258S-480; 3030-1626S;$$

$$D. III. \quad 14S-26; 110-59S; 306S-570; 3510-1884S;$$

$$D. IV. \quad 136-73S; 365S-680; 4080-2190S.$$

XI. Hinc differentiae tertiae nondum negotium conficiunt, quia inde orientur diversi valores pro S ; at ex differentiis quartis omnibus idem eruitur valor $S = \frac{136}{73}$, qui ergo est verus valor fractionis continuae, quem si secundum primam methodum explorare velimus, egregie cum

limitibus praescriptis convenire deprehendetur. Casus autem jam evoluti, in ordinem digesti, erunt:

$$\begin{array}{c|c|c|c} n = 2 & n = 3 & n = 4 & n = 5 \\ \hline S = 1 & S = \frac{4}{3} & S = \frac{21}{13} & S = \frac{136}{73} \end{array}$$

XII. Quia autem inter hos valores nullus ordo observatur, et methodus, qua sumus usi, pro casibus sequentibus nimis taediosas calculi operationes requireret, alias methodos sum traditurus, quae meliori successu ad scopum optatum, etiam pro majoribus numeris loco n assumtis, perdúcant.

Methodus secunda

summas harum serierum continuarum investigandi.

XIII. Quoniam vidimus valores litterarum S, A, B, C, D etc. semper secundum certam quandam legem uniformem progredi, dum litterae A, B, C, D valores expriment similiarum fractionum continuarum, vel uno, vel duobus, vel tribus membris truncatas, cum sit:

$$A = \frac{n+1}{2+n+2}; \quad B = \frac{n+2}{3+n+3}; \quad C = \frac{n+3}{4+n+4};$$

$$\frac{3+n+3}{4+etc.}; \quad \frac{4+n+4}{5+etc.}; \quad \frac{5+n+5}{6+n+6};$$

$$\frac{6+n+6}{7+etc.}$$

nullum est dubium, quin etiam nostra formula proposita, retro continuata, similem legem uniformem sit secutura. Sin autem nostra forma uno gradu retro continuetur, prodibit $\frac{n-1}{0+n-1}$, quam vocemus $=\alpha$, ita ut sit $\alpha = \frac{n-1}{s}$. At si

duobus gradibus retro continuemus, erit $\frac{n-2}{-1+\alpha} = \xi$. Si simili modo ultro retrogradiamur, nanciscemur has formulas:

$$\frac{n-3}{-2+\xi} = \gamma; \quad \frac{n-4}{-3+\gamma} = \delta; \quad \frac{n-5}{-4+\delta} = \varepsilon; \quad \frac{n-6}{-5+\varepsilon} = \zeta,$$

atque hic certo affirmare licet, inter has novas litteras.... $\zeta, \varepsilon, \delta, \gamma, \xi, \alpha, S, A, B$ etc. similem legem uniformitatis continuam deprehendi debere. Retro igitur hae formulae continuentur, donec perveniatur ad numeratorem $=0$, quo casu habebitur talis forma:

$$\frac{\sigma}{-\lambda+1} \\ \frac{\sigma}{-\lambda+1+z} \\ \frac{\sigma}{-\lambda+z+g} \\ \frac{\sigma}{-\lambda+3+\text{etc.}}$$

quae autem expressio, etiamsi numerator est $=0$, ideo non ipsa evanescere est censenda, quia evenire potest, ut etiam denominator evanescat. Atque hoc revera usu venit in nostris formis, quas sumus perscrutaturi; pro iis igitur erit:

$$0 = -\frac{\lambda+1}{-\lambda+1+z} \\ \frac{\lambda+1}{-\lambda+z+\text{etc.}}$$

quae ergo fractio continua si continuetur usque ad ipsam formam propositam S , inde elici poterit valor ipsius S , id quod pro singulis casibus ostendemus.

