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De transformatione functionum duas variabiles involventium dum earum loco aliae binae variabiles introducuntur

Leonhard Euler

DE TRANSFORMATIONE FUNCTIONUM, DUAS VARIABILES INVOLVENTIUM, DUM EARUM LOCO ALIAE BINAE VARIABILES INTRODUCUNTUR.

AUCTORE

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§. 1. Etsi hoc argumentum in tertio volumine Institutionum mearum calculi integralis jam fusius pertractavi, tamen hic methodum sum traditurus, cujus ope tales transformationes multo facilius expediri queant. Si igitur z fuerit functio quaecunque binarum variabilium x et y, harumque loco aliae binae variabiles quaecunque t et u in calculum introducantur, quaestio huc redit: quamadmodum omnes formulae differentiales, ex proposita functione z oriundae, cujusmodi sunt $\left(\frac{\partial z}{\partial x}\right)$; $\left(\frac{\partial z}{\partial y}\right)$; $\left(\frac{\partial \partial z}{\partial x^2}\right)$; $\left(\frac{\partial \partial z}{\partial x \partial y}\right)$; $\left(\frac{\partial \partial z}{\partial y^2}\right)$; etc. per binas novas variabiles t et u exprimantur?

§. 2. Quoniam relatio inter binas variabiles x et y, respectu novarum t et u, cognita assumitur, non solum binae priores x et y tanquam functiones binarum posteriorum t et u spectari poterunt, sed etiam istae t et u

erunt certae functiones binarum priorum x et y, quam relationem per sequentes formulas differentiales repraesentabo:

$$\partial t = P \partial x + Q \partial y \text{ et } \partial u = R \partial x + S \partial y,$$

quae ut sint determinatae, necesse est fieri: $\left(\frac{\partial P}{\partial y}\right) = \left(\frac{\partial Q}{\partial y}\right)$ et $\left(\frac{\partial R}{\partial y}\right) = \left(\frac{\partial S}{\partial x}\right)$; ubi litterae P, Q, R, S non solum tanquam functiones ipsarum x et y sed etiam ipsarum t et u spectari poterunt, ob cognitam rationem, quam hae binae variabiles inter se tenent.

§. 3. His positis primo investigemus valores formularum differentialium primi gradus, quae sunt $\left(\frac{\partial z}{\partial x}\right)$ et $\left(\frac{\partial z}{\partial y}\right)$, quos accipient per novas variabiles t et u. Ac primo quidem, cum sit tam $\partial z = \partial x \left(\frac{\partial z}{\partial x}\right) + \partial y \left(\frac{\partial z}{\partial y}\right)$ quam $\partial z = \partial t \left(\frac{\partial z}{\partial t}\right) + \partial u \left(\frac{\partial z}{\partial u}\right)$, loco ∂t et ∂u valores ante stabilitos scribendo prodibit ista aequatio:

$$\partial x \left(\frac{\partial z}{\partial x}\right) + \partial y \left(\frac{\partial z}{\partial y}\right) = P \partial x \left(\frac{\partial z}{\partial t}\right) + Q \partial y \left(\frac{\partial z}{\partial t}\right) + R \partial x \left(\frac{\partial z}{\partial u}\right) + S \partial y \left(\frac{\partial z}{\partial u}\right).$$

Hic evidens est utrinque terminos, eodem differentiali ∂x vel ∂y affectos, seorsim inter se aequari debere; unde colligimus has duas aequationes:

$$\text{I. } \left(\frac{\partial z}{\partial x}\right) = P \left(\frac{\partial z}{\partial t}\right) + R \left(\frac{\partial z}{\partial u}\right)$$

$$\text{II. } \left(\frac{\partial z}{\partial y}\right) = Q \left(\frac{\partial z}{\partial t}\right) + S \left(\frac{\partial z}{\partial u}\right),$$

ubi jam litterae P, Q, R, S tanquam functiones ipsarum t et u spectari poterunt, sicque formulae differentiales $\left(\frac{\partial z}{\partial x}\right)$ et $\left(\frac{\partial z}{\partial y}\right)$ per binas novas $\left(\frac{\partial z}{\partial t}\right)$ et $\left(\frac{\partial z}{\partial u}\right)$ exprimuntur.

