1805

Disquitiones analyticae super evolutione potestatis trinomialis $(1+x+xx)^n$

Leonhard Euler

conficitur $\sin 3(\vartheta - \Phi) = \frac{\alpha(1-3tt)-3t+t^3}{\sqrt{1+\alpha\alpha(1+tt)^3}}$, ideoque:

$$(\sin 3(\vartheta - \Phi))^{\frac{2}{3}} = \frac{(\alpha(1-3tt)-3t+t^3)^{\frac{2}{3}}}{(1+\alpha\alpha)^{\frac{1}{3}}(1+tt)}.$$

Hisque valoribus substitutis nanciscemur.

$$\int P\partial v = -\alpha^{\frac{1}{3}}\int\frac{\partial t(1+tt)}{(3t-t^3)(\alpha(1-3tt)-3t+t^3)^{\frac{2}{3}}}$$

$$\int Q\partial v = +\alpha^{\frac{2}{3}}\int\frac{t\partial t(1+tt)}{(3t-t^3)(\alpha(1-3tt)-3t+t^3)^{\frac{2}{3}}}.$$

Certo igitur affirmare licet, has formulas ab irrationalitate penitus liberari posse, etiamsi mihi quidem nulla via patere videatur hoc praestandi; unde Geometris amplissimus campus aper[illegible]ur suam sagacitatem exercendi.

§. 30. Si loco formulae $\int\frac{\partial z}{\sqrt[3]{1+z^3}}$ assumsissemus generalem $\int\frac{\partial z}{\sqrt[n]{(1+z^n)}}$, eamque simili modo tractavissemus, pervenissemus ad sequens theorema:

Integralia harum duarum formularum:

$$\int\frac{\partial\Phi\sin\Phi}{\sin n\ldots\sin(n\vartheta-n\Phi)^{\frac{n-1}{n}}}\text{ et }\int\frac{\partial\Phi\cos\Phi}{\sin n\Phi(\sin(n\vartheta-n\Phi))^{\frac{n-1}{n}}}$$

certe per logarithmos et arcus circulares exprimi possunt, ideoque dabitur certa substitutio, cujus ope hae formulae ad rationalitatem perduci possunt; unde haec observatio eo majorem attentionem meretur.

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DISQUISITIONES ANALYTICAE

SUPER EVOLUTIONE POTESTATIS TRINOMIALIS

$(1+x+xx)^n$.

AUCTORE

L. EULERO.

Conventui exhibita die 17. Aug. 1775.

§. 1.

Cum olim in *Novorum Commentariorum Tomo XI*, sub titulo *observationum analyticarum*, istam potestatem trinomialem multo studio essem perscrutatus, in tam egregia symptomata incidi, quae majore attentione Geometrarum non indigna videbantur. Hanc ob rem nuper hoc idem argumentum denuo tractare suscepi, atque nonnullis artificiis analyticis usus, multo plura insignia phaenomena se mihi obtulerunt, quorum expositionem Geometris non ingratam fore confido.

§. 2. Incipio igitur ab ipsa evolutione hujus formulae: $(1+x+xx)^n$, quae pro singulis valoribus exponentis n sequentes praebet expressiones in tabula subjuncta repraesentatas:

n	$(1+x+xx)^n$
0	1
1	$1+x+xx$
2	$1+2x+3xx+2x^3+x^4$
3	$1+3x+6xx+7x^3+6x^4+3x^5+x^6$
4	$1+4x+10xx+16x^3+19x^4+16x^5+10x^6+4x^7+x^8$
5	$1+5x+15xx+30x^3+45x^4+51x^5+45x^6+30x^7+15x^8+5x^9+x^{10}$.
etc.	etc.

Hic

Investigatio relationis

vere non debere. Substituto igitur loco Π valore ante dato per Φ

Hic scilicet ex qualibet potestate facillime sequens deducitur; si enim pro quolibet valore exponentis n, quilibet coëfficiens cum binis praecedentibus in unam summam colligatur, obtinetur coëfficiens pro potestate sequente exponentis $n+1$ subscribenda.

§. 3. Hanc tabulam aspicienti statim patet, in qualibet evolutione coefficientes terminorum usque ad medium, qui dignitatem x^n refert, crescere, inde autem iterum eodem ordine decrescere, usque ad ultimum terminum, qui est x^{2n}. Deinde etiam haud difficulter perspicitur, pro potestate $(1+x+xx)^n$ in genere terminos initiales ita expressum iri:

$$1+nx+\frac{n(n+1)}{1.2}x^2+\frac{n(n-1)(n+4)}{1.2.3}x^3+\frac{n(n-1)(nn+7n-6)}{1.2.3.4}x^4$$
$$+\frac{n(n-1)(n+1)(n-2)(n+12)}{1.2.3.4.5}x^5+\text{etc.}$$

Hos autem terminos ulterius prosequi non attinet, quia in eorum coëfficientibus nullus ordo deprehenditur.

§. 4. Hic autem imprimis ad coëfficientem maximum seu medium respicio, quem pro potestate $(1+x+xx)^n$ in genere perpetuo statuam $=px^n$; tum vero terminos hunc sequentes ita repraesentabo: qx^{n+1}; rx^{n+2}; sx^{n+3}; tx^{n+4}; etc. unde termini medium praecedentes erunt ordine retrogrado qx^{n-1}; rx^{n-2}; sx^{n-3}; tx^{n-4}; etc. Deinde vero pro potestate sequenti $(1+x+xx)^{n+1}$ easdem litteras apice sum notaturus; scilicet: p', q', r', s', etc. quas porro pro potestate denuo sequente $(1+x+xx)^{n+2}$ apici duplici designabo; pro sequentibus, apice triplici, quadruplici, et ita porro.

§. 5. His praemissis, in hac dissertatione ex seriebus superioris tabulae potissimum terminos medios maximis coëfficientibus affectos, sum contemplaturus, qui sunt

sunt: $1, x, 3x^2, 7x^3, 19x^4, 51x^5$ etc. qui junctim sumti constituunt seriem, cujus summam littera P indicabo, ita ut

$$P=1+x+3x^2+7x^3+19x^4+51x^5\ldots+px^n+p'x^{n+1}+p''x^{n+2}+\text{etc.}$$

§. 6. Praeterea vero, quemadmodum isti termini ex tabula superiore secundum diagonalem sunt desumti, simili modo tales series formemus secundum diagonales ulteriores illi parallelas, quarum seriarum summas pariter peculiaribus litteris denotemus sequenti modo:

$$Q=x^2+2x^3+6x^4+16x^5+45x^6+\ldots+qx^{n+1}+q'x^{n+2}+q''x^{n+3}+\text{etc.}$$
$$R=x^4+3x^5+10x^6+30x^7+\ldots+rx^{n+2}+r'x^{n+3}+r''x^{n+4}+\text{etc.}$$
$$S=x^6+4x^7+15x^8+\ldots+sx^{n+3}+s'x^{n+4}+s''x^{n+5}+\text{etc.}$$
$$T=x^8+5x^9+\ldots+tx^{n+4}+t'x^{n+5}+t''x^{n+6}+\text{etc.}$$

etc. etc.

His constitutis propositum mihi est primo in valores litterarum minuscularum p, q, r, s, etc. earumque derivatarum p', q', r', s' etc. p'', q'', r'', s'', etc. inquirere, quo facto etiam valores litterarum majuscularum P, Q, R, S, etc. indagabo.

Investigatio litterarum p, q, r, s, etc.

§. 7. Cum p sit coëfficiens potestatis x^n ex evolutione formulae $(1+x+xx)^n$ oriundae, istam formulam hoc modo repraesentemus: $(x(1+x)+1)^n$, pro cujus evolutione utamur signandi modo jam aliquoties a me usitato, quo coëfficientes similis potestatis binomialis per hos characteres designare soleo: $(\frac{n}{1})$, $(\frac{n}{2})$, $(\frac{n}{3})$, $(\frac{n}{4})$, $(\frac{n}{5})$, etc. ita ut sit

$$(\tfrac{n}{1})=n$$
$$(\tfrac{n}{2})=\frac{n(n-1)}{1.2}$$
$$(\tfrac{n}{3})=\frac{n(n-1)(n-2)}{1.2.3}$$
$$(\tfrac{n}{4})=\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}$$
$$(\tfrac{n}{5})=$$

$(\frac{n}{5}) = \frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}$

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$(\frac{n}{\lambda}) = \frac{n(n-1)(n-2)(n-3) \ldots \ldots (n-\lambda+1)}{1.2.3 \ldots \ldots \lambda}$

Circa quos characteres hic annotasse juvabit, in genere ſemper esse $(\frac{n}{\lambda}) = (\frac{n}{n-\lambda})$, quandoquidem hi coëfficientes retro eundem ordinem servant, et quia coëfficientes extremi ſunt unitas, erit $(\frac{n}{0}) = (\frac{n}{n}) = 1$. Deinde, quia ex lege progressionis tam omnes termini primum antecedentes quam termini ultimum sequentes evanescunt, erit ut ſequitur:

$(\frac{n}{-1}) = (\frac{n}{n+1}) = 0$
$(\frac{n}{-2}) = (\frac{n}{n+2}) = 0$
$(\frac{n}{-3}) = (\frac{n}{n+3}) = 0$
etc.

§. 8. His praemissis formula noſtra $(x(1+x)+1)^n$, more ſolito tanquam binomium evoluta, dabit hanc ſeriem:

$x^n(1+x)^n + (\frac{n}{1})x^{n-1}(1+x)^{n-1} + (\frac{n}{2})x^{n-2}(1+x)^{n-2} + (\frac{n}{3})x^{n-3}(1+x)^{n-3} +$ etc.

ubi notetur esse in genere:

$(1+x)^\lambda = 1 + (\frac{\lambda}{1})x + (\frac{\lambda}{2})x^2 + (\frac{\lambda}{3})x^3 +$ etc.

Ex ſingulis igitur membris illius formae expoſitae depromi debent termini poteſtatem x^n continentes, quippe qui conjunctim ſumti component terminum medium px^n.

§. 9. Primum autem membrum $x^n(1+x)^n$ tantum terminum hujus formae praebet x^n. Ex membro autem ſecundo hanc formam habebit terminus ſecundus, qui eſt $(\frac{n}{1})(\frac{n-1}{1})x^n$. Ex tertio membro poteſtas x^n oritur ex termi-

no

no tertio, qui eſt $(\frac{n}{2})(\frac{n-2}{2})x^n$. Simili modo ex membro quarto deducitur $(\frac{n}{3})(\frac{n-3}{3})x^n$. Ex quinto oritur $(\frac{n}{4})(\frac{n-4}{4})x^n$ et ita porro. Hinc igitur verus valor litterae p ita colligitur

$p = 1 + (\frac{n}{1})(\frac{n-1}{1}) + (\frac{n}{2})(\frac{n-2}{2}) + (\frac{n}{3})(\frac{n-3}{3}) + (\frac{n}{4})(\frac{n-4}{4}) +$ etc.

§. 10. Simili modo ex eadem evolutione colligere licet coëfficientes poteſtatis x^{n+1}, qui junctim ſumti dabunt valorem litterae q. Talis autem poteſtas ex primo membro orta erit $(\frac{n}{1})x^{n+1}$. Ex ſecundo membro oritur $(\frac{n}{1})(\frac{n-1}{2})x^{n+1}$; ex tertio membro $(\frac{n}{2})(\frac{n-2}{3})x^{n+1}$; ex quarto $(\frac{n}{3})(\frac{n-3}{4})x^{n+1}$; et ita porro, quocirca verus valor litterae q hoc modo exprimetur:

$q = (\frac{n}{1}) + (\frac{n}{1})(\frac{n-1}{2}) + (\frac{n}{2})(\frac{n-2}{3}) + (\frac{n}{3})(\frac{n-3}{4}) +$ etc.

ubi ob analogiam primus terminus $(\frac{n}{1})$ ita repraesentatus eſt intelligendus $(\frac{n}{0})(\frac{n}{1})$. Sic enim, cum quilibet terminus duobus conſtet factoribus, priores factores conſtituunt hanc ſeriem: $(\frac{n}{0}), (\frac{n}{1}), (\frac{n}{2}), (\frac{n}{3}), (\frac{n}{4})$, etc. poſteriores vero iſtam: $(\frac{n}{1}), (\frac{n-1}{2}), (\frac{n-2}{3}), (\frac{n-3}{4})$, etc.