XIV. Sit igitur $n=2$, et forma fractionis continuæ, retro continuatae, erit $-1 + \frac{1}{\sigma+S}$, quae ergo nihilo aequata dabit $S=1$, ut ante invenimus. Pro casu $n=3$ orietur haec aequatio: $0 = -\frac{2+z}{-1+z} + \frac{1}{\sigma+S}$, unde fit $2 = \frac{z}{-1+z} \frac{\sigma+S}{S}$ ideo-

que $S = \frac{4}{3}$, ut ante. Pro casu $n = 4$ habebimus:

$$o = -\frac{3+1}{-2+2} = \frac{1}{0+S},$$

unde fit $S = \frac{21}{13}$. Hic autem calculus expeditius instituetur, si litteris ante introductis α, ξ, γ utamur; tum enim erit $o = -3 + \gamma$. Erat autem $\alpha = \frac{3}{8}$; $\xi = \frac{2}{-1+\alpha}$; $\gamma = \frac{1}{-2+\xi}$. Hic ergo erit $\gamma = 3$, ideoque $\xi = \frac{1}{-2+\xi}$, unde fit $\xi = \frac{7}{3} = \frac{2}{-1+\alpha}$. hincque colligitur $\alpha = \frac{13}{7} = \frac{3}{8}$, sicque tandem erit $S = \frac{21}{13}$. Hoc ergo artificio etiam in sequentibus utamur.

XV. Quo has operationes pro majoribus numeris n sublevemus, ex formulis pro litteris $\alpha, \xi, \gamma, \delta$ etc. ante assumtis derivemus reciprocas, quibus quaelibet littera per praecedentem definiatur, quas utrasque formulas in sequenti tabella exhibeamus:

Cum sit	erit
$\alpha = \frac{n-1}{S}$	$S = 0 + \frac{n-1}{\alpha}$
$\xi = \frac{n-2}{-1+\alpha}$	$\alpha = 1 + \frac{n-2}{\xi}$
$\gamma = \frac{n-3}{-2+\xi}$	$\xi = 2 + \frac{n-3}{\gamma}$
$\delta = \frac{n-4}{-3+\gamma}$	$\gamma = 3 + \frac{n-4}{\delta}$
$\varepsilon = \frac{n-5}{-4+\delta}$	$\delta = 4 + \frac{n-5}{\varepsilon}$
$\zeta = \frac{n-6}{-5+\varepsilon}$	$\varepsilon = 5 + \frac{n-6}{\zeta}$
	$\zeta = 6 + \frac{n-7}{\eta}$

XVI. Jam beneficio hujus tabulae facile erit omnes casus evolvere. Ac primo quidem, sumto $n=1$, quo casu fit:

$$S = \frac{1}{1+2} + \frac{2}{2+3} + \frac{3}{3+4} + \text{etc.}$$

videtur fieri $S=0$, cum tamen supra innuimus esse $S = \frac{1}{e-1}$. Verum probe notetur, istam conclusionem non valere, si etiam fuerit $\alpha=0$, quia tum erit $S=0+\frac{0}{0}$. Nihil vero repugnat, quo minus sit $\frac{0}{0} = \frac{1}{e-1}$, quae exceptio solo hoc casu locum habet. Progrediamur ergo ad reliquos casus; ac sumto $n=2$, ubi est $S = \frac{2}{1+3} + \frac{3}{2+4} + \text{etc.}$

quia hic ξ non est $=0$, manifesto erit $\alpha=1$, hincque $S=1$, uti jam ante invenimus. Sit jam $n=3$, ideoque:

$$S = \frac{1}{1+4} + \frac{2}{2+5} + \frac{3}{3+6} + \text{etc.}$$

erit $\xi=2$, quia non est $\gamma=0$; hinc ergo regrediendo erit $\alpha = 1 + \frac{1}{2} = \frac{3}{2}$ et $S = \frac{4}{3}$. Sumto porro $n=4$, quo casu erit:

$$S = \frac{4}{1+5} + \frac{5}{2+6} + \frac{6}{3+7} + \text{etc.}$$

erit $\gamma=3$, unde oriuntur sequentes valores:

$$\xi = 2 + \frac{1}{3} = \frac{7}{3};$$

$$\alpha = 1 + \frac{2 \cdot 3}{3} = \frac{13}{3} \text{ et}$$

$$S = \frac{21}{13}.$$

Sit $n=5$, quo casu fit: $S = \frac{5}{1 + \frac{7}{2 + \frac{8}{3 + \frac{9}{4 + \text{etc.}}}}}$

tum erit $\delta = 4$, unde sequentes oriuntur valores:

$$\begin{aligned}\gamma &= 3 + \frac{1}{4} = \frac{13}{4}, \\ \xi &= 2 + \frac{2 \cdot 4}{13} = \frac{34}{13}, \\ \alpha &= 1 + \frac{3 \cdot 13}{34} = \frac{73}{34} \text{ et} \\ S &= 0 + \frac{4 \cdot 34}{73} = \frac{136}{73}.\end{aligned}$$

XVII. Nunc ulterius progrediamur, ac ponamus $n=6$, quo casu erit:

$$S = \frac{6}{1 + \frac{8}{2 + \frac{9}{3 + \text{etc.}}}}$$

eritque $\varepsilon = 5$, unde sequentes oriuntur valores:

$$\begin{aligned}\delta &= 4 + \frac{1}{5} = \frac{21}{5}, \\ \gamma &= 3 + \frac{2 \cdot 5}{21} = \frac{73}{21}, \\ \xi &= 2 + \frac{3 \cdot 21}{73} = \frac{209}{73}, \\ \alpha &= 1 + \frac{4 \cdot 73}{209} = \frac{501}{209}, \\ S &= 0 + \frac{5 \cdot 209}{501} = \frac{1045}{501}.\end{aligned}$$

XVIII. Sit nunc $n=7$ et $S = \frac{7}{1 + \frac{9}{2 + \frac{10}{3 + \frac{11}{4 + \text{etc.}}}}}$

erit $\zeta = 6$, unde sequentes oriuntur valores:

$$\begin{aligned}\varepsilon &= 5 + \frac{1}{6} = \frac{31}{6}, \\ \delta &= 4 + \frac{2 \cdot 6}{31} = \frac{136}{31},\end{aligned}$$

$$\gamma = 3 + \frac{3 \cdot 136}{136} = \frac{501}{136},$$

$$\epsilon = 2 + \frac{4 \cdot 136}{501} = \frac{1546}{501},$$

$$\alpha = 1 + \frac{5 \cdot 501}{1546} = \frac{4051}{1546},$$

$$S = 0 + \frac{6 \cdot 1546}{4051} = \frac{9276}{4051}.$$

XIX. Sit nunc $n=8$ et $S = \frac{8}{1 + \frac{9}{2 + \frac{10}{3 + \text{etc.}}}}$, tum erit

$\eta = 7$, unde sequentes oriuntur valores:

$$\zeta = 6 + \frac{1}{7} = \frac{43}{7},$$

$$\epsilon = 5 + \frac{2 \cdot 7}{43} = \frac{229}{43},$$

$$\delta = 4 + \frac{3 \cdot 43}{229} = \frac{1045}{229},$$

$$\gamma = 3 + \frac{4 \cdot 229}{1045} = \frac{4051}{1045},$$

$$\epsilon = 2 + \frac{5 \cdot 1045}{4051} = \frac{13327}{4051},$$

$$\alpha = 1 + \frac{6 \cdot 4051}{13327} = \frac{37633}{13327},$$

$$S = 0 + \frac{7 \cdot 13327}{37633} = \frac{93289}{37633}.$$

XX. Hos jam valores pro littera S inventos in ordinem disponamus, quo eorum progressio facilius considerari possit, quos ergo sequenti modo repraesentemus:

n	2	3	4	5	6	7	8
S	1	$\frac{4}{3}$	$\frac{21}{13}$	$\frac{136}{73}$	$\frac{1045}{501}$	$\frac{9276}{4051}$	$\frac{93289}{37633}$

