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Solutio problematis mechanici

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SOLUTIO PROBLEMATIS MECHANICI

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§. 1.

Tab. I. Fig. I. Referat circulus XTGF, cujus centrum in C, globum sive cylindrum, aliudve corpus rotundum, plano inclinato AH incumbens, cujus inclinatio ad horizontem sit angulus $AHD = \zeta$; circuli autem radius ponatur $CX = a$. Corporis, hoc circulo repraesentati, axis maneat perpetuo horizontalis, ejusque massa statuatur $= M$, qua simul ejus pondus exprimatur. Momentum vero inertiae respectu axis sit $= Mkk$.

§. 2. Huic circulo, seu corpori rotundo, circumvolutum intelligatur filum, cujus terminus extremus fixus sit in puncto E, a plano inclinato distante intervallo $AE = a$, hoc est radio circuli aequali, cujus situs initio tangebat punctum E, filo scilicet penitus involuto. Hinc autem elapso tempore t pervenerit in situm figurâ repraesentatum, quem jam filum deserat in puncto T, ejusque portio ulterior circulo sit circumvoluta, indefinitae longitudinis, ut, quamdiu corpus descendat, evolvi queat.

§. 3.

§. 3. Nunc igitur circulus tangat planum inclinatum AH in puncto X, ac ponatur ſpatium $AX = x$, ita ut, ſumpto intervallo $AB = a$, ſit $BX = x - a$, eritque BX ſpatium motu corporis progreſſivo deſcriptum. Tum igitur, ducta recta EC, ea erit plano inclinato AH parallela, ideoque aequalis ipſi x; unde ducto radio CT erit anguli ECT cosinus $= \frac{a}{x}$, at anguli XCT ſive CET ſinus $= \frac{a}{x}$. Ponatur autem iſte angulus $XCT = CET = \vartheta$, ita ut ſit $\sin.\vartheta = \frac{a}{x}$.

§. 4. Hinc jam longitudo fili evoluti colligitur fore $ET = x \cos.\vartheta$. cui ſi aequalis ſtatuatur arcus TGF, erit F punctum circuli, quod initio applicatum erat puncto E, ubi ergo radius CF erat plano inclinato AH parallelus ſive cadebat in EG. Statuatur angulus motu rotatorio jam confectus $GCF = \Phi$, erit angulus $TCF = 90° - \vartheta + \Phi$, ex quo ipſe arcus $TGF = a(90° - \vartheta + \Phi)$, qui cum aequalis ſit rectae $TE = a \cot.\vartheta$, hinc concluditur angulus $\Phi = \cot.\vartheta - 90° + \vartheta$. Unde patet, tam ſpatium x quam angulum Φ per angulum ϑ definiri poſſe, cum ſit $X = \frac{a}{\sin.\vartheta}$ et $\Phi = \cot.\vartheta - 90° + \vartheta$.

§. 5. Ut jam in motum corporis noſtri ſuper plano inclinato deſcendentis inquiramus, obſervandum eſt, corpus primo ſollicitari a propria gravitate $= M$; unde oritur vis motum progreſſivum accelerans $= M \sin.\zeta$. Deinde vero corpus etiam urgetur a tenſione fili TE, quae autem tenſio etiamnunc penitus eſt incognita. Ponatur ea $= T$, et cum corpus urgeat ſecundum directionem TE, ea producit vim directioni motus progreſſivi contrariam ſecundum CE ſollicitantem $= T \cos.\vartheta$, ita ut motus progreſſivus centri C, ubi ſimul centrum gravitatis totius corporis existere assumimus, vis fit $= M \sin.\zeta - T \cos.\vartheta$.

§. 6. Ex primis igitur motus principiis, sumendo elementum temporis dt constans, ac denotante g altitudinem lapsus uno minuto secundo peracti, sequitur fore $\frac{\partial\partial x}{2g\partial t^2} = \frac{M \sin.\zeta - T \cos.\vartheta}{M}$. Hac scilicet aequatione motus corporis progressivus determinatur, unde autem nihil adhuc concludere licet, propterea quod vis T est incognita. Hanc ergo aequationem combinari oportet cum ea, qua motus corporis gyratorius determinatur.