§. 11. Pari modo ex poteſtatibus x^{n+2}, quae ex ſingulis membris deducuntur, formabitur terminus rx^{n+2}; at vero primum membrum pro hac poteſtate praebet $1 \cdot (\frac{n}{2})x^{n+2}$, sive analogiae gratia $(\frac{n}{0})(\frac{n}{2})x^{n+2}$. Ex membro ſecundo oritur eadem poteſtas $(\frac{n}{1})(\frac{n-1}{3})x^{n+2}$; ex tertio membro $(\frac{n}{2})(\frac{n-2}{4})x^{n+2}$; ex quarto $(\frac{n}{3})(\frac{n-3}{5})x^{n+2}$ et ita porro; ex quibus ergo collectis nanciscimur valorem litterae r hoc modo expressum:

$r = (\frac{n}{0})(\frac{n}{2}) + (\frac{n}{1})(\frac{n-1}{3}) + (\frac{n}{2})(\frac{n-2}{4}) + (\frac{n}{3})(\frac{n-3}{5}) +$ etc.

§. 12.

ante per Φ expreſſo nanciſcemur ſequentem aequationem a fractionibus jam liberatam:

$A(n - 2\alpha - \lambda - 1)(n - 2\alpha - \lambda) + B(n - 2\alpha - \lambda)$

§. 37. Subſtituamus nunc loco litterarum A, B, C valores modo inventos, et aequatio inter has ternas litteras erit:

§. 12. Superfluum foret eandem deductionem pro litteris sequentibus apponere, quandoquidem jam satis perspicuum est fore:

$$s = \left(\frac{n}{0}\right)\left(\frac{n}{3}\right) + \left(\frac{n}{1}\right)\left(\frac{n-1}{4}\right) + \left(\frac{n}{2}\right)\left(\frac{n-2}{5}\right) + \left(\frac{n}{3}\right)\left(\frac{n-3}{6}\right) + \text{etc.}$$

$$t = \left(\frac{n}{0}\right)\left(\frac{n}{4}\right) + \left(\frac{n}{1}\right)\left(\frac{n-1}{5}\right) + \left(\frac{n}{2}\right)\left(\frac{n-2}{6}\right) + \left(\frac{n}{3}\right)\left(\frac{n-3}{7}\right) + \text{etc.}$$

$$u = \left(\frac{n}{0}\right)\left(\frac{n}{5}\right) + \left(\frac{n}{1}\right)\left(\frac{n-1}{6}\right) + \left(\frac{n}{2}\right)\left(\frac{n-2}{7}\right) + \left(\frac{n}{3}\right)\left(\frac{n-3}{8}\right) + \text{etc.}$$

etc. etc.

atque in genere si potestati $x^{\nu+\lambda}$ tribuamus litteram z, erit

$$z = \left(\frac{n}{0}\right)\left(\frac{n}{\lambda}\right) + \left(\frac{n}{1}\right)\left(\frac{n-1}{\lambda+1}\right) + \left(\frac{n}{2}\right)\left(\frac{n-2}{\lambda+2}\right) + \left(\frac{n}{3}\right)\left(\frac{n-3}{\lambda+3}\right) + \text{etc.}$$

§. 13. Manifestum hic est omnes terminos harum serierum contineri in hac forma generali: $\left(\frac{n}{\alpha}\right)\left(\frac{n-\alpha}{\beta}\right)$, quam observo semper huic esse aequalem; $\left(\frac{n}{\beta}\right)\left(\frac{n-\beta}{\alpha}\right)$, ita ut litterae α et β permutationem patiantur. Cum enim facta evolutione sit:

$$\frac{n}{\alpha} = \frac{n(n-1)(n-2)(n-3)\ldots\ldots(n-\alpha+1)}{1.2.3\ldots\ldots\ldots\alpha},$$

et $\frac{n-\alpha}{\beta} = \frac{(n-\alpha)(n-\alpha-1)(n-\alpha-2)\ldots\ldots(n-\alpha-\beta+1)}{1.2.3.4\ldots\ldots\ldots\beta}$;

facta multiplicatione erit

$$\left(\frac{n}{\alpha}\right)\left(\frac{n-\alpha}{\beta}\right) = \frac{n(n-1)(n-2)(n-3)\ldots\ldots(n-\alpha-\beta+1)}{1.2.3\ldots\alpha.1.2.3\ldots\beta}$$

ubi permutabilitas litterarum α et β in oculos incurrit.

§. 14. Quodsi jam series ante inventae hoc modo immutentur, prima quidem pro p inventa nullam mutationem patitur, reliquae vero sequenti modo referentur:

$$q = \left(\frac{n}{1}\right)\left(\frac{n-1}{0}\right) + \left(\frac{n}{2}\right)\left(\frac{n-2}{1}\right) + \left(\frac{n}{3}\right)\left(\frac{n-3}{2}\right) + \left(\frac{n}{4}\right)\left(\frac{n-4}{3}\right) + \text{etc.};$$

$$r = \left(\frac{n}{2}\right)\left(\frac{n-2}{0}\right) + \left(\frac{n}{3}\right)\left(\frac{n-3}{1}\right) + \left(\frac{n}{4}\right)\left(\frac{n-4}{3}\right) + \left(\frac{n}{5}\right)\left(\frac{n-5}{4}\right) + \text{etc.}$$

$$s = \left(\frac{n}{3}\right)\left(\frac{n-3}{0}\right) + \left(\frac{n}{4}\right)\left(\frac{n-4}{1}\right) + \left(\frac{n}{5}\right)\left(\frac{n-5}{4}\right) + \left(\frac{n}{6}\right)\left(\frac{n-6}{5}\right) + \text{etc.}$$

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$$z = \left(\frac{n}{\lambda}\right)\left(\frac{n-\lambda}{0}\right) + \left(\frac{n}{\lambda+1}\right)\left(\frac{n-\lambda-1}{1}\right) + \left(\frac{n}{\lambda+2}\right)\left(\frac{n-\lambda-2}{2}\right) + \text{etc.}$$

§. 15.

§. 15. Praeterea vero maxime memorabilis est haec conversio, qua est $\left(\frac{n}{\alpha}\right)\left(\frac{n-\alpha}{\beta}\right) = \left(\frac{\alpha+\beta}{\alpha}\right)\left(\frac{n}{\alpha+\beta}\right)$. Cum enim sit

$$\left(\frac{\alpha+\beta}{\alpha}\right) = \frac{(\alpha+\beta)(\alpha+\beta-1)(\alpha+\beta-2)\ldots(\beta+1)}{1.2.3\ldots\ldots\ldots\alpha}, \text{ et}$$

$$\left(\frac{n}{\alpha+\beta}\right) = \frac{n(n-1)(n-2)\ldots\ldots(n-\alpha-\beta+1)}{1.2.3\ldots\alpha\times(\alpha+1)(\alpha+2)\ldots(\alpha+\beta)}, \text{ sive}$$

$$\left(\frac{n}{\alpha+\beta}\right) = \frac{n(n-1)(n-2)\ldots\ldots(n-\alpha-\beta+1)}{1.2.3\ldots\beta.(\beta+1)\ldots(\beta+\alpha)}, \text{ productum erit}$$

$$\left(\frac{\alpha+\beta}{\alpha}\right)\left(\frac{n}{\alpha+\beta}\right) = \frac{n(n-1)(n-2)\ldots\ldots(n-\alpha-\beta+1)}{1.2.3\ldots\beta\times1.2.3\ldots\alpha},$$

in quam eandem formam resolvitur formula $\left(\frac{n}{\alpha}\right)\left(\frac{n-\alpha}{\beta}\right)$.

§. 16. Per hanc igitur transformationem superiores series sequenti modo exprimi poterunt:

$$p = 1 + \left(\frac{3}{0}\right)\left(\frac{n}{2}\right) + \left(\frac{4}{2}\right)\left(\frac{n}{4}\right) + \left(\frac{6}{3}\right)\left(\frac{n}{6}\right) + \left(\frac{8}{4}\right)\left(\frac{n}{8}\right) + \text{etc.}$$

$$q = \left(\frac{1}{0}\right)\left(\frac{n}{1}\right) + \left(\frac{3}{1}\right)\left(\frac{n}{3}\right) + \left(\frac{5}{2}\right)\left(\frac{n}{5}\right) + \left(\frac{7}{3}\right)\left(\frac{n}{7}\right) + \text{etc.}$$

$$q = \left(\frac{2}{0}\right)\left(\frac{n}{2}\right) + \left(\frac{4}{1}\right)\left(\frac{n}{4}\right) + \left(\frac{6}{2}\right)\left(\frac{n}{6}\right) + \left(\frac{8}{3}\right)\left(\frac{n}{8}\right) + \text{etc.}$$

$$s = \left(\frac{3}{0}\right)\left(\frac{n}{3}\right) + \left(\frac{5}{1}\right)\left(\frac{n}{5}\right) + \left(\frac{7}{2}\right)\left(\frac{n}{7}\right) + \left(\frac{9}{3}\right)\left(\frac{n}{9}\right) + \text{etc.}$$

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$$z = \left(\frac{\lambda}{0}\right)\left(\frac{n}{\lambda}\right) + \left(\frac{\lambda+2}{1}\right)\left(\frac{n}{\lambda+2}\right)\left(\frac{\lambda+4}{2}\right)\left(\frac{n}{\lambda+4}\right) + \left(\frac{\lambda+6}{3}\right)\left(\frac{n}{\lambda+6}\right) + \text{etc.}$$

§. 17. Notari adhuc meretur alia transformatio, quae ad calculum numericum imprimis est accommodata. Cum enim ex prima forma sit $z = \left(\frac{n}{\lambda}\right) + \left(\frac{n}{1}\right)\left(\frac{n-1}{\lambda+1}\right) + \left(\frac{n}{2}\right)\left(\frac{n-2}{\lambda+2}\right) +$ etc. quilibet terminus hujus seriei est $\left(\frac{n}{\alpha}\right)\left(\frac{n-\alpha}{\lambda+\alpha}\right)$, qui dicatur $= \Pi$, eritque facta evolutione:

$$\Pi = \frac{n(n-1)(n-2)\ldots(n-\alpha+1)\ldots(n-2\alpha-\lambda+1)}{1.2.3\ldots\ldots\alpha.1.2.3\ldots\ldots\lambda+\alpha}.$$

Quodsi jam hic loco α scribamus $\alpha+1$, ut oriatur terminus sequens, qui ergo erit $= \frac{n(n-1)(n-2)\ldots\ldots\ldots n-2\alpha-\lambda-1}{1.2.3\ldots(\alpha+1)\times1.2.3(\lambda+\alpha+1)}$,

hic per illum divisus praebet quotum
$\frac{(n-2\alpha-\lambda)(n-2\alpha-\lambda-1)}{(\alpha+1)(\lambda+\alpha+1)}$. Hinc ergo erit terminus sequens
$\Pi . \frac{(n-2\alpha-\lambda)(n-2\alpha-\lambda-1)}{(\alpha+1)(\lambda+\alpha+1)}$.

§. 18. Quodsi ergo in hac serie, more Newtoniano, littera Π denotet quemlibet terminum praecedentem, sequens semper erit $\Pi . \frac{(n-2\alpha-\lambda)(n-2\alpha-\lambda-1)}{(\alpha+1)(\lambda+\alpha+1)}$; unde cum primus terminus sit $(\frac{n}{\lambda})$, ubi est $\alpha = 0$, si hic designetur per Π, erit terminus secundus
$= \Pi \frac{(n-\lambda)(n-\lambda-1)}{(\lambda+1)}$, qui si denuo vocetur Π, erit terminus tertius
$= \Pi \frac{(n-\lambda-2)(n-\lambda-3)}{2(\lambda+2)}$, qui si denuo vocetur Π, erit termin. quartus
$= \Pi \frac{(n-\lambda-4)(n-\lambda-5)}{3(\lambda+3)}$, et ita porro. Hoc modo nostra series pro z hanc induet formam:
$$z = (\tfrac{n}{\lambda}) + \Pi \frac{(n-\lambda)(n-\lambda-1)}{1.(\lambda+1)} + \Pi \frac{(n-\lambda-2)(n-\lambda-3)}{2(\lambda+2)} + \Pi \frac{(n-\lambda-4)(n-\lambda-5)}{3.(\lambda+3)} + \text{etc.}$$
ubi scilicet perpetuo Π designat terminum praecedentem.