§. 4. Multo autem difficilius est hinc valores formularum differentialium secundi gradus, quae sunt $(\frac{\partial\partial z}{\partial x^2})$; $(\frac{\partial\partial z}{\partial x\partial y})$; $(\frac{\partial\partial z}{\partial y^2})$ elicere, id quod tamen sequenti modo satis commode praestari poterit. Incipiamus a prima harum formularum $(\frac{\partial\partial z}{\partial x^2})$, quae oritur ex formula $(\frac{\partial z}{\partial x})$, si ea differentietur, sumto $\partial y = o$, et differentiale denuo per ∂x dividatur. At vero, sumto $\partial y = o$, ex formulis principalibus erit $\partial t = P\partial x$ et $\partial u = R\partial x$, unde fit $(\frac{\partial t}{\partial x}) = P$ et $(\frac{\partial u}{\partial x}) = R$. Hinc tantum opus est ut formulae $P(\frac{\partial z}{\partial t}) + R(\frac{\partial z}{\partial u})$ differentiale per ∂x dividatur, pro casu scilicet $\partial y = o$. Cum autem jam P et Q sint functiones binarum t et u, earum differentialia talem habebunt formam: $M\partial t + N\partial u$; unde ergo, ob $(\frac{\partial t}{\partial x}) = P$ et $(\frac{\partial u}{\partial x}) = R$, pro $\frac{\partial P}{\partial x}$ habebimus $MP + NR$. Simili modo etiam $\frac{\partial Q}{\partial x}$ ad functionem ipsarum t et u reducetur, quae reductio cum per se sit manifesta, in calculo retineamus $\frac{\partial P}{\partial x}$ et $\frac{\partial Q}{\partial x}$. Interim tamen, cum sit $M = (\frac{\partial P}{\partial t})$ et $N = (\frac{\partial P}{\partial u})$, erit $\frac{\partial P}{\partial x} = P(\frac{\partial P}{\partial t}) + R(\frac{\partial P}{\partial u})$; similique modo erit $(\frac{\partial Q}{\partial x}) = P(\frac{\partial P}{\partial t}) + R(\frac{\partial Q}{\partial u})$.

§. 5. Superest ergo ut etiam formulas $(\frac{\partial z}{\partial t})$ et $(\frac{\partial z}{\partial u})$ eadem lege tractemus. Cum igitur in genere sit:

$$\partial . (\frac{\partial z}{\partial t}) = \partial t (\frac{\partial\partial z}{\partial t^2}) + \partial u (\frac{\partial\partial z}{\partial t\partial u}),$$

hoc per ∂x divisum, ob $(\frac{\partial t}{\partial x}) = P$ et $(\frac{\partial u}{\partial x}) = R$, evadet

$$P(\frac{\partial\partial z}{\partial t^2}) + R(\frac{\partial\partial z}{\partial t\partial u}) = \frac{\partial . \frac{\partial z}{\partial t}}{\partial x}.$$

Simili modo $\frac{1}{\partial x}\partial\,.\left(\frac{\partial z}{\partial u}\right) = P\left(\frac{\partial\partial z}{\partial t\,\partial u}\right) + R\left(\frac{\partial\partial z}{\partial u^2}\right)$. His ergo observatis erit:

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = \frac{\partial P}{\partial x}\left(\frac{\partial z}{\partial t}\right) + PP\left(\frac{\partial\partial z}{\partial t^2}\right) + PR\left(\frac{\partial\partial z}{\partial t\,\partial u}\right)$$
$$+ \frac{\partial R}{\partial x}\left(\frac{\partial z}{\partial u}\right) + RP\left(\frac{\partial\partial z}{\partial t\,\partial u}\right) + RR\left(\frac{\partial\partial z}{\partial u^2}\right),$$

quae formula contrahitur in hanc:

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = \frac{\partial P}{\partial x}\left(\frac{\partial z}{\partial t}\right) + \frac{\partial R}{\partial x}\left(\frac{\partial z}{\partial u}\right) + PP\left(\frac{\partial\partial z}{\partial t^2}\right)$$
$$+ 2PR\left(\frac{\partial\partial z}{\partial t\,\partial u}\right) + RR\left(\frac{\partial\partial z}{\partial u^2}\right).$$

§. 6. Aggrediamur jam secundam formulam $\left(\frac{\partial\partial z}{\partial x\,\partial y}\right)$, quae primo ex formula $\frac{\partial z}{\partial x}$ derivari potest, eam scilicet differentiando, sola y pro variabili sumta, ita ut sit $\partial x = 0$. Deinde etiam illa formula derivari potest ex formula $\left(\frac{\partial z}{\partial y}\right)$, eam differentiando, sumta sola x variabili, ideoque $\partial y = 0$. Evolvamus primo hoc modo formulam $\left(\frac{\partial z}{\partial x}\right)$, et quia sumto $\partial x = 0$ fit $\partial t = Q\partial y$ et $\partial u = S\partial y$, ideoque $\left(\frac{\partial t}{\partial y}\right) = Q$ et $\left(\frac{\partial u}{\partial y}\right) = S$, hinc ex quantitatibus P et R oriuntur formulae $\frac{\partial P}{\partial y}$ et $\frac{\partial R}{\partial y}$, quarum valores, uti casu praecedente, per se erunt cogniti. Erit scil. $\frac{\partial P}{\partial y} = Q\left(\frac{\partial P}{\partial t}\right) + S\left(\frac{\partial P}{\partial u}\right)$; similique modo erit $\frac{\partial R}{\partial y} = Q\left(\frac{\partial R}{\partial t}\right) + S\left(\frac{\partial R}{\partial u}\right)$, quarum autem loco retineamus formas $\frac{\partial P}{\partial y}$ et $\frac{\partial R}{\partial y}$. Porro vero habebimus:

$$\frac{1}{\partial y}\partial\,.\left(\frac{\partial z}{\partial t}\right) = Q\left(\frac{\partial\partial z}{\partial t^2}\right) + S\left(\frac{\partial\partial z}{\partial t\,\partial u}\right)$$
$$\frac{1}{\partial y}\partial\,.\left(\frac{\partial z}{\partial u}\right) = Q\left(\frac{\partial\partial z}{\partial t\,\partial u}\right) + S\left(\frac{\partial\partial z}{\partial u^2}\right).$$