Inter has autem fractiones primo intuitu nulla certa lex regnare videtur; verum re attentius perpensa haud difficulter quandam progressionis legem observare licet. Si enim quemlibet numeratorem cum summa numeratoris ac deno-

minatoris praecedentis fractionis comparemus, ordinem maxime memorabilem deprehendemus, cum sit:

$$\begin{aligned} 4 &= 2 \quad (1 + 1) = 2 \cdot 2 \quad , \\ 21 &= 3 \quad (4 + 3) = 3 \cdot 7 \quad , \\ 136 &= 4 \quad (21 + 13) = 4 \cdot 34 \quad , \\ 1045 &= 5 \quad (136 + 73) = 5 \cdot 209 \quad , \\ 9276 &= 6 \quad (1045 + 501) = 6 \cdot 1546 \quad , \\ 93289 &= 7 \quad (9276 + 4051) = 7 \cdot 13327 \quad . \end{aligned}$$

Pro denominatoribus autem haud absimilis ratio observatur, cum quilibet sit summa praecedentis numeratoris certo multiplo denominatoris aucti, qui ordo sequenti modo in oculos incurret:

$$\begin{aligned} 3 &= 1 + 2 \quad , \quad 1 = 1 + 2 \quad , \\ 13 &= 4 + 3 \quad , \quad 3 = 4 + 9 \quad , \\ 73 &= 21 + 4 \quad , \quad 13 = 21 + 52 \quad , \\ 501 &= 136 + 5 \quad , \quad 73 = 136 + 365 \quad , \\ 4051 &= 1045 + 6 \quad , \quad 501 = 1045 + 3006 \quad , \\ 37633 &= 9276 + 7 \quad , \quad 4051 = 9276 + 28357 \quad . \end{aligned}$$

XXI. Hinc ergo si pro quolibet numero n inventa fuerit fractio $S = \frac{p}{q}$, pro sequente numero $n+1$ fiet $S = \frac{n(p+q)}{p+nq}$, cujus formulae ope ex quolibet casu sequens multo expeditius reperiri poterit, quam per praecedentem methodum. Ita cum pro casu $n=8$ repertus fuerit valor $S = \frac{93289}{37633}$, pro sequente casu $n=9$ erit: $S = \frac{9(93289 + 37633)}{93289 + 8 \cdot 37633} = \frac{1047376}{294353}$. Ve-

rum quia haec eximia regula hactenus sola inductione innitur, ejus demonstrationem in sequente articulo dabimus. Ante autem quam hoc argumentum deseramus, observasse juvabit, posteriorem columnam tabulae (§. XV.) datae nobis insignem novam proprietatem suppeditare. Si enim loco litterarum α , ξ , γ , etc. earum valores successive substituamus, pro S novam fractionem continuam nanciscemur, quae ita se habet:

$$S = \frac{n-1}{1+n} \frac{n-2}{2+n} \frac{n-3}{3+n} \frac{n-4}{4+n} \frac{n-5}{5+n} \text{ etc.}$$

quae forma semper abrumpitur, unde operae pretium erit hanc transformationem in sequenti theoremate ante oculos posuisse.

Theorema.

XXII. Si fuerit $S = \frac{n}{1+n} \frac{n+r}{2+n} \frac{n+2}{3+n} \frac{n+3}{4+n} \text{ etc.}$

erit etiam semper

$$S = \frac{n-r}{1+n} \frac{n-2}{2+n} \frac{n-3}{3+n} \frac{n-4}{4+n} \frac{n-5}{5+n} \text{ etc.}$$

siquidem n fuerit numerus integer positivus, excepta unitate, ob rationem supra allegatam.