§. 19. Hinc igitur si loco λ successive scribamus valores 1, 2, 3, 4 etc. pro nostris litteris p, q, r, s, etc. sequentes nanciscemur series:
$$p = 1 + \Pi \frac{n(n-1)}{1.1} + \Pi \frac{(n-2)(n-3)}{2.2} + \Pi \frac{(n-4)(n-5)}{3.3} + \Pi \frac{(n-6)(n-7)}{4.4} + \text{etc.}$$
$$q = (\tfrac{n}{1}) + \Pi \frac{(n-1)(n-2)}{1.2} + \Pi \frac{(n-3)(n-4)}{2.3} + \Pi \frac{(n-5)(n-6)}{3.4} + \Pi \frac{(n-7)(n-8)}{4.5} + \text{etc.}$$
$$r = (\tfrac{n}{2}) + \Pi \frac{(n-2)(n-3)}{1.3} + \Pi \frac{(n-4)(n-5)}{2.4} + \Pi \frac{(n-6)(n-7)}{3.5} + \Pi \frac{(n-8)(n-9)}{4.6} + \text{etc.}$$
$$s = (\tfrac{n}{3}) + \Pi \frac{(n-3)(n-4)}{1.4} + \Pi \frac{(n-5)(n-6)}{2.5} + \Pi \frac{(n-7)(n-8)}{3.6} + \Pi \frac{(n-9)(n-10)}{4.7} + \text{etc.}$$
etc. etc.

§. 20. Istae formae ad calculum numericum imprimis sunt accommodatae, quod pro sola littera p ostendisse sufficiet. Quaeramus scilicet exempli gratia valorem ipsius p pro

pro casu $n = 6$, ac singulae ejus partes sequenti modo reperientur:

$$\begin{aligned} \text{I.} &= . &&= 1 \\ \text{II.} &= 1 . \tfrac{6.5}{1.1} &&= 30 \\ \text{III.} &= 30 . \tfrac{4.3}{2.2} &&= 90 \\ \text{IV.} &= 90 . \tfrac{2.1}{3.3} &&= 20 \end{aligned}$$

ergo Summa $= p = 141$

§. 21. Simili modo quaeramus valorem ipsius p pro casu $n = 12$, cujus singulae partes sequenti modo colligentur:

$$\begin{aligned} \text{I.} &= 1 \; . \; . \; . \; . &&= 1 \\ \text{II.} &= 1 . \tfrac{12.11}{1.1} &&= 132 \\ \text{III.} &= 132 . \tfrac{10.9}{2.2} &&= 2970 \\ \text{IV} &= 2970 . \tfrac{8.7}{3.3} &&= 18480 \\ \text{V} &= 18480 . \tfrac{6.5}{4.4} &&= 34650 \\ \text{VI} &= 34650 . \tfrac{4.3}{5.5} &&= 16632 \\ \text{VII} &= 16632 . \tfrac{2.1}{6.6} &&= 924 \end{aligned}$$

ergo Summa $= p = 73789$.

§. 22. Mox autem trademus modum multo expeditiorem singulos terminos harum serierum ex binis praecedentibus eliciendi, unde facili calculo omnes valores pro litteris p, q, r, etc. pro singulis exponentibus n exhiberi poterunt, sicque omnes istos valores, quousque libuerit, continuare licebit. Hanc autem relationem primo seorsim pro numeris sub littera p contentis instituamus.

L 2 Inve-

Investigatio relationis inter ternos valores consecutivos p, p', p''.

§. 23. Cum sit $p = 1 + (\frac{n}{1})(\frac{n-1}{1}) + (\frac{n}{2})(\frac{n-2}{2}) + (\frac{n}{3})(\frac{n-3}{3}) + \text{etc.}$ hujus feriei confideremus terminum quemcunque $(\frac{n}{\alpha})(\frac{n-\alpha}{\alpha})$, quem vocemus $= \Pi$, ita ut facta evolutione fit $\Pi = \frac{n(n-1)(n-2) \ldots (n-2\alpha+1)}{1 \cdot 2 \cdot 3 \ldots \alpha \times 1 \cdot 2 \cdot 3 \ldots \alpha}$; terminum autem, qui hunc sequitur, defignemus per Φ, ut fit $\Phi = (\frac{n}{\alpha+1})(\frac{n-\alpha-1}{\alpha+1})$, ideoque facta evolutione $\Phi = \frac{n(n-1)(n-2) \ldots (n-2\alpha-1)}{1 \cdot 2 \cdot 3 \ldots (\alpha+1) \times 1 \cdot 2 \cdot 3 \ldots (\alpha+1)}$ hincque ergo habebitur:

$$\frac{\Phi}{\Pi} = \frac{(n-2\alpha)(n-2\alpha-1)}{(\alpha+1)(\alpha+1)} \text{ ideoque } \Pi = \frac{(\alpha+1)(\alpha+1)\Phi}{(n-2\alpha)(n-2\alpha-1)}.$$

§. 24. Jam pro valoribus fequentibus p' et p'' defignemus valores ipfius Φ respondentes per Φ' et Φ'', qui, quoniam oriuntur ex valore Φ, fi loco n fcribatur $n+1$ et $n+2$, erit facta evolutione

$\Phi' = \frac{(n+1)n(n-1) \ldots (n-2\alpha)}{1 \cdot 2 \cdot 3 \ldots (\alpha+1) \times 1 \cdot 2 \cdot 3 \ldots (\alpha+1)}$, unde patet fore

$\Phi' : \Phi = \frac{n+1}{n-2\alpha-1}$, hincque $\Phi' = \frac{n+1}{n-2\alpha-1}\Phi$. Simili modo, fi hic quoque loco n fcribamus $n+1$, habebimus

$$\Phi'' = \frac{n+2}{n-2\alpha}\Phi', \text{ five } \Phi'' = \frac{(n+1)(n+2)\Phi}{(n-2\alpha-1)(n-2\alpha)}.$$

§. 25. Hinc jam formemus hanc expressionem: $A\Phi + \frac{B}{n+1}\Phi' + \frac{C}{(n+2)(n+1)}\Phi''$, cujus ergo valor per ipsam litteram Φ ita exprimetur:

$\Phi(A + \frac{B}{n-2\alpha-1} + \frac{C}{(n-2\alpha-1)(n-2\alpha)})$, ubi litteras A, B, C ita definire conemur, ut forma ista aequalis evadat ipfi termino praecedenti Π. Evidens autem eft, iftas litteras, ut ad omnes terminos pariter pertinere queant, litteram α involvere

non debere. Subftituto igitur loco Π valore ante dato per Φ expresso, nanciscemur fequentem aequationem per Φ divisam:

$$A + \frac{B}{n-2\alpha-1} + \frac{C}{(n-2\alpha-1)(n-2\alpha)} = \frac{(\alpha+1)(\alpha+1)}{(n-2\alpha-1)(n-2\alpha)},$$

quae a fractionibus liberata evadit

$$A(n-2\alpha-1)(n-2\alpha) + B(n-2\alpha) + C = (\alpha+1)(\alpha+1).$$

§. 26. Cum in hac aequatione littera α ad fecundam dimenfionem afcendat, ternae litterae A, B, C praecife fufficient, ut ex hac aequatione determinari queant. Primo igitur coaequemus utrinque terminos quadratum $\alpha\alpha$ involventes, unde orietur ifta aequatio: $4A\alpha\alpha = \alpha\alpha$, ideoque $A = \frac{1}{4}$. Eodem modo coaequemus terminos ipsam litteram α involventes, unde perducimur ad hanc aequationem: $2\alpha(1-2n)A - 2\alpha \cdot B = 2\alpha$, unde fit $B = -\frac{2n-1}{4} - 1 = -\frac{2n+3}{4}$. Denique termini ab α immunes dant hanc aequationem: $(nn-n)A + nB + C = 1$, unde reperitur $C = \frac{(n+2)^2}{4}$.

§. 27. His igitur valoribus inventis, pro fingulis terminis femper erit $A\Phi + \frac{B}{n+1} \cdot \Phi' + \frac{C}{(n+2)(n+1)}\Phi'' = \Pi$. Quod fi ergo hinc computemus hanc formulam: $A \cdot p + \frac{B}{n+1}p' + \frac{C}{(n+2)(n+1)}p''$, ex primis terminis pro Φ assumtis orietur praecedens feriei p qui eft $= 0$; ex fecundis autem terminis pro Φ assumtis orietur terminus primus qui eft 1; ex terminis autem tertiis conficitur terminus fecundus, qui eft $(\frac{n}{1})(\frac{n-1}{1})$; ex terminis quartis pro Φ assumtis conficitur tertius qui eft $(\frac{n}{2})(\frac{n-2}{2})$ et ita porro; ficque omnes tres feries hoc modo collectae, producent hanc feriem: $0 + 1 + (\frac{n}{1})(\frac{n-1}{1}) + (\frac{n}{2})(\frac{n-2}{2}) +$ etc. quae eft ipsa feries pro p data

gralia intra praescriptos terminos fuerint assumta, fcilicet ab $x = a$ usque ad $x = b$, qui quidem termini nondum

tertio, casu quo $x = -2$, ex quibus ergo binos terminos illos a et b defumi oportet. Ita autem hos binos terminos data. Hinc habebimus inter ternas litteras p, p', p'' hanc aequationem:

$$A \cdot p + \frac{B}{n+1} p' + \frac{C}{(n+2)(n+1)} p'' = p.$$

§. 28. Substituamus nunc loco litterarum A, B, C valores modo inventos, et nostra aequatio inter has ternas litteras erit $\frac{1}{4} p - \frac{(2n+3)}{4(n+1)} p' + \frac{n}{4(n+1)} p'' = p$, quae reducitur ad hanc: $\frac{n+2}{n+1} p'' - \frac{(2n+3)}{n+1} p' = 3p$, unde fit $p'' = p' + (\frac{n+1}{n+2})(p' + 3p)$.

§. 29. Hinc igitur facile pro singulis valoribus exponentis n omnes numeri littera p designati definiri poterunt, dum quilibet ex duobus praecedentibus componitur. Ita sumto $n = 0$, erit $p = 1$ et $p' = 1$, ideoque tertius $p'' = 1 + \frac{1}{2}(1 + 3 \cdot 1)$. Sumto $n = 1$, erit $p = 1$, $p' = 3$, ideoque terminus quartus $p'' = 3 + \frac{2}{3}(3 + 3 \cdot 1) = 7$. Sumto $n = 2$, ob $p = 3$ et $p = 7$, erit terminus quintus $p'' = 7 + \frac{3}{4}(7 + 3 \cdot 3) = 19$. Si sumatur $n = 3$, ob $p = 7$ et $p' = 19$, erit terminus sextus $p'' = 19 + \frac{4}{5}(19 + 3 \cdot 7) = 51$.

§. 30. Si hoc modo ulterius progrediamur, poterimus hanc progressionem continuare, quousque libuerit, ope formae $p' + \frac{n+1}{n+2}(p' + 3p) = p''$, quae nobis suppeditat sequentes determinationes:

$$51 + \tfrac{5}{6}(51 + 3 \cdot 19) = 141$$
$$141 + \tfrac{6}{7}(141 + 3 \cdot 51) = 393$$
$$393 + \tfrac{7}{8}(393 + 3 \cdot 141) = 1107$$
$$1107 + \tfrac{8}{9}(1107 + 3 \cdot 393) = 3139$$
$$3139 + \tfrac{9}{10}(3139 + 3 \cdot 1107) = 8953$$
$$8953 + \tfrac{10}{11}(8953 + 3 \cdot 3139) = 25653$$
$$25653 + \tfrac{11}{12}(25653 + 3 \cdot 8953) = 73789.$$

etc.

§. 31.

§. 31. Simili methodo etiam relatio inter ternos terminos consecutivos pro sequentibus litteris q, r, s, etc. investigari poterit, quem laborem quo generaliter expediamus, inquiramus in relationem, inter ternos terminos z, z' et z'', quibus respondere assumsimus litteram λ.

Investigatio Relationis
inter ternos terminos consecutivos z, z', z''.

§. 32. Cum sit $z = (\frac{n}{\lambda}) + (\frac{n}{1})(\frac{n-1}{\lambda+1}) + (\frac{n}{2})(\frac{n-2}{\lambda+2}) + (\frac{n}{3})(\frac{n-3}{\lambda+3}) +$ etc. hujus seriei consideremus terminum quemcunque $\Pi = (\frac{n}{\alpha})(\frac{n-\alpha}{\lambda+\alpha})$, cujus valor evolutus est $\Pi = \frac{n(n-1)(n-2) \ldots \ldots (n-2\alpha-\lambda+1)}{1 \cdot 2 \cdot 3 \ldots \alpha \times 1 \cdot 2 \cdot 3 \ldots (\lambda+\alpha)}$. Terminum jam hunc sequentem $(\frac{n}{\alpha+1})(\frac{n-\alpha-1}{\lambda+\alpha+1}) = \Phi$ evolvamus, unde fit

$\Phi = \frac{n(n-1)(n-2) \ldots \ldots \ldots (n-2\alpha-\lambda-1)}{1 \cdot 2 \cdot 3 \ldots (\alpha+1) \times 1 \cdot 2 \cdot 3 \ldots (\lambda+\alpha+1)}$. Hinc ergo colligimus $\frac{\Phi}{\Pi} = \frac{(n-2\alpha-\lambda)(n-2\alpha-\lambda-1)}{(\alpha+1)(\lambda+\alpha+1)}$, ideoque $\Pi = \frac{(\alpha+1)(\lambda+\alpha+1)}{(n-2\alpha-\lambda)(n-2\alpha-\lambda-1)}$.