His igitur colligendis reperiemus:

$$\left(\frac{\partial\partial z}{\partial x\partial y}\right) = \frac{\partial P}{\partial y}\left(\frac{\partial z}{\partial t}\right) + PQ\left(\frac{\partial\partial z}{\partial t^2}\right) + PS\left(\frac{\partial\partial z}{\partial t\partial u}\right)$$
$$+ \frac{\partial R}{\partial y}\left(\frac{\partial z}{\partial u}\right) + QR\left(\frac{\partial\partial z}{\partial t\partial u}\right) + RS\left(\frac{\partial\partial z}{\partial u^2}\right),$$

quam etiam hoc modo exprimere licet:

$$\left(\frac{\partial\partial z}{\partial x\partial y}\right) = \frac{\partial P}{\partial y}\left(\frac{\partial z}{\partial t}\right) + \frac{\partial R}{\partial y}\left(\frac{\partial z}{\partial u}\right)$$
$$+ PQ\left(\frac{\partial\partial z}{\partial t^2}\right) + (PS + QR)\left(\frac{\partial\partial z}{\partial t\partial u}\right) + RS\left(\frac{\partial\partial z}{\partial u^2}\right).$$

§. 7. Eundem autem valorem etiam ex altera formula $\left(\frac{\partial z}{\partial y}\right) = Q\left(\frac{\partial z}{\partial t}\right) + S\left(\frac{\partial z}{\partial u}\right)$ elicere licebit, eam differentiando, sumta sola y variabili, ideoque $\partial y = 0$; unde fit $\left(\frac{\partial t}{\partial x}\right) = P$ et $\left(\frac{\partial u}{\partial x}\right) = R$. Hinc igitur primo habemus $P\left(\frac{\partial Q}{\partial t}\right) + R\left(\frac{\partial Q}{\partial u}\right) = \frac{\partial Q}{\partial x}$, similique modo $\frac{\partial S}{\partial x} = P\left(\frac{\partial S}{\partial t}\right) + R\left(\frac{\partial S}{\partial u}\right)$. Deinde erit uti in primo casu:

$$\frac{1}{\partial x}\partial . \left(\frac{\partial z}{\partial t}\right) = P\left(\frac{\partial\partial z}{\partial t^2}\right) + R\left(\frac{\partial\partial z}{\partial t\partial u}\right) \text{ et}$$
$$\frac{1}{\partial x}\partial . \left(\frac{\partial z}{\partial u}\right) = P\left(\frac{\partial\partial z}{\partial t\partial u}\right) + R\left(\frac{\partial\partial z}{\partial u^2}\right),$$

quibus collectis orietur:

$$\left(\frac{\partial\partial z}{\partial x\partial y}\right) = \frac{\partial Q}{\partial x}\left(\frac{\partial z}{\partial t}\right) + PQ\left(\frac{\partial\partial z}{\partial t^2}\right) + QR\left(\frac{\partial\partial z}{\partial t\partial u}\right)$$
$$+ \frac{dS}{\partial x}\left(\frac{\partial z}{\partial u}\right) + PS\left(\frac{\partial\partial z}{\partial t\partial u}\right) + RS\left(\frac{\partial\partial z}{\partial u^2}\right),$$

quae etiam hoc modo repraesentari potest:

$$\left(\frac{\partial\partial z}{\partial x\partial y}\right) = \frac{\partial Q}{\partial x}\left(\frac{\partial z}{\partial t}\right) + \frac{\partial S}{\partial x}\left(\frac{\partial z}{\partial u}\right)$$
$$+ PQ\left(\frac{\partial\partial z}{\partial t^2}\right) + (QR + PS)\left(\frac{\partial\partial z}{\partial t\partial u}\right) + RS\left(\frac{\partial\partial z}{\partial u^2}\right),$$

quae formula cum praecedente egregie convenit: initio enim jam notavimus esse $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ et $\frac{\partial R}{\partial y} = \frac{\partial S}{\partial x}$.