§. 7. Jam observavimus, corpus motu gyratorio jam absolvisse angulum $GCF = \Phi$, in sensum GF, qui ergo motus unice producitur a tensione fili $ET = T$, cujus momentum, respectu axis gyrationis, est Ta, quod divisum per momentum inertiae Mkk dabit accelerationem motus gyratorii, quae ex principiis motus est $\frac{\partial\partial\Phi}{2g\partial t^2}$. Habebimus ergo hanc aequationem: $\frac{\partial\partial\Phi}{2g\partial t^2} = \frac{Ta}{Mkk}$, ex qua colligitur ipsa vis incognita, seu tensio $T = \frac{Mkk\partial\partial\Phi}{2ag\partial t^2}$.

§. 8. Substituatur igitur iste valor in aequatione ante inventa et facta divisione per massam M prodibit ista aequatio: $\frac{\partial\partial x}{2g\partial t^2} = \sin.\zeta - \frac{kk\partial\partial\Phi \cos.\vartheta}{2ag\partial t^2}$, unde fit $\sin.\zeta = \frac{kk\partial\partial\Phi \cos.\vartheta + a\partial\partial x}{2ag\partial t^2}$, atque in hujus aequationis integratione erit elaborandum. Verum si ambas incognitas x et Φ per angulum ϑ exprimere velimus, ope formularum $x = \frac{a}{\sin.\vartheta}$ et $\Phi = \cot.\vartheta - 90^0 + \vartheta$, ob differentialia secundi gradus delaberemur in aequationem tantopere complicatam, ut nihil prorsus inde concludi posset.

§. 9.

§. 9. Singulari autem artificio haec insignis difficultas superari potest, cujus ope potius ipse angulus ϑ ex calculo extrudi poterit, ita ut, etiamsi binae variabiles x et Φ in ea relinquantur, totum tamen negotium facile confici queat. Cum enim sit $\partial x = -\frac{a\partial\vartheta\, cos.\vartheta}{sin.\vartheta^2}$ et $\partial\Phi = -\frac{\partial\vartheta}{sin.\vartheta^2} + \partial\vartheta = -\frac{\partial\vartheta\, cos\,\vartheta^2}{sin.\vartheta^2}$, colligitur $\frac{\partial\Phi}{\partial x} = \frac{cos.\vartheta}{a}$, sive $cos.\vartheta = \frac{a\partial\Phi}{\partial x}$.

§. 10. In aequatione ergo differentio-differentiali ante inventa loco $cos.\vartheta$ iste valor $\frac{a\partial\Phi}{\partial x}$ substitutus hanc producit aequationem integrationem admittens: $\partial x\, sin.\zeta = \frac{kk\partial\Phi\partial\partial\Phi + \partial x\partial\partial x}{2g\partial t^2}$, cujus integrale est $C + x\, sin.\zeta = \frac{\partial x^2 + kk\partial\Phi^2}{4g\partial t^2}$. Quoniam autem motus initium assumsimus ubi $x = a$, hoc integrale, habita constantis ratione, erit $(x-a)\, sin.\zeta = \frac{\partial x^2 + kk\partial\Phi^2}{4g\partial t^2}$.

§. 11. Ex hac jam aequatione ipsum tempus elapsum t determinari poterit. Cum enim sit

$$4g(x-a)\partial t^2\, sin.\zeta = \partial x^2 + kk\partial\Phi^2, \text{ ob } x = \frac{a}{sin.\vartheta},$$

$$\partial x = -\frac{a\partial\vartheta\, cos.\vartheta}{sin.\vartheta^2} \text{ et } \partial\Phi = -\frac{\partial\vartheta\, cos.\vartheta^2}{sin.\vartheta^2},$$

his valoribus substitutis habebitur ista aequatio:

$$\frac{4ag\partial t^2\, sin.\zeta\,(1-sin.\vartheta)}{sin.\vartheta} = \frac{\partial\vartheta^2\, cos.\vartheta^2}{sin.\vartheta^2}(aa + kk\, cos.\vartheta^2), \text{ unde fit}$$

$$\partial t = \frac{\partial\vartheta\, cos.\vartheta}{2\, sin.\vartheta}\sqrt{\frac{aa + kk\, cos.\vartheta^2}{ag\, sin.\zeta\, sin.\vartheta(1-sin.\vartheta)}}, \text{ unde colligitur}$$

$$t = -\int\frac{\partial\vartheta\, cos.\vartheta}{2\, sin.\vartheta}\sqrt{\frac{aa + kk\, cos.\vartheta^2}{ag\, sin.\zeta\, sin.\vartheta(1-sin.\vartheta)}}.$$

Hic scilicet signum negativum praefigimus, quoniam angulus ϑ continuo decrescit, qui angulus initio, ubi $t = 0$, erat $\vartheta = 90°$.

§. 12. Hinc igitur patet, tempus t pro quovis angulo ϑ definiri posse per formulam quidem integralem, quam autem

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neque per logarithmos neque per arcus circulares evolvere licet. Quo haec expreſſio aliquanto ſimplicior evadat, ponatur $\sin.\vartheta = s$, eritque tempus

$$t = \frac{-1}{2\sqrt{ag\sin.\zeta}}\int\frac{\partial s}{s}\sqrt{\frac{aa+kk(1-ss)}{s(1-s)}},$$

ac ſi ponatur $s = \frac{1}{z}$ erit

$$t = \frac{+1}{2\sqrt{ag\sin.\zeta}}\int\frac{\partial z}{z}\sqrt{\frac{(aa+kk)zz-kk}{z-1}}.$$

Viciſſim ergo, conceſſis quadraturis, ad quodvis tempus t angulus ϑ reperiri poterit, quo invento etiam ſpatium x cum angulo Φ habebitur.

§. 13. Longe autem difficilius eſt tenſionem fili ET per expreſſionem finitam aſſignare, quippe quae hactenus fuerat incognita atque hanc ob cauſſam ex calculo expulſa. Erat autem ex motu gyratorio $\frac{Ta}{Mkk} = \frac{\partial\partial\Phi}{2g\partial t^2}$, unde neceſſe eſt differentiale ſecundum $\partial\partial\Phi$ ad differentiale primi gradus revocare, quod praeſtari poterit ope poſtremae aequationis differentio-differentialis, unde integrale hauſimus, quaeque erat $\partial x\partial\partial x + kk\partial\Phi\partial\partial\Phi = 2g\partial t^2\partial x\sin.\zeta$. Fuerat autem $\partial x\cos\vartheta = a\partial\Phi$, hinc differentiando colligitur $a\partial\partial\Phi = \partial\partial x\cos.\vartheta - \partial\vartheta\partial x\sin.\vartheta$, ex qua aequatione oritur $\partial\partial x = \frac{a\partial\partial\Phi}{\cos.\vartheta} + \partial x\partial\vartheta\,tag\,\vartheta$, qui valor in illa ſubſtitutus praebet hanc: $\partial\partial\Phi\left(\frac{a}{\cos.\vartheta} + \frac{kk\cos.\vartheta}{a}\right) = 2g\partial t^2\sin.\zeta - \partial x\partial\vartheta\,tag.\vartheta$. Sicque nunc differentio-differentiale $\partial\partial\Phi$ per differentialia primi gradus definitur. Atque ſi loco $2g\partial t^2$ et ∂x valores ſupra per $\partial\vartheta$ expreſſi ſubſtituantur, orietur

$$\frac{\partial\partial\Phi(aa+kk\cos.\vartheta^2)}{a\cos.\vartheta} = \partial\vartheta^2\left(\frac{aa(1+\sin.\vartheta^2-2\sin.\vartheta^3)+kk\cos.\vartheta^4}{2a\sin.\vartheta^3(1-\sin.\vartheta)}\right)$$

§. 14.