§. 33. Jam pro valoribus sequentibus z' et z'' designemus valores ipsi Φ respondentes per Φ' et Φ'', qui quoniam oriuntur ex valore Φ, si loco n scribatur $(n+1)$ et $(n+2)$, erit facta evolutione

$\Phi' = \frac{(n+1)n(n-1)(\ldots \ldots \ldots (n-2\alpha-\lambda)}{1 \cdot 2 \cdot 3 \ldots (\alpha+1) \times 1 \ldots (\alpha+\lambda+1)}$, unde patet fore $\frac{\Phi'}{\Phi} = \frac{n+1}{n-2\alpha-\lambda-1}$, hincque $\Phi' = \frac{(n+1)\Phi}{n-2\alpha-\lambda-1}$. Eodemque modo erit $\Phi'' = \frac{n+2}{n-2\alpha-\lambda} \Phi' = \frac{(n+2)(n+1)\Phi}{(n-2\alpha-\lambda)(n-2\alpha-\lambda-1)}$.

§. 34. Hinc prorsus ut supra formemus hanc expressionem: $A\Phi + \frac{B}{n+1} \Phi' + \frac{C}{(n+2)(n+1)} \Phi''$, cujus valor per Φ ita exprimitur: $\Phi(A + \frac{B}{n-2\alpha-\lambda-1} + \frac{C}{(n-2\alpha-\lambda)(n-2\alpha-\lambda-1)})$, ubi iterum litteras A, B, C ita definiri oportet, ut formula aequalis evadat ipsi termino praecedenti Π. Substituto igitur loco Π valore

ante

ante per Φ expresso nancifcemur fequentem aequationem a fractionibus jam liberatam:

$$A(n-2\alpha-\lambda-1)(n-2\alpha-\lambda)+B(n-2\alpha-\lambda)+C=(\alpha+1)(\alpha+\lambda+1).$$

§. 35. Facta igitur evolutione et coaequatis primo utrinque terminis $\alpha\alpha$ involventibus prodit haec aequatio pro determinatione litterae A: $4\alpha\alpha A=\alpha\alpha$, ideoque $A=\frac{1}{4}$. Eodem modo fi coaequentur termini fimplicem litteram α involventes, perducimur ad fequentem aequationem:

$(4\alpha\lambda-4n\alpha+2\alpha)A-2\alpha B=(\lambda+2)\alpha$, unde colligitur $B=-\frac{2n+3}{4}$. Denique coaequatis terminis ab α liberis prodit aequatio:

$$\frac{nn-2n\lambda-n+\lambda\lambda+\lambda}{4}-\frac{(n-\lambda)(2n+3)}{4}+C=\lambda+1,\text{ unde fit}$$
$$C=\frac{(n+2)^2}{4}-\frac{\lambda\lambda}{4}.$$

§. 36. His igitur valoribus inventis, pro fingulis terminis femper erit $A\Phi+\frac{B}{n+1}\Phi'+\frac{C}{(n+2)(n+1)}\Phi''=\Pi$. Quod fi igitur hinc computemus iftam formulam:

$Az+\frac{B}{n+1}z'+\frac{C}{(n+2)(n+1)}z''$, ex primis terminis pro Φ assumtis orietur praecedens feriei z, qui eft o. Ex fecundis autem terminis pro Φ assumtis orietur terminus primus $\left(\frac{n}{\lambda}\right)$; ex tertiis terminis conficitur terminus fecundus $\left(\frac{n}{1}\right)\left(\frac{n-1}{\lambda+1}\right)$; ex quartis terminis pro Φ assumtis conficitur tertius qui eft $\left(\frac{n}{2}\right)\left(\frac{n-2}{\lambda+2}\right)$, et ita porro; quibus collectis oritur ipfa feries pro z data

$$z=\left(\frac{n}{\lambda}\right)+\left(\frac{n}{1}\right)\left(\frac{n-1}{\lambda+1}\right)+\left(\frac{n}{2}\right)\left(\frac{n-2}{\lambda+2}\right)+\left(\frac{n}{3}\right)\left(\frac{n-3}{\lambda+3}\right)+\text{etc.}$$

Relatio igitur inter z, z′, z″ erit

$$Az+\frac{B}{n+1}z'+\frac{C}{(n+2)(n+1)}z''=z.$$

§. 37.

§. 37. Subftituamus nunc loco litterarum A, B, C valores modo inventos, et aequatio inter has ternas litteras erit:

$$\frac{1}{4}z-\frac{(2n+3)}{4(n+1)}z'+\frac{(n+2)^2-\lambda\lambda}{4(n+2)(n+1)}z''=z,$$

quae reducitur ad hanc formam:

$$\frac{(n+2)^2-\lambda\lambda}{(n+2)(n+1)}z''=\frac{2n+3}{n+1}z'+3z,\text{ unde colligitur}$$
$$z''=\frac{n+2}{(n+2)^2-\lambda\lambda}((2n+3)z'+3(n+1)z).$$

§. 38. Tribuamus nunc litterae λ fucceffive valores o, 1, 2, 3, 4, etc. ac reperiemus fequentes relationes pro fingulis litteris:

$$\frac{(n+2)^2-0^2}{(n+2)(n+1)}p''=\frac{2n+3}{n+1}p'+3p$$
$$\frac{(n+2)^2-1^2}{(n+2)(n+1)}q''=\frac{2n+3}{n+1}q'+3q$$
$$\frac{(n+2)^2-2^2}{(n+2)(n+1)}r''=\frac{2n+3}{n+1}r'+3r$$
$$\frac{(n+2)^2-3^2}{(n+2)(n+1)}s''=\frac{2n+3}{n+1}s'+3s$$
etc. etc.

§. 39. Cum igitur pro littera q habeamus hanc aequationem: $q''=\frac{n+2}{(n+1)(n+3)}((2n+3)q'+3(n+1)q)$, casu $n=0$ erit $q=0$ et $q'=1$, unde fit $q''=\frac{2}{3}(3\cdot1+3\cdot0)=2$. Nunc pro $n=1$, ob $q=1$ et $q'=2$, erit $q''=\frac{3}{2\cdot4}(5\cdot2+6\cdot1)=6$. Tum pro casu $n=2$, ob $q=2$ et $q'=6$, erit $q''=\frac{4}{3\cdot5}(7\cdot6+9\cdot2)=16$. Jam fumto $n=3$, ob $q=6$ et $q'=16$, erit $q''=\frac{5}{4\cdot6}(9\cdot16+12\cdot6)=45$. At casu $n=4$, ob $q=16$ et $q'=45$, erit $q''=\frac{6}{5\cdot7}(11\cdot45+15\cdot16)=126$.

§. 40. Hic autem calculus multo laboriofior et taediofior eft quam praecedens pro valoribus litterae p expofitus. Verum alia methodus multo facilior inde derivari poterit, qua omnes litteras q, r, s per folam litteram p cum suis deri-

derivatis p', p'' determinare licebit; tum enim postquam series numerorum p jam satis longe fuerit computata, inde etiam valores litterarum q, r, s, etc. multo leviori labore colligi poterunt, id quod in sequenti articulo ostendemus:

Determinatio litterarum
q, r, s, t, etc.
per solam primam p cum suis derivatis.

§. 41. Posito brevitatis gratia nostro trinomio $1+x+xx=X$, ejus binas potestates X^n et X^{n+1} evolutas ita disponamus, ut pares potestates ipsius x sibi invicem subscriptae appareant, hoc modo:

$$X^n = 1 + nx + \ldots + qx^{n-1} + px^n + qx^{n+1} + rx^{n+2} + sx^{n+3} + \text{etc.}$$
$$X^{n+1} = 1 + (n+1)x \ldots + rx^{n-1} + qx^n + p'x^{n-1} + qx^{n+2} + r'x^{n+3} + \text{etc.}$$

quo facto supra jam notavimus quemlibet coëfficientem inferioris seriei aequari superiori cum binis praecedentibus.

§. 42. Per hanc igitur legem sequentes nanciscemur aequalitates:

$$p' = q + p + q = 2q + p$$
$$q' = r + q + p$$
$$r' = s + r + q$$
etc.

unde colligimus sequentes determinationes

$$q = \tfrac{p'-p}{2};\; r = q' - q - p;\; s' = r' - r - q;\; t' = s' - s - r;\; \text{etc.}$$

§. 43. Manifestum est hic formulam $p' - p$ exprimere incrementum quantitatis p, dum exponens n unitate augetur, quod cum per Δp exprimi soleat, aequalitates inventae sequenti modo succinctius exhiberi poterunt:

$$q = \tfrac{1}{2}\Delta p, \text{ sive } 2q = \Delta p;\; 2r = 2\Delta q - 2p;\; 2s = 2\Delta r - 2q; \text{ etc.}$$

§. 44

§. 44. Charactere autem hoc differentiali Δ in usum vocato, cum sit $2q = \Delta p$, erit $2\Delta q = \Delta\Delta p$ ideoque

$2r = \Delta\Delta p - 2p$, hincque $2\Delta r = \Delta^3 p - 2\Delta p$, ex quo porro fit $2s = \Delta^3 p - 3\Delta p$ ideoque

$2\Delta s = \Delta^4 p - 3\Delta\Delta p$, ergo $2t = \Delta^4 p - 4\Delta\Delta p + 2p$, ideoque

$2\Delta t = \Delta^5 p - 4\Delta^3 p + 2\Delta p$. Hinc porro fit

$2u = \Delta^5 p - 5\Delta^3 p + 5\Delta p$, ideoque

$2\Delta u = \Delta^6 p - 5\Delta^4 p + 5\Delta\Delta p$, unde deducitur

$2v = \Delta^6 p - 6\Delta^4 p + 9\Delta\Delta p + 2p$, et ita porro.

§. 45. Quod si hos coëfficientes numericos attentius consideremus, lex progressionis convenire deprehenditur cum serie Geometris satis nota, unde pro valore z, cui ordinis index positus est λ, obtinebimus sequentem formam:

$$2z = \Delta^\lambda p - \lambda\Delta^{\lambda-2} p + \tfrac{\lambda(\lambda-3)}{1.2}\Delta^{\lambda-4} p - \tfrac{\lambda(\lambda-4)(\lambda-5)}{1.2.3}\Delta^{\lambda-6} p$$
$$+ \tfrac{\lambda(\lambda-5)(\lambda-6)(\lambda-7)}{1.2.3.4}\Delta^{\lambda-8} p - \tfrac{\lambda(\lambda-6)(\lambda-7)(\lambda-8)(\lambda-9)}{1.2.3.4.5}\Delta^{\lambda-10} p + \text{etc.}$$

quam seriem eo usque tantum continuari oportet, quamdiu indices ipsius Δ non evadunt negativi. Ita si sumamus $\lambda = 6$, quo casu fit $z = v$, ex hac lege generali utique prodit

$$2v = \Delta^6 p - 6\Delta^4 p + 9\Delta^2 p - 2p.$$

§. 46. Quo indoles hujus seriei clarius perspiciatur, meminisse oportet, hanc formam $\frac{(x+\sqrt{xx-4})^n}{2^n} + \frac{(x-\sqrt{xx-4})^n}{2^n}$ in sequentem seriem resolvi:

$$x^n - nx^{n-2} + \tfrac{n(n-3)}{1.2}x^{n-4} - \tfrac{n(n-4)(n-5)}{1.2.3}x^{n-6} + \text{etc.}$$

Hoc igitur modo nostro scopo jam est satisfactum, cum omnes litteras q, r, s, t. etc. per solam primam p suasque derivatas p', p'', p''', etc. expressas elicuerimus.

 De-

§. 69. Quod si jam hanc aequationem per $\partial\Phi \cos\lambda\Phi$ multiplicemus et integremus, omnia haec integralia, ter-

§. 72. Cum jam series nostra fit

Determinatio quantitatis p per formulam finitam integralem.