§. 8. Tertia denique formula $\left(\frac{\partial\partial z}{\partial y^2}\right)$ derivari debet ex formula $\left(\frac{\partial z}{\partial y}\right) = Q\left(\frac{\partial z}{\partial t}\right) + S\left(\frac{\partial z}{\partial u}\right)$, sumendo $\partial x = 0$, unde fit

$\frac{\partial t}{\partial y} = Q$ et $\frac{\partial u}{\partial y} = S$. Hinc ergo fiet

$$\frac{\partial Q}{\partial y} = Q\left(\frac{\partial Q}{\partial t}\right) + S\left(\frac{\partial Q}{\partial u}\right) \text{ et } \frac{\partial S}{\partial y} = Q\left(\frac{\partial S}{\partial t}\right) + S\left(\frac{\partial S}{\partial u}\right).$$

Deinde erit $\frac{1}{\partial y}\partial . \frac{\partial z}{\partial t} = Q\left(\frac{\partial\partial z}{\partial t^2}\right) + S\left(\frac{\partial\partial z}{\partial t\partial u}\right)$

$$\frac{1}{\partial y}\partial . \frac{\partial z}{\partial u} = Q\left(\frac{\partial\partial z}{\partial t\partial u}\right) + S\left(\frac{\partial\partial z}{\partial u^2}\right),$$

quibus collectis fiet:

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \frac{\partial Q}{\partial y}\left(\frac{\partial z}{\partial t}\right) + QQ\left(\frac{\partial\partial z}{\partial t^2}\right) + QS\left(\frac{\partial\partial z}{\partial t\partial u}\right)$$
$$+ \frac{\partial S}{\partial y}\left(\frac{\partial z}{\partial u}\right) + QS\left(\frac{\partial\partial z}{\partial t\partial u}\right) + SS\left(\frac{\partial\partial z}{\partial u^2}\right),$$

sive concinnius:

$$\frac{\partial\partial z}{\partial y^2} = \frac{\partial Q}{\partial y}\left(\frac{\partial z}{\partial t}\right) + \frac{\partial S}{\partial y}\left(\frac{\partial z}{\partial u}\right)$$
$$+ QQ\left(\frac{\partial\partial z}{\partial t^2}\right) + 2QS\left(\frac{\partial\partial z}{\partial t\partial u}\right) + SS\left(\frac{\partial\partial z}{\partial u^2}\right).$$

§. 9. Istos jam valores pro formulis differentialibus secundi gradus inventos heïc uni obtutui exponamus:

$$\text{I.} \quad \left(\frac{\partial\partial z}{\partial x^2}\right) = \frac{\partial P}{\partial x}\left(\frac{\partial z}{\partial t}\right) + \frac{\partial R}{\partial x}\left(\frac{\partial z}{\partial u}\right)$$
$$+ PP\left(\frac{\partial\partial z}{\partial s^2}\right) + 2PR\left(\frac{\partial\partial z}{\partial t\partial u}\right) + RR\left(\frac{\partial\partial z}{\partial u^2}\right),$$

$$\text{II.} \quad \left(\frac{\partial\partial z}{\partial x\partial y}\right) = \frac{\partial P}{\partial y}\left(\frac{\partial z}{\partial t}\right) + \frac{\partial R}{\partial y}\left(\frac{\partial z}{\partial u}\right)$$
$$+ PQ\left(\frac{\partial\partial z}{\partial t^2}\right) + (PS + QR)\left(\frac{\partial\partial z}{\partial t\partial u}\right) + RS\left(\frac{\partial\partial z}{\partial u^2}\right),$$

$$\text{III.} \quad \left(\frac{\partial\partial z}{\partial y^2}\right) = \frac{\partial Q}{\partial y}\left(\frac{\partial z}{\partial t}\right) + \frac{\partial S}{\partial y}\left(\frac{\partial z}{\partial u}\right)$$
$$+ QQ\left(\frac{\partial\partial z}{\partial t^2}\right) + 2QS\left(\frac{\partial\partial z}{\partial t\partial u}\right) + SS\left(\frac{\partial\partial z}{\partial u^2}\right),$$

quibus jungantur formulae differentiales primi gradus:

$$\left(\frac{\partial z}{\partial x}\right) = P\left(\frac{\partial z}{\partial t}\right) + R\left(\frac{\partial z}{\partial u}\right)$$
$$\left(\frac{\partial z}{\partial y}\right) = Q\left(\frac{\partial z}{\partial t}\right) + S\left(\frac{\partial z}{\partial u}\right).$$

Manifestum autem est eadem hac methodo inveniri posse valores formularum differentialium tertii gradus, quae sunt $\frac{\partial^3 z}{\partial t^3}$; $\frac{\partial^3 z}{\partial t^2 \partial u}$; $\frac{\partial^3 z}{\partial t \partial u^2}$; $\frac{\partial^3 z}{\partial u^3}$. Atque adeo ulterius progredi liceret; verum quia formulae nimis complexae essent prodlturae, sufficiat methodum tantum exposuisse.

Problema.