§. 14. Quod ſi ergo hinc valor pro $\partial\partial\Phi$ in formula pro tenſione fili T inventa ſubſtituatur, prodibit

$$T = \frac{Mkk}{2ag\partial t^2} + \frac{\partial\vartheta^2 \cos.\vartheta\,(aa(1+\sin.\vartheta^2-2\sin.\vartheta^3)+kk\cos.\vartheta^4)}{2\sin.\vartheta^3\,(1-\sin.\vartheta)(aa+kk\cos.\vartheta^2)}$$

Est vero $2ag\partial t^2 = \frac{\partial\vartheta^2 \cos.\vartheta^2\,(aa+kk\cos.\vartheta^2)}{2\sin.\vartheta^2\ \sin.\zeta\ \sin.\vartheta'(1-\sin.\vartheta)^2}$, unde fit tenſio

$$T = \frac{Mkk\sin.\zeta\,(aa(1+\sin.\vartheta^2-2\sin.\vartheta^3)+kk\cos.\vartheta^4}{\cos.\vartheta\,(aa+kk\cos.\vartheta^2)^2}$$

quae expreſſio quia ſolum angulum ϑ involvit, ad quodvis tempus elapſum t tenſio T abſolute aſſignari poterit et cum pondere M comparari.

§. 15. Denique etiam tam celeritas corporis progreſſiva, quae eſt $\frac{\partial x}{\partial t}$, quam gyratoria $\frac{\partial\Phi}{\partial t}$, ſatis eleganter definiri poteſt. Si enim ponamus $\frac{\partial x}{\partial t} = v$ et $\frac{\partial\Phi}{\partial t} = u$, erit aequatio $vv + kkuu = 4g(x-a)\sin\zeta$. Unde ſimul patet principium virium vivarum hic egregie conſervari. Cum enim ſit Mvv vis viva motus progreſſivi et $Mkkuu$ vis viva motus gyratorii, erit ſumma utriusque $M(vv+kkuu) = 4gM(x-a)\sin\zeta$, cujus aequationis membrum poſtremum exprimit deſcenſum verticalem centri gravitatis corporis in maſſam M ductum, cui conſtat ſemper aequalem eſſe debere totam vim vivam hoc deſcenſu generatam.

§. 16. Quin etiam ex ſolo principio conſervationis virium vivarum iſta aequatio facile deduci potuiſſet. Neque vero patet quomodo hinc ipſa tenſio fili, qua inſigne momentum perfectae ſolutionis eſt conſtituendum, derivari potuiſſet. Quam ob cauſſam operam dedi, ut univerſam hujus Problematis ſolutionem ex primis motus principiis repeterem.

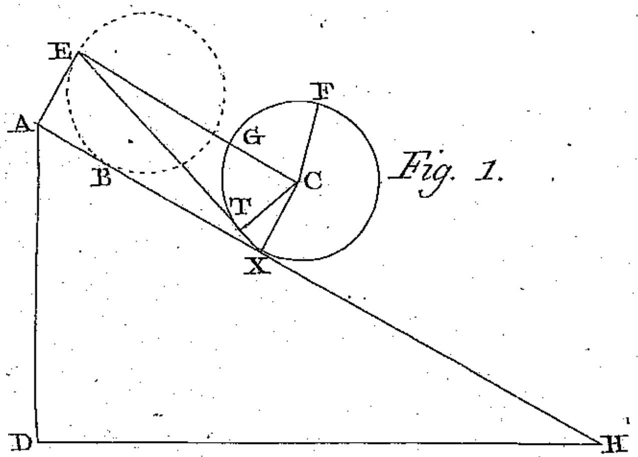


Fig. 1.