§. 47. Cum ſit per tertiam formam ſupra expoſitam $p = 1 + \left(\frac{2}{1}\right)\left(\frac{n}{2}\right) + \left(\frac{4}{2}\right)\left(\frac{n}{4}\right) + \left(\frac{6}{3}\right)\left(\frac{n}{6}\right) + \text{etc.}$ quilibet terminus in genere erit $\left(\frac{2\alpha}{\alpha}\right)\left(\frac{n}{2\alpha}\right)$, quem excipit iſte ſequens: $\left(\frac{2\alpha+2}{\alpha+1}\right)\left(\frac{n}{2\alpha+2}\right)$ Cum igitur facta evolutione ſit

$$\left(\frac{2\alpha}{\alpha}\right) = \frac{2\alpha(2\alpha-1)(2\alpha-2)\ldots\ldots\ldots(\alpha+1)}{1.2.3\ldots\ldots\ldots\alpha},$$ ſimilique modo

$$\left(\frac{2\alpha+2}{\alpha+1}\right) = \frac{(2\alpha+2)(2\alpha+1)2\alpha\ldots\ldots\ldots(\alpha+2)}{1.2.3\ldots\ldots(\alpha+1)},$$

haec poſterior forma per priorem diviſa dat quotum

$$\frac{(2\alpha+2)(2\alpha+1)}{(\alpha+1)^2} = \frac{2(2\alpha+1)}{\alpha+1},$$ ſicque erit $\left(\frac{2\alpha+2}{\alpha+1}\right) = \frac{4\alpha+2}{\alpha+1}\left(\frac{2\alpha}{\alpha}\right)$.

§. 48. Hac ergo reductione adhibita, ſumto $\alpha = 1$ erit $\left(\frac{4}{2}\right) = \frac{6}{2}\left(\frac{2}{1}\right)$; ſumto $\alpha = 2$ erit $\left(\frac{6}{3}\right) = \frac{10}{3}\left(\frac{4}{2}\right) = \frac{10}{3}\cdot\frac{6}{2}\cdot\frac{2}{1}$; ſi $\alpha = 3$ ſit $\left(\frac{8}{4}\right) = \frac{14}{4}\left(\frac{6}{3}\right) = \frac{14}{4}\cdot\frac{10}{3}\cdot\frac{6}{2}\cdot\frac{2}{1}$; tum ſi $\alpha = 4$ fiet $\left(\frac{10}{5}\right) = \frac{18}{5}\left(\frac{8}{4}\right) = \frac{18}{5}\cdot\frac{14}{4}\cdot\frac{10}{3}\cdot\frac{6}{2}\cdot\frac{2}{1}$; et ita porro. His igitur valoribus introductis erit per factores ordinarios numericos

$$p = 1 + \frac{2}{1}\left(\frac{n}{2}\right) + \frac{2.6}{1.2}\left(\frac{n}{4}\right) + \frac{2.6.10}{1.2.3}\left(\frac{n}{6}\right) + \frac{2.6.10.14}{1.2.3.4}\left(\frac{n}{8}\right) + \frac{2\ldots\ldots18}{1\ldots\ldots5}\left(\frac{n}{10}\right) + \text{etc.}$$

§. 49. Nunc igitur videamus, quomodo formam finitam integralem inveſtigari oporteat, cujus integrale intra datos terminos incluſum ad hanc ipſam ſeriem perducat. Hunc in finem contemplari conveniet iſtam formulam $(1+x)^n$, quippe cujus evolutio praebet hanc ſeriem: $1 + \left(\frac{n}{1}\right)x + \left(\frac{n}{2}\right)xx + \left(\frac{n}{3}\right)x^3 + \text{etc.}$, cujus termini alterni jam continent characteres noſtros litterae n.

§. 50. Hanc igitur ſeriem in duas partes diſcerpamus, ſecundum terminos alternos, ac ponamus

$$M = 1 + \left(\frac{n}{2}\right)xx + \left(\frac{n}{4}\right)x^4 + \left(\frac{n}{6}\right)x^6 + \text{etc.}$$
$$N = \left(\frac{n}{1}\right)x + \left(\frac{n}{3}\right)x^3 + \left(\frac{n}{5}\right)x^5 + \left(\frac{n}{7}\right)x^7 + \text{etc.}$$

ita

ita ut ſit $(1+x)^n = M + N$. Nunc autem inquiramus, quomodo ſeriem priorem M per operationes analyticas tractari oporteat, ut ipſa ſeries propoſita, ſeu valor ipſius p, inde exoriatur.

§. 51. Ad hoc efficiendum ducamus quantitatem M in certum quoddam differentiale ∂v cujuspiam functionis ipſius x, atque ſequentes integrationes ita determinemus, ut intra certos terminos, veluti ab $x = a$ usque ad $x = b$, includantur, quas conditiones ita comparatas eſſe oportet, ut ſequentibus conditionibus ſatisfiat:

1) $\int xx\partial v = \frac{2}{1}v$
2) $\int x^4\partial v = \frac{2.6}{1.2}v$
3) $\int x^6\partial v = \frac{2.6.10}{1.2.3}v$
4) $\int x^8\partial v = \frac{2.6.10.14}{1.2.3.4}v$

etc.

hoc enim modo integrale $\int M\partial v$ producet hanc ſeriem:

$$v + \frac{2}{1}\left(\frac{n}{2}\right)v + \frac{2.6}{1.2}\left(\frac{n}{4}\right)v + \frac{2.6.10}{1.2.3}\left(\frac{n}{6}\right)v + \text{etc.}$$

ita ut hoc modo id quod quaerimus nanciscamur $p = \int\frac{M\partial v}{v}$

§. 52. Formularum igitur integralium, quas hic expoſuimus, quaelibet ita a praecedente pendet, ut ſit

$$\int xx\partial v = \frac{2}{1}\int\partial v$$
$$\int x^4\partial v = \frac{6}{2}\int xx\partial v$$
$$\int x^6\partial v = \frac{10}{3}\int x^4\partial v$$
$$\int x^8\partial v = \frac{14}{4}\int x^6\partial v$$

etc.

ſicque in genere effici debet ut fiat $\int x^{2m}\partial v = \frac{4m-2}{m}\int x^{2m-2}\partial v$. Haec ſcilicet reductio locum habere debet, poſtquam integralia

gralia intra praescriptos terminos fuerint assumta, scilicet ab $x = a$ usque ad $x = b$, qui quidem termini nondum sunt cogniti, sed ad ipsam conditionem praescriptam accommodari debent.

§. 53. Cum igitur pro his terminis integrationis esse debeat

$$m\int x^{2m}\partial v = (4m-2)\int x^{2m-2}\partial v,$$

ponamus esse generatim

$$m\int x^{2m}\partial v = (4m-2)\int x^{2m-2}\partial v + \Pi x^{2m-1},$$

ubi scilicet Π ejusmodi sit functio, ut pars subnexa x^{2m-1} utroque termino tam $x = a$ quam $x = b$ in nihilum abeat. Haec jam aequatio differentiata et per x^{2m-1} divisa dat

$$mxx\partial v = (4m-2)\partial v + (3m-1)\Pi\partial x + x\partial\Pi,$$

quae aequatio subsistere debet pro omnibus numeris m.

§. 54. Hinc igitur ista aequatio in duas discerpi debebit, quarum altera contineat solos terminos littera m affectos, altera vero reliquos, quae ergo duae aequationes erunt

$$xx\partial v = 4\partial v + 2\Pi\partial x \quad \text{et} \quad 0 = -u\partial v - \Pi\partial x - x\partial\Pi.$$

Ex priori fit $\partial v = \frac{2\Pi\partial x}{xx-4}$; ex altera vero fit $\partial v = \frac{x\partial\Pi - \Pi\partial x}{2}$, qui ambo valores inter se coaequati praebent hanc aequationem: $4\Pi\partial x = (xx-4)(x\partial\Pi - \Pi\partial x) = x^3\partial\Pi - xx\Pi\partial x - 4x\partial\Pi + 4\Pi\partial x$ hincque colligitur $\frac{\partial\Pi}{\Pi} = \frac{x\partial x}{xx-4}$, unde integrando fit $l\Pi = l\sqrt{xx-4}$ ideoque $\Pi = C\sqrt{xx-4}$, vel etiam $\Pi = C\sqrt{4-xx}$, quo valore invento assequimur nostrum differentiale assumtum $\partial v = \frac{2C\partial x}{\sqrt{4-xx}}$, unde fit $v = 2CA\sin\frac{x}{2}$.

§. 55. Consideremus nunc formulam subfixam $\Pi x^{2m-1} = Cx^{2m-1}\sqrt{4-xx}$, quam deprehendimus triplici modo in nihilum abire posse: primo scilicet quando $x = 0$, casu excepto quo $m = 0$; secundo, casu quo $x = 2$; ac tertio

tertio, casu quo $x = -2$, ex quibus ergo binos terminos illos a et b desumi oportet. Ita autem hos binos terminos eligi conveniet, ut etiam altera integrationis pars $\int N\partial v$ commode exprimatur. Quia enim posuimus $(1+x)^n = M+N$, etiam ad integrale $\int N\partial v$ est respiciendum, quod si penitus evanesceret, pro terminis integrationis sine dubio id esset commodissimum, tum enim foret $\int(M+N)\partial v$, sive $\int\partial v(1+x)^n = \int M\partial v$, consequenter haberemus $p = \frac{\int M\partial v}{v}$.

§. 56. Supra autem posuimus

$$N = (\tfrac{n}{1})x + (\tfrac{n}{3})x^3 + (\tfrac{n}{5})x^5 + (\tfrac{n}{7})x^7 + \text{etc.}$$

unde conficitur $\int N\partial v = \frac{n}{1}\int x\partial v + \frac{n}{3}\int x^3\partial v + (\frac{n}{5})\int x^5\partial v +$ etc. ubi per easdem reductiones, quas pro littera M instituimus, quaelibet formula integralis ad praecedentem, ope reductionis $\int x^{2m}\partial v = \frac{4m-2}{m}\int x^{2m-2}\partial v$ reduci potest. Sumto enim $m = \frac{3}{2}$ erit $\int x^3\partial v = \frac{8}{3}\int x\partial v$. Sumto $m = \frac{5}{2}$ erit $\int x^5\partial v = \frac{16}{5}\int x^3\partial v$. Sumto $n = \frac{7}{2}$ erit $\int x^7\partial v = \frac{24}{7}\int x^5\partial v$, etc. unde patet, si modo $\int x\partial v$ evanesceret etiam sequentia omnia esse evanitura.

§. 57. Quoniam igitur invenimus $dv = \frac{2C\partial x}{\sqrt{(4-xx)}}$, erit $x\partial v = \frac{2Cx\partial x}{\sqrt{(4-xx)}}$, hincque $\int x\partial v = 2C\sqrt{4-xx}$, quae expressio binis casibus vel $x = +2$ vel $x = -2$ evanescit. Quam obrem si terminos integrationis constituamus $x = 2$, et $x = -2$, non solum partes illae subnexae Πx^{2n-1}, verum etiam totus valor integralis $\int N\partial v$ evanescet, atque adeo hoc casu quaesito nostro perfecte satisfecimus, cum sit $p = \frac{\int\partial v(1+x)^n}{v}$

§. 58. Cum igitur invenerimus $\partial v = \frac{2C\partial x}{\sqrt{(4-xx)}}$, ejus integrale, ita sumtum ut evanescat posito $x = 2$, erit

$v = 2$

$v = 2\,CA \sin\frac{\pi}{2} - 2\,C\frac{\pi}{2}$, quae expressio reducitur ad hanc: $v = -2\,CA\cos\frac{\pi}{2}$; unde hoc integrali usque ad alterum terminum $x = -2$ extenso prodit $v' = - C\pi$. His igitur valoribus substitutis erit formula quaesita $p = -\frac{1}{\pi}\int\frac{(1+x)^n\,\partial x}{\sqrt{4-xx}}$. Haec scil formula integralis a termino $x = 2$ usque ad terminum $x = -2$ extensa, verum praebebit valorem ipsius p.

§. 59. Quo hanc formulam concinniorem reddamus, statuamus $x = 2\cos\Phi$, ubi evidens est casu $x = 2$ fieri angulum $\Phi = 0$; casu vero $x = -2$ fieri $\Phi = \pi$, ita ut hoc angulo introducto integrale capi debeat a termino $\Phi = 0$ usque ad $\Phi = \pi$; tum vero erit $\partial x = -2\,\partial\Phi\sin\Phi$ et $\sqrt{4-xx} = 2\sin\Phi$, qua substitutione facta nanciscemur hanc aequationem:

$$p = +\frac{1}{\pi}\int(1+2\cos\Phi)^n\,\partial\Phi\left[\begin{array}{l}\text{a } \Phi = 0\\ \text{ad } = \pi\end{array}\right].$$

Determinatio reliquarum litterarum per formulas finitas integrales.