Investigare casus, quibus hanc aequationem differentio-differentialem: $(\frac{\partial\partial z}{\partial x^2}) = vv\,(\frac{\partial\partial z}{\partial y^2})$ *generaliter integrare liceat, ope transformationis ante explicatae.*

Solutio.

Si hic loco formularum $(\frac{\partial\partial z}{\partial x^2})$ et $(\frac{\partial\partial z}{\partial y^2})$ valores modo inventos substituamus, orietur sequens aequatio:

$$\frac{\partial P}{\partial x}(\frac{\partial z}{\partial t}) + \frac{\partial R}{\partial x}(\frac{\partial z}{\partial u}) + PP(\frac{\partial\partial z}{\partial t^2}) + 2PR(\frac{\partial\partial z}{\partial t\partial u}) + RR(\frac{\partial\partial z}{\partial u^2})$$
$$= vv\,[\frac{\partial Q}{\partial y}(\frac{\partial z}{\partial t}) + \frac{\partial S}{\partial y}(\frac{\partial z}{\partial u}) + QQ(\frac{\partial\partial z}{\partial t^2}) + 2QS(\frac{\partial\partial z}{\partial t\partial u}) + SS(\frac{\partial\partial z}{\partial u^2})].$$

Nunc igitur quaeritur, quomodo novae variabiles t et u accipi oporteat, ut haec aequatio integrationem admittat. Hunc in finem efficiamus primo ut partes $(\frac{\partial\partial z}{\partial t^2})$ se mutuo destruant, quod eveniet si fuerit $PP = QQvv$, ideoque $P = \pm Qv$. Simili modo partes $(\frac{\partial\partial z}{\partial u^2})$ se distruent casu $RR = SSvv$, ideoque $R = \pm Sv$. Sumto autem $P = +Qv$, necessario sumi debet $R = -Sv$, quio alioquin etiam partes $(\frac{\partial\partial z}{\partial t\partial u})$ se mutuo destruerent.

§. 11. Sumamus igitur $P = Qv$ et $R = -Sv$, atque nostra aequatio induet hanc formam:

$$\frac{\partial . Qv}{\partial x}\left(\frac{\partial z}{\partial t}\right) - \frac{\partial . Sv}{\partial x}\left(\frac{\partial z}{\partial u}\right) - 2QS\,vv\left(\frac{\partial\partial z}{\partial t\,\partial u}\right)$$
$$= vv\frac{\partial Q}{\partial y}\left(\frac{\partial z}{\partial t}\right) + vv\frac{\partial S}{\partial y}\left(\frac{\partial z}{\partial u}\right) + 2QS\,vv\left(\frac{\partial\partial z}{\partial t\,\partial u}\right),$$

sive

$$\frac{\partial . Qv}{\partial x}\left(\frac{\partial x}{\partial t}\right) - \frac{\partial . Sv}{\partial x}\left(\frac{\partial z}{\partial u}\right) = vv\frac{\partial Q}{\partial y}\left(\frac{\partial z}{\partial t}\right) + vv\frac{\partial S}{\partial y}\left(\frac{\partial z}{\partial u}\right)$$
$$+ 4QSvv\left(\frac{\partial\partial z}{\partial t\,\partial u}\right),$$

quae aeqatio tribus tantum membris principalibus constat, scilicet:

$$4QSvv\left(\frac{\partial\partial z}{\partial t\,\partial u}\right) + \left(vv\frac{\partial Q}{\partial y} - \frac{\partial . Qv}{\partial x}\right)\left(\frac{\partial z}{\partial t}\right) + \left(vv\frac{\partial S}{\partial y} + \frac{\partial . Sv}{\partial x}\right)\left(\frac{\partial z}{\partial u}\right) = 0.$$

§. 12. Cum igitur sit $P = Qv$ et $R = -Sv$, erit $\partial t = Q\,(v\partial x + \partial y)$ et $\partial u = S\,(\partial y - v\partial x)$; unde patet quantitates Q et S ita accipi debere, ut hae duae formulae integrationem admittant, id quod a valore v potissimum pendet. Quo igitur a simplicissimis incipiamus, sumamus $v = a$, capique poterit tam $Q = 1$ quam $S = 1$, unde fit $P = a$ et $R = -a$; quibus positis aequatio nostra erit $4aa\left(\frac{\partial\partial z}{\partial t\,\partial u}\right) = 0$, sive $\left(\frac{\partial\partial z}{\partial t\,\partial u}\right) = 0$; qua ergo bis integrata habebimus $t = ax + y$ et $u = y - ax$.