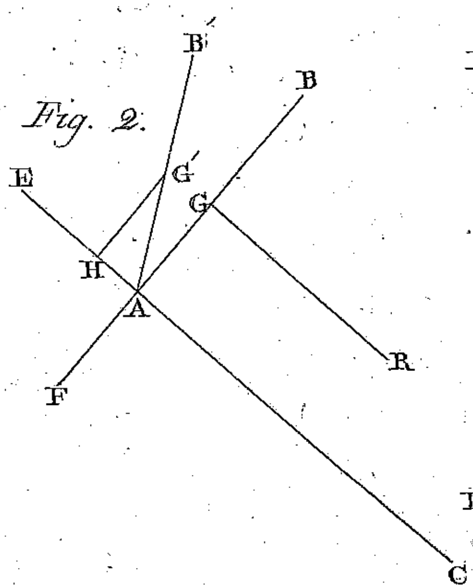


Fig. 2.

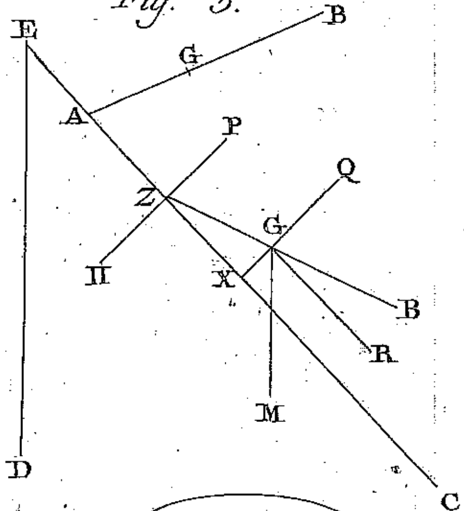
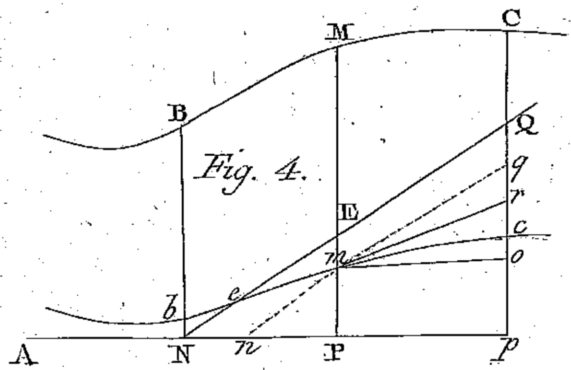


Fig. 3.



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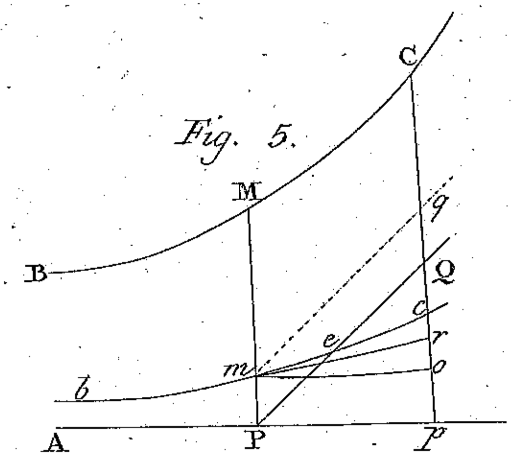
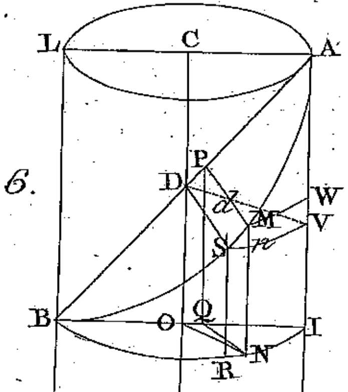


Fig. 5.



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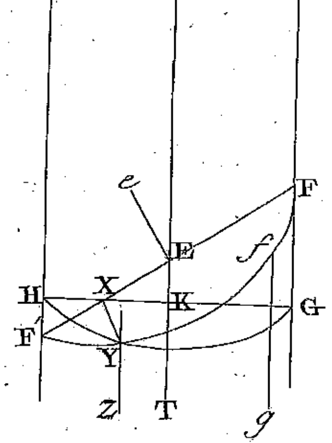

Fig. 7.