§. 60. Hoc facile praestari poterit per relationes quas supra inter has litteras tradidimus. Primo scilicet habuimus $2q = \Delta p = p' - p$, ubi p' nascitur ex p, si loco n scribatur $n+1$. Quoniam igitur modo invenimus

$$p = \frac{1}{\pi}\int(1+2\cos\Phi)^n\,\partial\Phi,\ \text{erit}\ p' = \frac{1}{\pi}\int(1+2\cos\Phi)^{n+1}\,\partial\Phi,$$

hincque ergo erit $p' - p = \frac{2}{\pi}\int\cos\Phi\,(1+2\cos\Phi)^n\,\partial\Phi$, quo valore substituto reperietur

$$q = \frac{1}{\pi}\int\partial\Phi\cos\Phi\,(1+2\cos\Phi)^n\left[\begin{array}{l}\text{a } \Phi = 0\\ \text{ad } \Phi = \pi\end{array}\right]$$

hinc ergo porro erit

$$q' = \frac{1}{\pi}\int\partial\Phi\cos\Phi\,(1+2\cos\Phi)^{n+1}.$$

§. 61.

§. 61. Supra autem vidimus esse $r = q' - q - p$, nunc vero erit $q' - q = \frac{2}{\pi}\int\partial\Phi\cos\Phi^2\,(1+2\cos\Phi)^n$. Hinc ergo si subtrahatur p, ob $2\cos\Phi^2 - 1 = \cos 2\Phi$, elicimus litteram

$r = \frac{1}{\pi}\int\partial\Phi\cos 2\Phi\,(1+2\cos\Phi)^n$, unde iterum fit

$r' = \frac{1}{\pi}\int\partial\Phi\cos 2\Phi\,(1+2\cos\Phi)^{n+1}$.

§. 62. Quoniam igitur supra invenimus $s = r' - r - q$, habebimus hic primo $r' - r = \frac{2}{\pi}\int\partial\Phi\cos\Phi\cos 2\Phi\,(1+2\cos\Phi)^n$. Hinc ergo si subtrahatur q, ob $2\cos\Phi\cdot\cos 2\Phi - \cos\Phi = \cos 3\Phi$, erit $s = \frac{1}{\pi}\int\partial\Phi\cos 3\Phi\,(1+2\cos\Phi)^n$. Simili modo jam evidens est fore $t = \frac{1}{\pi}\int\partial\Phi\cos 4\Phi\,(1+2\cos\Phi)^n$; eodemque modo reperietur fore $u = \frac{1}{\pi}\int\partial\Phi\cos 5\Phi\,(1+2\cos\Phi)^n$; atque adeo in genere erit $z = \frac{1}{\pi}\int\partial\Phi\cos\lambda\Phi\,(1+2\cos\Phi)^n$.

§. 63. Quoniam Analysis, qua hic usi sumus, prorsus est singularis et parum consueta, haud abs re erit veritatem harum formularum demonstratione analytica muniri, quam de singulis uno quasi labore sequenti modo instituere licebit. Inchoandum erit ab evolutione formulae $(1+2\cos\Phi)^n$, quae perducit ad hanc seriem:

$$1 + \left(\tfrac{n}{1}\right)2\cos\Phi + \left(\tfrac{n}{2}\right)4\cos\Phi^2 + \left(\tfrac{n}{3}\right)8\cos\Phi^3 + \left(\tfrac{n}{4}\right)16\cos\Phi^4 + \text{etc.}$$

Per notas autem angulorum reductiones constat fore

$$2\cos\Phi = 2\cos\Phi$$
$$4\cos\Phi^2 = 2\cos 2\Phi + 2$$
$$8\cos\Phi^3 = 2\cos 3\Phi + 6\cos\Phi$$
$$16\cos\Phi^4 = 2\cos 4\Phi + 8\cos 2\Phi + 6$$
$$32\cos\Phi^5 = 2\cos 5\Phi + 10\cos 3\Phi + 20\cos\Phi$$

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$$2^\alpha\cos\Phi^\alpha = 2\cos\alpha\Phi + 2\left(\tfrac{\alpha}{1}\right)\cos(\alpha-2)\Phi + 2\left(\tfrac{\alpha}{2}\right)\cos(\alpha-4)\Phi + 2\left(\tfrac{\alpha}{3}\right)\cos(\alpha-6)\Phi + \text{etc.}$$

ubi

ponamus:

ubi probe notandum est, quando terminus ultimus est absolutus, tum tantum simplum capi debere; praeterea vero etiam cosinus angulorum negativorum prorsus omitti debent.

§. 64. His igitur rite dispositis erit

$$(1+2\cos\Phi)^n = 1 + (\tfrac{n}{1})\,2\cos\Phi + (\tfrac{n}{2})\,2(\cos 2\Phi + 1) + 2(\tfrac{n}{3})(\cos 3\Phi + 3\cos\Phi)$$
$$+ 2(\tfrac{n}{4})(\cos 4\Phi + 4\cos 2\Phi + 3) + 2(\tfrac{n}{5})(\cos 5\Phi + 5\cos 3\Phi + 10\cos\Phi)$$
$$+ 2(\tfrac{n}{6})(\cos 6\Phi + 6\cos 4\Phi + 15\cos 2\Phi + 10) + \text{etc.}$$

unde sequentes integrationes sunt petendae.

§. 65. Incipiamus a prima litera p, ubi hanc seriem ducere in $\partial\Phi$ et integrari oportet. Cum igitur in genere sit $\int\partial\Phi\cos m\Phi = \frac{1}{m}\sin m\Phi$, iste valor jam evanescit posito $\Phi = 0$, pro altero integrationis termino $\Phi = \pi$ manifesto evanescit, si quidem omnes numeri n sunt integri. Ad integrationem igitur soli termini absoluti relinquuntur, tum vero integrali rite sumto erit $\int\partial\Phi = \pi$, quo observato erit nostrum integrale

$$\int\partial\Phi(1+2\cos\Phi)^n = \pi + 2(\tfrac{n}{2})\pi + 6(\tfrac{n}{4})\pi + 20(\tfrac{n}{6})\pi + \text{etc.}$$

Quod si hic forma generalis supra data consulatur, hi coëfficientes numerici revocentur ad formas $(\frac{2}{1})$, $(\frac{4}{2})$, $(\frac{6}{3})$, etc. prorsus uti veritas formulae postulat. Erit enim

$$p = \tfrac{1}{\pi}\int\partial\Phi(1+2\cos\Phi)^n = 1 + (\tfrac{2}{1})(\tfrac{n}{2}) + (\tfrac{4}{2})(\tfrac{n}{4}) + (\tfrac{6}{3})(\tfrac{n}{6}) + \text{etc.}$$

§. 66. Pergamus ad secundam litteram q, ubi superiorem seriem per $\partial\Phi\cos\Phi$ multiplicari et integrari oportet. Ad hoc observetur esse in genere

$$\int\partial\Phi\cos\Phi\cos m\Phi = \tfrac{1}{2(m+1)}\sin(m+1)\Phi + \tfrac{1}{2(m-1)}\sin(m-1)\Phi$$

quae expressio posito $\Phi = \pi$ in nihilum abit solo casu excepto quo $m = 1$, quippe quo fit $\int\partial\Phi\cos\Phi\cos\Phi = \frac{1}{2}\Phi = \frac{\pi}{2}$.

Ex

Ex quo intelligitur, ex superiori serie alios terminos hic non in computum venire, nisi qui contineant $\cos\Phi$, qui sunt:

$$2(\tfrac{n}{1})\cos\Phi + 2(\tfrac{3}{1})(\tfrac{n}{3})\cos\Phi + 2(\tfrac{5}{2})(\tfrac{n}{5})\cos\Phi + 2(\tfrac{7}{3})(\tfrac{n}{7})\cos\Phi + \text{etc.}$$

Hi autem termini ducti in $\partial\Phi\cos\Phi$ et integrati, ob $\int 2\partial\Phi\cos\Phi^2 = \pi$, dabunt, per π divisi, ipsum valorem

$$q = (\tfrac{n}{1}) + (\tfrac{3}{1})(\tfrac{n}{3}) + (\tfrac{5}{2})(\tfrac{n}{5}) + (\tfrac{7}{3})(\tfrac{n}{7}) + \text{etc.}$$

§. 67. Pro littera r superior series multiplicari debet per $\partial\Phi\cos 2\Phi$. Cum igitur in genere sit $\cos 2\Phi\cos m\Phi = \frac{1}{2}\cos(m+2)\Phi + \frac{1}{2}\cos(m-2)\Phi$, per $\partial\Phi$ multiplicando integrale pro termino $\Phi = \pi$ semper evanescit, excepto solo casu $m = 2$, quippe quo fit $\int\partial\Phi\cos 2\Phi^2 = \frac{\pi}{2}$. Hic igitur ex serie superiori soli termini per $\cos 2\Phi$ affecti in computum veniunt, qui sunt

$$2\cos 2\Phi\left((\tfrac{n}{2}) + (\tfrac{4}{1})(\tfrac{n}{4}) + (\tfrac{6}{2})(\tfrac{n}{6}) + (\tfrac{8}{3})(\tfrac{n}{8}) + \text{etc.}\right)$$

Quia igitur $\int 2\partial\Phi\cos 2\Phi^2 = \pi$, omnibus terminis colligendis et per π dividendo reperitur

$$r = (\tfrac{n}{2}) + (\tfrac{4}{1})(\tfrac{n}{4}) + (\tfrac{6}{2})(\tfrac{n}{6}) + (\tfrac{8}{3})(\tfrac{n}{8}) + \text{etc.}$$

§. 68. Quo haec clariora reddantur ac facilius ad valorem generalem z accommodari queant, evolutionem potestatis $(1+2\cos\Phi)^n$ statim secundum cosinus multiplorum anguli Φ disponamus hoc modo:

$$(1+2\cos\Phi)^n = 1 + (\tfrac{2}{1})(\tfrac{n}{2}) + (\tfrac{4}{2})(\tfrac{n}{4}) + (\tfrac{6}{3})(\tfrac{n}{6}) + (\tfrac{8}{4})(\tfrac{n}{8}) + \text{etc.}$$
$$+ 2\cos\Phi\left((\tfrac{n}{1}) + (\tfrac{3}{1})(\tfrac{n}{3}) + (\tfrac{5}{2})(\tfrac{n}{5}) + (\tfrac{7}{3})(\tfrac{n}{7}) + (\tfrac{9}{4})(\tfrac{n}{9}) + \text{etc.}\right)$$
$$+ 2\cos 2\Phi\left((\tfrac{n}{2}) + (\tfrac{4}{1})(\tfrac{n}{4}) + (\tfrac{6}{2})(\tfrac{n}{6}) + (\tfrac{8}{3})(\tfrac{n}{8}) + (\tfrac{10}{4})(\tfrac{n}{10}) + \text{etc.}\right)$$
$$+ 2\cos 3\Phi\left((\tfrac{n}{3}) + (\tfrac{5}{1})(\tfrac{n}{5}) + (\tfrac{7}{2})(\tfrac{n}{7}) + (\tfrac{9}{3})(\tfrac{n}{9}) + (\tfrac{11}{4})(\tfrac{n}{11}) + \text{etc.}\right)$$

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$$+ 2\cos\lambda\Phi\left((\tfrac{n}{\lambda}) + (\tfrac{\lambda+2}{1})(\tfrac{n}{\lambda+2}) + (\tfrac{\lambda+4}{2})(\tfrac{n}{\lambda+4}) + (\tfrac{\lambda+6}{3})(\tfrac{n}{\lambda+6}) + \text{etc.}\right)$$

N 2 §. 69.

...iam igitur pro nostris terminis integratio-

§. 69 Quod si jam hanc aequationem per $\partial\Phi \cos\lambda\Phi$ multiplicemus et integremus, omnia haec integralia, terminis praescriptis inclusa, evanescent, excepto membro $2\cos\lambda\Phi(\ldots)$, propterea quod productum $2\cos\lambda\Phi^2$ continet partem absolutam, unde per integrationem oritur π, ita ut sit $\int\partial\Phi\cos\lambda\Phi(1+2\cos\Phi)^n = \pi\left(\left(\frac{n}{\lambda}\right)+\left(\frac{\lambda+2}{1}\right)\left(\frac{n}{\lambda+2}\right)+\frac{\lambda+4}{2}\left(\frac{n}{\lambda+4}\right)+\text{etc.}\right)$ qui valor per π divisus ipsum valorem ipsius z supra inventum praebet: unde veritas harum novarum expressionum luculenter est demonstrata.