§. 13. Ad aequationem inventam $\left(\frac{\partial\partial z}{\partial t\,\partial u}\right) = 0$ integrandam statuamus $\left(\frac{\partial z}{\partial u}\right) = s$, fietque $\left(\frac{\partial s}{\partial t}\right) = 0$, ubi sola t variabilis accipitur, existente u constante, quare integrando ponamus $s = \Gamma' : u$, existente $\int \partial u \Gamma' : u = \Gamma : u$. Cum igitur sit $\left(\frac{\partial z}{\partial u}\right) = s = \Gamma' : u$, (ubi jam sola u variabilis sumitur, ita ut t pro constante habeatur), integrando habebimus $z = \Gamma : u + \Delta : t$, consequenter, loco t et u scriptis eorum valoribus, habebimus hujus aequationis: $\left(\frac{\partial \partial z}{\partial x^2}\right) = a\,a \left(\frac{\partial \partial z}{\partial y^2}\right)$ integrale completum $z = \Gamma : (y - a x) + \Delta : (y + a x)$, prouti quidem jam dudum constat.

§. 14. Tentemus nunc solutionem generaliorem, sumendo $v = \frac{Y}{X}$, existente X functione ipsius x, et Y functione ipsius y, eritque $\partial t = \frac{Q(Y \partial x + X \partial y)}{X}$ et $\partial u = \frac{S(X \partial y - Y \partial y)}{X}$, quae ambae formulae integrabiles redduntur, sumendo $Q = \frac{1}{Y}$ et $S = \frac{1}{Y}$; tum enim fit $t = \int \frac{\partial x}{X} + \int \frac{\partial y}{Y}$ et $u = \int \frac{\partial y}{Y} - \int \frac{\partial x}{X}$; porro vero $P = \frac{1}{X}$ et $R = -\frac{1}{X}$.

§. 15. His ergo positis aequatio nostra hanc induet formam:

$$\frac{4}{X^2}\left(\frac{\partial \partial z}{\partial t \partial u}\right) + \left(\frac{X' - Y'}{Y^2}\right)\left(\frac{\partial z}{\partial t}\right) - \left(\frac{X' + Y'}{X^2}\right)\left(\frac{\partial z}{\partial u}\right) = 0.$$

quae ducta in X^2 reducitur ad hanc:

$$4\left(\frac{\partial \partial z}{\partial t \partial u}\right) + (X' - Y')\left(\frac{\partial z}{\partial t}\right) - (X' + Y')\left(\frac{\partial z}{\partial u}\right) = 0$$

de qua aequatione observandum est, eam in genere integrabilem esse non posse, nisi alterutra forma $\left(\frac{\partial z}{\partial t}\right)$ vel $\left(\frac{\partial z}{\partial u}\right)$ evanescat. Statuamus igitur $X' - Y' = 0$, id quod tantum duplici modo fieri potest: 1°) scilicet quando vel $X' = 0$ et $Y' = 0$, hoc est quando utraque functio X et Y est constans, ideoque etiam v constans, quem casum modo ante expedivimus; 2°) vero quando $X' = b$ et $Y' = b$, unde fit $X = bx$ et $Y = by$, hincque $v = \frac{y}{x}$, ideoque aequatio nostra proposita est $\left(\frac{\partial\partial z}{\partial x^2}\right) = \frac{yy}{xx}\left(\frac{\partial\partial z}{\partial y^2}\right)$, sive

$$xx\left(\frac{\partial\partial z}{\partial x^2}\right) = yy\left(\frac{\partial\partial z}{\partial y^2}\right),$$

quem ergo casum hic evolvamus.

§. 16. Cum igitur sit $X = bx$ et $Y = by$, erit $t = \frac{1}{b} lxy$ et $u = \frac{1}{b} l\frac{y}{x}$. Porro vero erit $P = \frac{1}{bx}$; $Q = \frac{1}{by}$; $R = -\frac{1}{bx}$; $S = \frac{1}{by}$. Aequatio autem inter t et u erit $2\left(\frac{\partial t\partial u}{\partial\partial z}\right) - b\left(\frac{\partial z}{\partial u}\right) = 0$, a cujus ergo integratione tota nostra solutio pendet. Faciamus igitur, ut ante, $\left(\frac{\partial z}{\partial u}\right) = s$, et aequatio nostra erit: $2\left(\frac{\partial s}{\partial t}\right) - bs = 0$; ubi sola t est variabilis, ideoque u constans; quo notato erit $2\partial s - bs\partial t = 0$, sive $2.\frac{\partial s}{s} - b\partial t = 0$, cujus integrale est $ls - \frac{1}{2}bt = \text{Const.} = l\Gamma' : u$, et ad numeros transeundo: $se^{-\frac{1}{2}bt} = \Gamma' : u$, ideoque $s = \left(\frac{\partial z}{\partial u}\right) = e^{\frac{1}{2}bt}\Gamma' : u$, sive posito $b = 2c$ erit $\left(\frac{\partial z}{\partial u}\right) = e^{ct}\Gamma' : u$. In hac autem aequatione jam sola u est variabilis, unde fit $\partial z = e^{ct}\partial u\Gamma' : u$, cujus integrale manifesto est $z = e^{ct}\Gamma : u + \Delta : t$.