Methodus tertia

in summas harum fractionum continuarum inquirendi.

XXIII. Si evolutiones casuum ante tractatorum contem-
plemur, animadvertemus in fractionibus pro litteris α ,
 β , γ , etc. datis inverso ordine cujuslibet numerator datae
denominatorum sequentis; quamobrem omnes terminos or-
dine disponamus, ac differentias tam primas quam secundas
et sequentes subjiciamus. Ita pro casu $n=2$ termini harum
fractionum, ab unitate incipiendo, erunt sequentes:

1. 2. 3. 4.

D. I. 1. 1. 1.

Simili modo pro $n=4$ habebimus hos terminos:

1. 3. 7. 13. 21.

D. I. 2. 4. 6. 8.

D. II. 2. 2. 2. /

Casus vero $n=5$ praebet sequens schema:

1. 4. 13. 34. 73. 136.

D. I. 3. 9. 21. 39. 63.

D. II. 6. 12. 18. 24.

D. III. 6. 6. 6.

Casus porro $n=6$ dat sequens schema:

1. 5. 21. 73. 209. 501. 1045.

D. I. 4. 16. 52. 136. 292. 544.

D. II. 12. 36. 84. 156. 242.

D. III. 24. 48. 72. 86.

D. IV. 24. 24. 24.

Pro casu $n = 7$ habebimus :

1. 6. 31. 136. 501. 1546. 4051. 9276.

D. I. 5. 25. 105. 365. 1045. 2505. 5225.

D. II. 20. 180. 260. 680. 1460. 2720.

D. III. 60. 180. 420. 780. 1260.

D. IV. 120. 240. 360. 480.

D. V. 120. 120. 120.

Casus denique $n = 8$ dat sequens schema :

1. 7. 43. 229. 1045. 4051. 13327. 37633. 93289.

D. I. 6. 36. 186. 816. 3006. 9276. 24306. 55656.

D. II. 30. 150. 630. 2190. 6270. 15030. 31350.

D. III. 120. 480. 1560. 4080. 8760. 16320.

D. IV. 360. 1080. 2520. 4680. 7560.

D. V. 720. 1440. 2160. 2880.

D. VI. 720. 720. 720.

XXIV. Consideratio horum casuum nobis sequentes conclusiones suppeditat :

1°. Quia omnes hi casus tandem ad differentias constantes perducunt, hinc discimus, omnes istas series esse

algebraicas, quarum scilicet terminus generalis algebraice exhiberi queat.

2°. Porro etiam videmus differentias constantes constituere progressionem hypergeometricam, scilicet:

1. 2. 6. 24. 120. 720. etc.

3°. Constat autem terminos generales cujusque progressionis ex terminis primis singularum differentiarum formari, qui ergo termini primi se habent ut sequens tabella indicat:

$n = 2$	1.
$n = 3$	1. 1.
$n = 4$	1. 2. 2.
$n = 5$	1. 3. 6. 6.
$n = 6$	1. 4. 12. 24. 24.
$n = 7$	1. 5. 20. 60. 120. 120.
$n = 8$	1. 6. 30. 120. 360. 720. 720.

Evidens autem est hanc postremam seriem hoc modo representari posse:

1; 6; 6. 5; 6. 5. 4; 6. 5. 4. 3; 6. 5. 4. 3. 2; 6. 5. 4. 3. 2. 1.

4°. Cum ista progressio referatur ad casum $n = 8$, hinc tuto concludere licet, in genere terminos primos tam ipsius seriei, quam ipsarum differentiarum, hanc constitutos esse progressionem:

1; $n - 2$; $(n - 2)(n - 3)$; $(n - 2)(n - 3)(n - 4)$; etc.