§. Ceterum si singulas series paragraphi penultimi vel leviter consideremus, deprehendimus eas ipsis litteris nostris p, q, r, s, etc. esse aequales, ita ut nunc sit
$(1+2\cos\Phi)^n = p + q\cos\Phi + 2r\cos 2\Phi + 2s\cos 3\Phi + 2t\cos 4\Phi + \text{etc.}$
ubi simul ratio est manifesta, cur litterae q, r, s, etc duplicentur, quippe quae in hoc est posita, quod in evolutione formulae $(1+x+xx)^n$ littera p semel tantum in medio, reliquae vero litterae bis, a medio aequidistantes, occurrunt. Ex quo haec egregia affinitas inter illas binas potestates $(1+x+xx)^n$ et $(1+2\cos\Phi)^n$, summa attentione digna est censenda.

Investigatio summae seriei

$$P = 1+x+3xx+7x^3+19x^4+ \cdots px^n + p'x^{n+1} + \text{etc.}$$

§. 71. Quoniam hujus seriei terminus generalis est px^n, quem sequuntur $p'x^{n+1}$ et $p''x^{n+2}$, inter has ternas quantitates p, p', p'' invenimus supra hanc relationem:

$$(n+2)p'' = (2n+3)p' + 3(n+1)p$$

quam hoc modo ad usum nostrum accommodatam referamus:

$$3(n+1)p + (n+1)p' + (n+2)p' - (n+2)p'' = 0.$$

§. 72.

§. 72. Cum jam series nostra sit

$$1+x+3xx+7x^3+19x^4+ \cdots + px^n + p'x^{n+1} + p''x^{n+2} + \text{etc.}$$

ejusmodi operationes instituamus, quibus relatio modo allata obtineatur, id quod sequenti modo commodissime fiet:

$$\frac{3\partial.Px}{\partial x} = 3+6x+27xx \cdots + 3(n+1)px^n + \text{etc.}$$
$$+\frac{\partial.P}{\partial x} = 1+6x+21xx \cdots + (n+1)p'x^n + \text{etc.}$$
$$+\frac{\partial.Px}{x\partial x} = \frac{1}{x}+2+9x+28xx \cdots + (n+2)p'x^n + \text{etc.}$$
$$-\frac{\partial P}{x\partial x} = -\frac{1}{x}-6-21x-76xx \cdots - (n+2)p''x^n + \text{etc.}$$

Colligantur jam hae quatuor series in unam summam atque obtinebimus sequentem aequationem:

$$\frac{3\partial.Px}{\partial x} + \frac{\partial P}{\partial x} + \frac{\partial.Px}{x\partial x} - \frac{\partial P}{x\partial x} = 0,$$

quandoquidem omnes termini se mutuo destruunt.

§. 73. Hoc ergo modo deducti sumus ad aequationem finitam differentialem primi gradus, quae per $x\partial x$ multiplicata et in ordinem redacta ita se habebit:

$$P\partial x(3x+1) + \partial P(3xx+2x-1) = 0,$$

unde ergo fit $\frac{\partial P}{P} = \frac{\partial x(1+3x)}{1-2x-3xx}$, quae aequatio integrata praebet $l\,P = -\frac{1}{2}l(1-2x-3xx)+lC$. consequenter $P = \frac{C}{\sqrt{(1-2x-3xx)}}$. Hic ad constantem C determinandam notetur tantum nostram seriem propositam casu $x=0$ praebere $P=1$, unde patet sumi debere $C=1$, ita ut fit summa seriei $P = \frac{1}{\sqrt{(1-2x-3xx)}}$.

§. 74. Praeter ergo expectationem pertigimus ad summam algebraicam, quae expressio etiam ita est comparata, ut in seriem conversa ipsam nostram seriem reproducat, id quod ostendisse operae erit pretium. Cum igitur sit

P =

$$\int \frac{\partial\Phi\cos 3\Phi}{1+bb-2b\cos\Phi} = \frac{\pi b^3}{1-bb}$$

$P = (1 - 2x - 3xx)^{-\frac{1}{2}}$, si hujus trinomii partes posteriores conjunctim spectentur, evolutio nobis dabit

$P = 1 + \frac{1}{2}(2x + 3xx) + \frac{1 \cdot 3}{2 \cdot 4}(2x + 3xx)^2 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}(2x + 3xx)^3 + \text{etc.}$

quam sufficiet ad potestatem tantum tertiam usque evolvisse. Hoc modo nanciscemur

$$P = 1 + x + \tfrac{3}{2}xx + \tfrac{9}{2}x^3 + \text{etc.} = 1 + x + 3xx + 7x^3 + \text{etc.}$$
$$+ \tfrac{3}{2}xx + \tfrac{5}{2}x^3$$

quae igitur perfecte congruit.

§. 75. At vero haec eadem summa adhuc alio modo investigari potest, ex formula scilicet integrali, quam pro valore litterae p invenimus:

$$p = \tfrac{1}{\pi}\int \partial\Phi (1 + 2\cos\Phi)^n \left[\begin{smallmatrix} a\ \Phi = 0 \\ ad\ \Phi = \pi \end{smallmatrix}\right].$$

Haec enim formula, sumto $n = 0$, per x multiplicata, dat terminum secundum x; casu porro $n = 2$, per xx multiplicata, dat terminum tertium $3xx$; quo observato summa quaesita ita poterit repraesentari:

$$P = \tfrac{1}{\pi}\int \partial\Phi\, (1 + x(1 + 2\cos\Phi) + xx(1 + 2\cos\Phi)^2$$
$$+ x^3(1 + 2\cos\Phi)^3 + \text{etc.})$$

ubi probe est observandum, in hac integratione quantitatem x tanquam constantem spectari, siquidem solus angulus Φ est variabilis.

§. 76. Evidens autem est, seriem infinitam, in quam elementum $\partial\Phi$ duci oportet, esse geometricam, cujus ergo summa erit $\frac{1}{1 - x(1 + 2\cos\Phi)} = \frac{1}{1 - x - 2x\cos\Phi}$, sicque adeo pro P jam habemus hanc expressionem finitam:

$$P = \tfrac{1}{\pi}\int \frac{\partial\Phi}{1 - x - 2x\cos\Phi} \left[\begin{smallmatrix} a\ \Phi = 0 \\ ad\ \Phi = \pi \end{smallmatrix}\right],$$

quae

aequatio ita poterit exhiberi:

$$P = \frac{1}{\pi(1-x)}\int \frac{\partial\Phi}{1 - \frac{2x}{1-x}\cos\Phi} \left[\begin{smallmatrix} a\ \Phi = 0 \\ ad\ \Phi = \pi \end{smallmatrix}\right],$$

ubi jam brevitatis gratia statuamus $\frac{2x}{1-x} = k$, ut habeamus

$$P = \frac{1}{\pi(1-x)}\int \frac{\partial\Phi}{1 - k\cos\Phi}.$$

§ 77. Constat autem hujus formulae $\frac{\partial\Phi}{1 - n\cos\Phi}$ integrale esse $\frac{1}{\sqrt{(1-nn)}}\, A\cos\frac{\cos\Phi - n}{1 - n\cos\Phi}$; unde si loco n scribamus $-k$, adipiscimur pro nostro casu $P = \frac{1}{\pi(1-x)\sqrt{(1-kk)}}\, A\cos\frac{\cos\Phi + k}{1 - k\cos\Phi}$, ubi constantis additione non est opus, quia haec expressio casu $\Phi = 0$ sponte evanescit. Faciamus igitur pro altero termino $\Phi = \pi$, unde fit $\cos\Phi = -1$ et $A\cos\frac{\cos\Phi - k}{1 - k\cos\Phi} = A\cos -1 = \pi$, ergo habebimus $P = \frac{1}{(1-x)\sqrt{(1-kk)}}$, quae expressio ob $k = \frac{2x}{1-x}$, abit in hanc: $P = \frac{1}{\sqrt{(1 - 2x - 3xx)}}$, prorsus ut ante.

§. 78 Cum sit $1 - 2x - 3xx = (1 - x)^2 - 4xx = (1 + x)(1 - 3x)$, sequitur seriem nostram summandam duobus casibus fieri infinite magnam, scilicet altero casu quo $x = -1$, altero vero quo $x = \frac{1}{3}$. Tum vero nostra series habebit summam finitam, quando x continetur intra hos limites: -1 et $\frac{1}{3}$; sin autem x extra hos limites accipiatur, tum summa semper erit imaginaria. Ita sumto $x = \frac{1}{4}$, habebitur haec summatio:

$$1 + \tfrac{1}{4} + \tfrac{3}{4^2} + \tfrac{7}{4^3} + \tfrac{19}{4^4} + \tfrac{51}{4^5} + \tfrac{141}{4^6} + \text{etc.} = \tfrac{4}{\sqrt{5}}.$$

Investigatio summae reliquarum serierum Q, R, S, etc. supra §. 6. expositarum.

§. 79. Incipiamus a serie Q, quae est

$$Q = xx + 2x^3 + 6x^4 + \ldots + qx^{n+1} + q'x^{n+2} + \text{etc.}$$

cujus

cujus primus terminus xx ex potestate $n = 1$ oritur; ubi perinde ac si seriem ab $n = 0$ inchoare velimus, praefigi debet terminus $0x$. Pro hac autem serie ostendimus supra esse $q = \frac{1}{2}(p' - p)$, unde hujus seriei summa ex serie prima p sequenti modo elici poterit.

§ 80. Cum sit $P = 1 + x + 3xx + \ldots + px^n + p'x^{n+1} + \text{etc.}$
erit $Px = \ldots\ldots\ x + xx + \ldots + px^{n+1} + \text{etc.}$
quae posterior series a priori subtracta relinquit
$P(1-x) = 1 + 2xx + \ldots + (p' - p)x^{n+1} + \text{etc.}$
Quare cum sit $p' - p = 2q$, erit
$P(1-x) = 1 + 2Q$; sicque innotescit hujus seriei summa, cum sit $Q = \frac{P(1-x) - 1}{2}$. Modo ante autem vidimus esse $P = \frac{1}{\sqrt{(1-2x-3xx)}}$
sicque habebimus $Q = \frac{1 - x - \sqrt{(1 - 2x - 3xx)}}{2\sqrt{(1 - 2x - 3xx)}}$.

§. 81. Procedamus ad seriem R, quae ita se habebat:
$R = x^4 + 3x^5 + 10x^6 + \ldots + rx^{n+2} + r'x^{n+3} + \text{etc.}$
cujus primus terminus x^4 ex potestate $n = 2$ est ortus, unde praefixi concipiendi sunt bini termini $0x^2 + 0x^3$, ad cujus summam inveniendam notetur esse $r = q' - q - p$. Hinc si sequentes operationes instituantur:

$Q = xx + 2x^3 + \ldots + qx^{n+1} + q'x^{n+2} + \text{etc.}$
$-Qx = \quad - x^3 \ldots\ldots - qx^{n+2} - \text{etc.}$
$-Px^2 = -xx - x^3 \ldots\ldots - px^{n+2} - \text{etc.}$

inde conjunctim fiet
$$Q(1-x) - Pxx = (q' - q - p)x^{n+2} = R.$$

§. 82. Hoc igitur modo summam R determinavimus per binas series praecedentes Q et P, quae cum jam sint cognitae, etiam summam seriei R algebraice per certam formationem ipsius x expressam sumus adepti, quae quomodo commode evolvi queat, deinceps ostendemus.

§. 83.

§. 83. Pro serie S, quae ita erat proposita:
$S = x^6 + 4x^7 + 15x^8 + \ldots + sx^{n+3} + s'x^{n+4} + \text{etc.}$
ei tres termini evanescentes praefigi sunt censendi, scilicet $0x^3 + 0x^4 + 0x^5$, siquidem a potestate $n = 0$ incipere velimus. Supra autem invenimus esse $s = r' - r - q$, unde sequentes operationes instituamus:

$R = x^4 + 3x^5 + 10x^6 + \ldots\ldots + rx^{n+2} + r'x^{n+3} + \text{etc.}$
$-Rx = \quad - x^5 - 3x^6 - \ldots\ldots - rx^{n+3} - \text{etc.}$
$-Qxx = -x^4 - 2x^5 - 6x^6 - \ldots\ldots - qx^{n+3} - \text{etc.}$

quibus tribus seriebus collectis oritur haec series:
$x^6 + \ldots\ldots + sx^{n+3} + \text{etc.}$
quae est ipsa series S. Quocirca summa hujus seriei per binas praecedentes Q et R ita determinatur ut sit $S = R(1-x) - Qxx$, cujus evolutio etiam satis simpliciter expediri poterit, uti mox ostendetur.