§. 17. Cum igitur sit $t = \frac{1}{2c} lxy$, ideoque $e^{ct} = \sqrt{xy}$ et $u = \frac{1}{2c} l\frac{y}{x}$, hinc erit $\Gamma : u =$ funct. cuicunque $\frac{y}{x}$, similique modo $\Delta : t =$ funct. cuicunque ipsius xy, consequenter integrale nostrum completum erit $z = \sqrt{xy}\,\Sigma : \frac{y}{x} + \Theta : xy$; ubi prius membrum multiplicari potest per $\sqrt{\frac{y}{x}}$, quo facto erit $z = y\Sigma : \frac{y}{x} + \Theta : xy$. Ubi observasse juvabit, cum $\Sigma : \frac{y}{x}$ comprehendat omnes functiones nullius dimensionis ipsarum x et y, prius membrum denotare omnes functiones ipsarum x et y unius dimensionis. Praeter hos autem duos casus modo tractatos haud patet alios exhiberi posse, quibus aequatio in problemate proposita complete integrare liceat.

Problema.

Investigare casus, quibus haec aequatio differentialis secundi gradus: $xx\left(\frac{\partial\partial z}{\partial x^2}\right) - fxy\left(\frac{\partial\partial y}{\partial x\partial y}\right) + gyy\left(\frac{\partial\partial z}{\partial y^2}\right) = 0$ *ope transformationis hic traditae complete integrari possit.*

Solutio.

§. 18. Loco binarum variabilium x et y, quarum functio est z, introducantur binae aliae t et u, quarum ad illas relatio his aequationibus exprimatur: $\partial t = P\partial x + Q\partial y$ et $\partial u = R\partial x + S\partial y$, atque ex formulis supra datis colligamus primo coëfficientem termini $\left(\frac{\partial\partial z}{\partial t^2}\right)$, qui est

$$PPxx - fPQxy + gQQyy.$$

Similique modo coëfficiens termini $\left(\frac{\partial\partial z}{\partial u^2}\right)$ erit

$$RRxx - fRSxy + gSSyy,$$

quos ambos evanescentes reddamus, quod quo commodius fieri queat, statuamus $f = a + b$ et $g = ab$; hocque modo prior coëfficiens resolvitur in hos factores: $(Px - aQy)(Px - bQy)$, qui ut evanescat ponamus $Px = aQy$. Alter vero coëfficiens in hos factores resolvitur: $(Rx - aSy)(Rx - bSy)$, qui ut evanescat faciamus $Rx = bSy$.

§. 19. Cum igitur sit $P = \frac{aQy}{x}$ et $R = \frac{bSy}{x}$, formulae principales pro ∂t et ∂u erunt: $\partial t = Q\left(\frac{ay\partial x + x\partial y}{x}\right)$ et $\partial u = S\left(\frac{by\partial x + x\partial y}{x}\right)$, quae ambae fient integrabiles sumendo $Q = \frac{1}{y}$ et $S = \frac{1}{y}$. Sic enim fiet $\partial t = a.\frac{\partial x}{x} + \frac{\partial y}{y}$ et $\partial u = b.\frac{\partial x}{x} + \frac{\partial y}{y}$; unde integralia erunt $t = alx + ly$ et $u = blx + ly$, sive $t = l\,x^a y$ et $u = l\,x^b y$. Sumtis autem $Q = \frac{1}{y}$ et $S = \frac{1}{y}$, erit $P = \frac{a}{x}$ et $R = \frac{b}{x}$.

§. 20. His jam valoribus substitutis, quia termini $\left(\frac{\partial\partial z}{\partial t^2}\right)$ et $\left(\frac{\partial\partial z}{\partial u^2}\right)$ jam ad nihilum sunt perducti, coëfficiens termini $\left(\frac{\partial\partial z}{\partial t\,\partial u}\right)$ reperietur

$$2PRxx - f(QR + PS)xy + 2gQSyy,$$

qui ob $f = a + b$ et $g = ab$, facta substitutione litterarum majuscularum, fit $4ab - (a+b)^2 = -(a-b)^2$, ita ut hic terminus jam sit $-(a-b)^2\left(\frac{\partial\partial z}{\partial t\,\partial u}\right)$. Porro vero termini $\left(\frac{\partial z}{\partial t}\right)$ coëfficiens erit $xx\frac{\partial P}{\partial x} - fxy\frac{\partial Q}{\partial x} + gyy\frac{\partial Q}{\partial y}$, qui ob $\frac{\partial P}{\partial x} = \frac{-a}{xx}$, $\frac{\partial Q}{\partial x} = 0$ et $\frac{\partial Q}{\partial y} = \frac{-1}{yy}$ abit in hanc formam: $-a(b+1)$. Similique modo coëfficiens termini $\left(\frac{\partial t}{\partial u}\right)$ colligitur fore $xx\left(\frac{\partial R}{\partial x}\right) - fxy\frac{\partial S}{\partial x} + gyy\frac{\partial S}{\partial y} = -b(a+1)$.