5°. Deinde vero ex doctrina progressionum constat terminum generalem cujusque seriei reperiri, si, manente termino ipsius seriei primo, primarum differentiarum terminus primus multiplicetur per x ; secundarum vero differentiarum per $\frac{x(x-1)}{1 \cdot 2}$; tertiarum per $\frac{x(x-1)(x-2)}{1 \cdot 2 \cdot 3}$ et ita porro, unde terminus generalis pro nostro casu hoc modo exprimetur:

$1 + (n-2)x + (n-2)(n-3)\frac{x(x-1)}{1 \cdot 2} + (n-2)(n-3)(n-4)\frac{x(x-1)(x-2)}{1 \cdot 2 \cdot 3}$ etc. unde sumto $x=1$, oritur terminus secundus $n=1$; at si loco x sumantur numeri 2, 3, 4 etc. orientur termini tertius, quartus, quintus, etc. Conveniet autem haec pro singulis casibus evolvere.

XXV. Hinc ergo si fuerit $n=2$, terminus generalis erit $=1$; at si $n=3$, terminus generalis erit $1+x$. Hoc autem casu ipsa series erat: 1. 2. 3. 4; ubi patet sumto $x=3$ prodire terminum ultimum 4, qui per praecedentem divisus dat valorem ipsius $S = \frac{4}{3}$. Sumatur nunc $n=4$ et seriei 1. 3. 7. 13. 21 erit terminus generalis:

$$= 1 + 2x + x(x-1) = 1 + x + xx,$$

unde sumto $x=4$ prodit terminus ultimus $=21$, qui per praecedentem 13 divisus dat $S = \frac{21}{13}$. Sit nunc $n=5$ et progressionis 1. 4. 13. 34. 73. 136 terminus generalis erit:

$$1 + 3x + 3 \cdot \frac{2 \cdot x(x-1)}{1 \cdot 2} + 3 \cdot 2 \cdot \frac{1 \cdot x(x-1)(x-2)}{1 \cdot 2 \cdot 3},$$

unde, sumto $x=5$, oritur ultimus terminus 136, qui per

penultimum divisus. dat valorem ipsius S. Simili modo, sumto $n = 6$, seriei 1. 5. 21. 73. 209. 501. 1045 terminus generalis erit:

$$1 + 4x - 4 \cdot 3 \frac{x(x-1)}{1 \cdot 2} + 4 \cdot 3 \cdot 2 \frac{x(x-1)(x-2)}{1 \cdot 2 \cdot 3} + 4 \cdot 3 \cdot 2 \cdot 1 \frac{x(x-1)(x-2)(x-3)}{1 \cdot 2 \cdot 3 \cdot 4},$$

qui posito $x = 6$ praebebat terminum ultimum, at posito $x = 5$ penultimum; quorum ille per hunc divisus praebebat valorem ipsius S.

XXVI. Quoniam autem hic tantum agitur de termino ultimo et penultimo, horum terminorum formam pro casu $n = 6$ accuratius perpendamus. Sumto igitur $x = 5$, erit terminus penultimus:

$$1 + 4 \cdot 5 + 4 \cdot 3 \cdot \frac{5 \cdot 4}{1 \cdot 2} + 4 \cdot 3 \cdot 2 \cdot \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} + 4 \cdot 3 \cdot 2 \cdot 1 \cdot \frac{5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4},$$

at sumto $x = 6$, habebitur terminus ultimus:

$$1 + 4 \cdot 6 + 4 \cdot 3 \cdot \frac{6 \cdot 5}{1 \cdot 2} + 4 \cdot 3 \cdot 2 \cdot \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} + 4 \cdot 3 \cdot 2 \cdot 1 \cdot \frac{6 \cdot 5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4},$$

unde si pro hoc casu ponatur $S = \frac{p}{q}$ et denominatores ad priores factores referantur, erit:

$$p = 1 + 4 \cdot 6 + 6 \cdot 6 \cdot 5 + 4 \cdot 6 \cdot 5 \cdot 4 + 1 \cdot 6 \cdot 5 \cdot 4 \cdot 3.$$

Eodemque modo erit:

$$q = 1 + 4 \cdot 5 + 6 \cdot 5 \cdot 4 + 4 \cdot 5 \cdot 4 \cdot 3 + 1 \cdot 5 \cdot 4 \cdot 3 \cdot 2,$$

ubi evidens est coefficientes priores ex potestate quarta binomii esse desumptos.