§. 84. Eodem modo series T per binas praecedentes R et S definietur hoc modo:

$S = x^6 + 4x^7 + 15x^8 \ldots\ldots + sx^{n+3} + s'x^{n+4} + \text{etc.}$
$-Sx = \quad - x^7 - 4x^8 \ldots\ldots - sx^{n+4} - \text{etc.}$
$-Rxx = -x^6 - 3x^7 - 10x^8 \ldots\ldots - rx^{n+4} - \text{etc.}$

Cum igitur $s' - s - r = t$, hae tres series collectae dabunt
$S(1-x) - Rxx = x^8 + \ldots + tx^{n+4} + \text{etc.}$
quae cum sit ipsa series T, erit $T = S(1-x) - Rxx$.

§. 85. Hinc igitur manifestum est singulas harum serierum satis simpliciter per binas praecedentes determinari posse, atque adeo per legem penitus uniformem. Eas conjunctim ob oculos pona-

ponamus:

$$Q = \frac{P(1 - x - \sqrt{1-2x-3xx})}{2}$$
$$R = Q(1-x) - Pxx$$
$$S = R(1-x) - Qxx$$
$$T = S(1-x) - Rxx$$
$$U = T(1-x) - Sxx$$
etc.

unde patet omnnes has ſummas ſecundum ſeriem recurrentem procedere, cujus ſcala relationis eſt $(1-x), -xx$. Verum mox patebit, hanc ſeriem adeo esse geometricam.

§. 86. Ad hoc ostendendum, cum facta evolutione ſit $\frac{Q}{P} = \frac{1-x-\sqrt{1-2x-3xx}}{2}$, vocemus brevitatis gratia $\frac{1-x-\sqrt{1-2x-3xx}}{2} = v$, ut habeamus $Q = Pv$; inde autem ſublata irrationalitate, cum ſit $\sqrt{1-2x-3xx} = 1-x-2v$, orietur haec aequatio: $(1-x)^2 - 4xx = (1-x)^2 - 4v(1-x) - 4vv$, quae reducitur ad iſtam: $v(1-x) - xx = vv$, quod probe notasse juvabit.

§. 87. Jam pro ſerie R, si loco Q hunc valorem Pv ſubſtituamus, orietur haec aequatio:
$R = P(v(1-x) - xx$, ideoque per relationem modo notatam $R = Pvv$. Si jam porro loco Q et R valores inventos ſcribamus, nanciscemur ſimili modo:

$$S = Pv(v(1-x) - xx) = Pv^3$$
$$T = Pvv(v(1-x) - xx) = Pv^4$$
$$U = Pv^3(v(1-x) - xx) = Pv^5$$
. . . .
. . . .
$$Z = Pv^{\lambda-1}(v(1-x) - xx) = Pv^\lambda.$$

§. 88.

§. 88. Quod ſi jam has determinationes ad formulas integrales, quas pro litteris p, q, r, etc. invenimus, transferamus, quoniam invenimus $z = \frac{1}{\pi}\int \partial\Phi \cos\lambda\Phi (1 + 2\cos\Phi)^n$, si exponenti n ſucceſsive valores tribuamus $0, 1, 2, 3, 4$ etc. quia ſeries Z a poteſtate x^λ inchoare eſt cenſenda, formula differentialis $\partial\Phi \cos\lambda\Phi$ per hanc ſeriem geometricam multiplicari debebit:

$$(1+2\cos\Phi)^0 x^\lambda + (1+2\cos\Phi)^1 x^{\lambda+1} + (1+2\cos\Phi)^2 x^{\lambda+2} + \text{etc.}$$

cujus ſumma eſt $\frac{x^\lambda}{1 - x - 2x\cos\Phi}$, qua ergo in calculum introducta ſumma quaeſita Z ita exprimetur:

$$Z = \frac{1}{\pi}\int \frac{x^\lambda \partial\Phi \cos\lambda\Phi}{1 - x - 2x\cos\Phi} \begin{bmatrix} a\ \Phi = 0 \\ ad\ \Phi = \pi \end{bmatrix},$$ ubi quantitas x eſt conſtans.

§. 89. Quoniam igitur hic invenimus iſtam ſummam ſcil. $Z = Pv^\lambda = \frac{v^\lambda}{\sqrt{(1-2x-3xx)}}$, existente
$v = \frac{1-x-\sqrt{(1-2x-3xx)}}{2}$,
nunc hujus ipſius formulae integralis valorem adeo algebraicum exhibere poterimus, quandoquidem nunc novimus eſſe.

$$\frac{1}{\pi}\int \frac{x^\lambda \partial\Phi \cos\lambda\Phi}{1 - x - 2x\cos\Phi} = \frac{v^\lambda}{\sqrt{(1-2x-3xx)}}$$

ſive multiplicando per $\frac{\pi}{x^\lambda}$ habebimus

$$\int \frac{\partial\Phi \cos\lambda\Phi}{1 - x - 2x\cos\Phi} = \frac{\pi}{\sqrt{(1-2x-3xx)}}\left(\frac{v}{x}\right)^\lambda.$$

§. 90.

§. 90. Quoniam haec integratio majori attentione digna videtur, eam in commodiorem formam transfundamus, et quoniam x et v hic ut constantes spectantur, ponamus $\frac{v}{x} = b$, atque ob $v = \frac{1 - x - \sqrt{1 - 2x - 3xx}}{2}$, erit $2bx = 1 - x - \sqrt{1 - 2x - 3xx}$, quae aequatio, sublata irrationalitate, praebet: $4bbxx - 4bx(1-x) + (1-x)^2 = (1-x)^2 - 4xx$, quae reducitur ad hanc: $bbx - b + bx = + x$, unde ipsa quantitas x satis commode determinatur, cum fiat $x = \frac{b}{bb + b - 1}$, ideoque $1 - x = \frac{bb - 1}{bb + b - 1}$, hincque porro cum esset

$$\sqrt{1 - 2x - 3xx} = 1 - x - 2bx, \text{ erit nunc}$$

$$\sqrt{1 - 2x - 3xx} = \frac{1 - bb}{1 + b + bb}.$$

§. 91. Quod si ergo loco quantitatis x litteram b in nostrum calculum introducamus, integratio inventa ad hanc formam reducetur simpliciorem:

$$\int \frac{\partial\Phi \cos\lambda\Phi}{1 - 2b\cos\Phi + bb} \begin{bmatrix} \text{a } \Phi = 0 \\ \text{ad } \Phi = \pi \end{bmatrix} = \frac{\pi b^\lambda}{1 - bb},$$

cujus veritas ex calculis hactenus expeditis est deducta; verum etiam immediate et directe demonstrari potest, quo ipso praecedentia omnia eo magis corroborabuntur.

§. 92. Ad hoc igitur demonstrandum in subsidium vocemus satis notam integrationem, qua est

$$\int \frac{\partial\Phi}{\alpha + \beta\cos\Phi} = \frac{1}{\sqrt{(\alpha\alpha - \beta\beta)}} \text{ A cos } \frac{\alpha\cos\Phi + \beta}{\alpha + \beta\cos\Phi}.$$

Fiat nunc $\alpha = 1 + bb$ et $\beta = -2b$ et habebimus

$$\int \frac{\partial\Phi}{1 - 2b\cos\Phi + bb} = \frac{1}{1 - bb} \text{ A cos } \frac{(1 + bb)\cos\Phi - 2b}{1 - 2b\cos\Phi + bb},$$

quod integrale jam evanescit posito $\Phi = 0$. Posito ergo pro altero termino $\Phi = \pi$, hoc integrale evadit $\frac{\pi}{1 - bb}$.

§. 93.

§. 93. Quoniam igitur pro nostris terminis integrationis invenimus $\int \frac{\partial\Phi}{1 - 2b\cos\Phi + bb} = \frac{\pi}{1 - bb}$, atque manifesto est $\int \partial\Phi = \pi$, ideoque

$$\int \frac{\partial\Phi(1 - 2b\cos\Phi + bb)}{1 - 2b\cos\Phi + bb} = \pi,$$

hanc formam in duas partes distribuendo habebimus

$$\pi = (1 + bb)\int \frac{\partial\Phi}{1 - 2b\cos\Phi + bb} - 2b\int \frac{\partial\Phi \cos\Phi}{1 - 2b\cos\Phi + bb},$$

unde colligimus

$$\int \frac{\partial\Phi \cos\Phi}{1 - 2b\cos\Phi + bb} = \frac{\pi b}{1 - bb}.$$

§. 94. Quoniam pro nostris terminis integrationis in genere est $\int \partial\Phi \cos i\Phi = 0$, siquidem i fuerit numerus integer, multiplicemus hanc formulam supra et infra per $1 + bb - 2b\cos\Phi$, atque obtinebimus

$$\int \frac{\partial\Phi((1 + bb)\cos i\Phi) - b\cos(i - 1)\Phi - b\cos(i + 1)\Phi}{1 + bb - 2b\cos\Phi} = 0.$$

Haec forma jam in tres partes secta nobis dabit

$$(1 + bb)\int \frac{\partial\Phi \cos i\Phi}{1 - 2b\cos\Phi + bb} = b\int \frac{\partial\Phi \cos(i - 1)\Phi}{1 - 2b\cos\Phi + bb} + b\int \frac{\partial\Phi \cos(i + 1)\Phi}{1 - 2b\cos\Phi + bb},$$

unde derivamus hanc reductionem generalem:

$$\int \frac{\partial\Phi \cos(i + 1)\Phi}{1 - 2b\cos\Phi + bb} = \frac{1 + bb}{b}\int \frac{\partial\Phi \cos i\Phi}{1 - 2b\cos\Phi + bb} - \int \frac{\partial\Phi \cos(i - 1)\Phi}{1 - 2b\cos\Phi + bb},$$

cujus ope ex integralibus binis pro angulis $i\Phi$ et $(i - 1)\Phi$ integrale pro angulo $(i + 1)\Phi$ determinari potest, unde sequentem tabulam conficere licebit:

$$\int \frac{\partial\Phi}{1 + bb - 2b\cos\Phi} = \frac{\pi}{1 - bb}$$

$$\int \frac{\partial\Phi \cos\Phi}{1 + bb - 2b\cos\Phi} = \frac{\pi b}{1 - bb}$$

$$\int \frac{\partial\Phi \cos 2\Phi}{1 + bb - 2b\cos\Phi} = \frac{\pi bb}{1 - bb}$$

$\int \partial\Phi$

$$\int \frac{\partial \Phi \cos 3\Phi}{1 + bb - 2b \cos \Phi} = \frac{\pi b^3}{1 - bb}$$

$$\int \frac{\partial \Phi \cos 4\Phi}{1 + bb - 2b \cos \Phi} = \frac{\pi b^4}{1 - bb}$$

. . . .

. . . .

$$\int \frac{\partial \Phi \cos \lambda\Phi}{1 + bb - 2b \cos \Phi} = \frac{\pi b^\lambda}{1 - bb}$$

prorsus uti fupra invenimus.

DE

DE INNUMERIS CURVIS ALGEBRAICIS

QUARUM LONGITUDINEM PER ARCUS HYPERBOLICOS METIRI LICET.

AUCTORE

NICOLAO FUSS.

Conventui exhibita die 28. Jun. 1798.

§. 1.

In Tomo quinto novorum Actorum Academiae, pro anno 1787, duae reperiuntur dissertationes viri immortalis Leonardi Euleri, fuper argumento plane novo et a nemine antea tractato; prior inscripta eft: *De innumeris curvis algebraicis, quarum longitudinem per arcus parabolicos metiri licet*; altera: *De innumeris curvis algebraicis, quarum longitudinem per arcus ellipticos metiri licet*. In calce pofterioris dissertationis auctor d clarat, fe nullo adhuc modo vel unicam faltem curvam algebraicam eruere potuisse, cujus fingulos arcus per arcus hyperbolicos metiri liceret. „Si talis quaestio" inquit Eulerus loco citato „circa arcus hyperbolicos proponatur, fa„teri cogor, nullo adhuc modo me vel unicam faltem cur„vam algebraicam eruere potuisse, cujus finguli arcus per „formulam $\int \frac{\partial v}{vv} \sqrt{1 + v^4}$ — quae arcum fiftit Hyperbolae ae„quilaterae — exprimerentur. Sin autem aequationem ge„neralem pro Hyperbola assumere velimus, pro qua eft $y = n\sqrt{1 + vv}$, elementum arcus inde nascitur $\partial S = \frac{\partial v \sqrt{1 + (nn + 1) vv}}{\sqrt{1 + vv}}$,

„quae