§. 21. Aequatio igitur resolvenda nunc hanc induet formam:

$$(a-b)^2\left(\frac{\partial\partial z}{\partial t\,\partial u}\right) + a(b+1)\left(\frac{\partial z}{\partial t}\right) + b(a+1)\left(\frac{\partial z}{\partial u}\right) = 0,$$

de qua autem ante omnia notari oportet, eam nullo modo adhuc cognito tractari posse, nisi alteruter posteriorum terminorum evanescat. Statuamus igitur $b = -1$; unde fit $f = a - 1$; $g = -a$ et $t = lx^a y$ et $u = l\frac{y}{x}$. Aequatio vero resolvenda erit $(a+1)^2 \left(\frac{\partial\partial z}{\partial t \partial u}\right) - (a+1)\left(\frac{\partial z}{\partial u}\right) = 0$. Hinc ergo si ponamus $\left(\frac{\partial z}{\partial u}\right) = v$, ob $\left(\frac{\partial\partial z}{\partial t \partial u}\right) = \left(\frac{\partial v}{\partial t}\right)$, erit $(a+1)^2 \left(\frac{\partial v}{\partial t}\right) = (a+1)v$, sive $(a+1)\left(\frac{\partial v}{\partial t}\right) = v$, ubi littera u tanquam constans est spectanda, quo observato erit $(a+1)\partial v = v\partial t$, ideoque $\frac{\partial v}{v} = \frac{\partial t}{a+1}$, unde fit $lv = \frac{t}{a+1} + lf:u$, sicque numeris sumtis erit: $v = e^{\frac{t}{a+1}} \Gamma' : u$.

§. 22. Cum igitur posuerimus $v = \frac{\partial z}{\partial u}$, ita ut nunc t pro constante sit habenda, erit: $\frac{\partial z}{\partial u} = e^{\frac{t}{a+1}} \Gamma' : u$, sive $\partial z = e^{\frac{t}{a+1}} \partial u \Gamma' : u$, unde ob $\int \partial u \Gamma' : u = \Gamma : u$ habebimus $z = e^{\frac{t}{a+1}} \Gamma : u + \Delta : t$, quae expressio, ob binas functiones arbitrarias, utique praebet integrale completum aequationis propositae, casu scilicet quo $f = a - 1$ et $g = -a$.

§. 23. Quo nunc hanc formam ad variabiles x et y transferamus, notemus primo esse: $t = l x^a y$, unde fit $e^t = x^a y$, hincque $e^{\frac{t}{a+1}} = x^{\frac{a}{a+1}} y^{\frac{1}{a+1}} = \sqrt[a+1]{x^a y}$; tum vero functio quaecunque ipsius t erit etiam functio quaecunque ipsius $x^a y$, unde pro $\Delta : t$ scribere licebit $\Delta : x^a y$. Deinde cum sit $u = l\frac{y}{x}$, ejus functio quaecunque etiam erit functio ipsius $\frac{y}{x}$, sicque loco $\Gamma : u$ nunc habebimus $\Gamma : \frac{y}{x}$. Hinc hujus aequationis differentio-differentialis:

$$xx\left(\frac{\partial\partial z}{\partial x^2}\right) - (a-1)\,xy\left(\frac{\partial\partial z}{\partial x\,\partial y}\right) - ayy\left(\frac{\partial\partial z}{\partial y^2}\right) = 0,$$

integrale completum erit: $z = \sqrt[a+1]{x^a y}\,\Gamma : \frac{y}{x} + \Delta : x^a y$. Quoniam igitur ista aequatio abit in eam quam praecedente problemate invenimus casu $a = 1$, eadem forma integralis prodibit, quam supra (§. 17.) invenimus, scilicet:

$$z = \sqrt{xy}\,\Gamma : \frac{y}{x} + \Delta : xy.$$

§. 24. Prius membrum illius formae integralis multo simplicius exprimi potest, dum scil. ejus factor prior per quandam functionem ipsius $\frac{y}{x}$ multiplicatur vel dividatur. Dividatur ergo per $\sqrt[a+1]{\frac{y}{x}}$, prodibit $z = x\,\Gamma : \frac{y}{x} + \Delta : x^a y$; ubi notandum est prius membrum $x\Gamma : \frac{y}{y}$ continere omnes functiones homogeneas unius dimensionis ipsarum x et y. Observetur hic, si etiam fuerit $x = -1$, ita ut aequatio proposita sit $xx\left(\frac{\partial\partial z}{\partial x^2}\right) - 2xy\left(\frac{\partial\partial z}{\partial x\,\partial y}\right) + yy\left(\frac{\partial\partial z}{\partial y^2}\right) = 0$, tum integrale completum fore $z = x\Gamma : \frac{y}{x} + \Delta : \frac{y}{x}$. Ubi notandum, etiamsi duae functiones ejusdem formae $\frac{y}{x}$ occurrant, eas in unam contrahi non posse, propterea quod prior multiplicata est per x.