XXVII. In omnibus igitur casibus optimo successu coefficientibus binomialibus uti poterimus; et quoniam, pro valore generali n , coefficientes ex potestate $n - 2$ sunt

desumendi, meminisse juvabit, me olim hos coefficients sequenti modo expressisse :

$$\left[\begin{smallmatrix} n-2 \\ 1 \end{smallmatrix} \right]; \left[\begin{smallmatrix} n-2 \\ 2 \end{smallmatrix} \right]; \left[\begin{smallmatrix} n-2 \\ 3 \end{smallmatrix} \right], \left[\begin{smallmatrix} n-2 \\ 4 \end{smallmatrix} \right] \text{ etc.}$$

Si ponamus ut hactenus $S = \frac{p}{q}$, erit :

$$p = 1 + \left[\begin{smallmatrix} n-2 \\ 1 \end{smallmatrix} \right] n + \left[\begin{smallmatrix} n-2 \\ 2 \end{smallmatrix} \right] n(n-1) + \left[\begin{smallmatrix} n-2 \\ 3 \end{smallmatrix} \right] n(n-1)(n-2) \\ + \left[\begin{smallmatrix} n-2 \\ 4 \end{smallmatrix} \right] n(n-1)(n-2)(n-3) + \text{etc. et}$$

$$q = 1 + \left[\begin{smallmatrix} n-2 \\ 1 \end{smallmatrix} \right] (n-1) + \left[\begin{smallmatrix} n-2 \\ 2 \end{smallmatrix} \right] (n-1)(n-2) \\ + \left[\begin{smallmatrix} n-2 \\ 3 \end{smallmatrix} \right] (n-1)(n-2)(n-3) \text{ etc.}$$

Ita si fuerit $n = 7$, erit hinc :

$$p = 1 + 5 \cdot 7 + 10 \cdot 7 \cdot 6 + 10 \cdot 7 \cdot 6 \cdot 5 + 5 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \\ + 1 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \text{ et}$$

$$q = 1 + 5 \cdot 6 + 10 \cdot 6 \cdot 5 + 10 \cdot 6 \cdot 5 \cdot 4 + 5 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \\ + 1 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2.$$

Tales igitur expressiones ad quosvis casus facile applicantur.

XXVIII. Quodsi jam istae formulae pro p et q inventae accuratius perpendantur, et cum sequente casu $n + 1$, pro quo sit $S = \frac{p'}{q'}$, ideoque :

$$p' = 1 + \left[\begin{smallmatrix} n-1 \\ 1 \end{smallmatrix} \right] (n+1) + \left[\begin{smallmatrix} n-1 \\ 2 \end{smallmatrix} \right] (n+1)n \\ + \left[\begin{smallmatrix} n-1 \\ 3 \end{smallmatrix} \right] (n+1)n(n-1) \text{ etc. et}$$

$$q' = 1 + \left[\begin{smallmatrix} n-1 \\ 1 \end{smallmatrix} \right] n + \left[\begin{smallmatrix} n-1 \\ 2 \end{smallmatrix} \right] n(n-1) \\ + \left[\begin{smallmatrix} n-1 \\ 3 \end{smallmatrix} \right] n(n-1)(n-2) + \text{etc.}$$

comparentur, haud difficulter inde deduci poterit insignis

illa relatio, quam jam ante in medium attulimus, scilicet semper esse:

$$p' = np + nq \text{ et}$$

$$q' = p + nq,$$

quae proprietas eo magis est notatu digna, quod ejus ope ex quolibet casu sequens facillime derivari potest, quemadmodum jam in praecedente articulo est ostensum.

