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Exempla quarundam memorabilium aequationum differentialium, quas adeo algebraice integrare licet, etiamsi nulla via pateat variabiles a se invicem separandi

Leonhard Euler

EXEMPLA
QUARUNDAM
MEMORABILIUM AEQUATIONUM DIFFERENTIALIUM,
quas adeo algebraice integrare licet, etiamſi nulla via pateat variabiles a ſe invicem ſeparandi.

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§. 1.

Facile quidem eſt hujusmodi aequationes, quotquot lubuerit, exhibere, quarum integralia aſſignari queant. Si enim pro V accipiatur quaecunque functio binarum variabilium x et y, ita vt ſit $\partial V = M\partial x + N\partial y$, evidens eſt huic aequationi differentiali $\partial x\,(PV + MS) + \partial y\,(QV + NS) = 0$ ſemper ſatisfacere aequationem finitam $V = 0$. Verum hoc integrale tantum eſt particulare. Praeterea vero ſi ejusmodi aequatio proponatur, plerumque haud difficulter iſta functio V vel divinando inveniri poteſt, ita vt hujusmodi aequationes parum in receſſu habere ſunt cenſendae. Hic autem tales aequationes in medium ſum allaturus, quarum integratio omnes methodos adhuc cognitas reſpuere videatur, cum tamen nihilominus earum integralia completa, atque adeo algebraica, exhiberi queant.

§. 2. Hujusmodi ſcilicet aequationes differentia deducere licet ex hac aequatione differentiali hactenus plrimum tractata: $\frac{\partial x}{\sqrt{X}} = \frac{\partial y}{\sqrt{Y}}$, in qua eſt $X = \alpha + 2\beta x$ $\gamma xx + \delta x^3 + \varepsilon x^4$ et $Y = \alpha + 2\beta y + \gamma yy + 2\delta y^3 + \varepsilon y$ cujus integrale completum hac aequatione finita exprimitu

I. $\frac{\sqrt{X} + \sqrt{Y}}{x - y} = \sqrt{2\lambda + \gamma + 2\delta(x+y) + \varepsilon(x+y)^2}$, vbi denotat conſtantem arbitrariam integratione ingreſſa quod ergo integrale etiam hoc modo exhiberi poteſt

II. $\sqrt{XY} = \lambda(x-y)^2 - \alpha - \beta(x+y) - \gamma xy - \delta xy(x+y)$ $\varepsilon xxyy$. Quin etiam irrationalitatem penitus tollen hoc integrale ſequentem induet formam:

III. $0 = \lambda\lambda(x-y)^2 - 2\lambda(\alpha + \beta(x+y) + \gamma x$ $+ \delta xy(x+y) + \varepsilon xxyy) + (\beta\beta - \alpha\gamma) - 2\alpha\delta(x+$ $- \alpha\varepsilon(x+y)^2 - 2\beta\delta xy - 2\beta\varepsilon xy(x+y) + (\delta\delta - \gamma\varepsilon)xxy$

Hinc jam ſequentia exempla evolvamus.

Exemplum I.

§. 3. Cum ex aequatione $\frac{\partial x}{\sqrt{X}} = \frac{\partial y}{\sqrt{Y}}$ fit $\frac{\partial x}{\partial y} = \sqrt{}$ habebimus $\frac{\partial x}{\partial y} = \frac{\sqrt{XY}}{Y}$, vbi, ſi valores pro Y et $\sqrt{XY}$ forma integralis ſecunda ſubſtituamus, prodibit

$$\frac{\partial x}{\partial y} = \frac{\lambda(x-y)^2 - \alpha - \beta(x+y) - \gamma xy - \delta xy(x+y) - \varepsilon xxyy}{\alpha + 2\beta y + \gamma yy + 2\delta y^3 + \varepsilon y^4}$$

quae more ſolito in ordinem redacta hanc induet formam

$$\partial x(\alpha + 2\beta y + \gamma yy + 2\delta y^3 + \varepsilon y^4) + \partial y(\alpha + \beta(x +$$
$$+ \gamma xy + \delta xy(x+y) + \varepsilon xxyy) = \lambda \partial y(x -$$

cujus aequationis ergo integrale eſt aequatio finita, qu ſub triplici forma exhibuimus. Quoniam autem in hoc tegrali nulla nova conſtans occurrit, quae in differenti non inſit, hoc integrale tantum pro particulari eſt habendu

§.

§. 4. Interim tamen haec aequatio differentialis jam ita est comparata, vt nemo certe ejus integrale divinando elicere potuerit, cum sex quantitates diversae ibi occurrant. Quin etiam si quatuor adeo litterae evanescant, tamen integrale adhuc satis absconditum deprehenditur. Veluti si sumamus $\beta = \gamma = \delta = \varepsilon = 0$, oritur haec aequatio differentialis $\alpha \partial x + \alpha \partial y = \lambda \partial y (x - y)^2$, cujus ergo integrale ex prima forma erit $\frac{2\sqrt{\alpha}}{x-y} = \sqrt{2\lambda}$, sive $x - y = \frac{\sqrt{2\alpha}}{\lambda}$, sive $x = y + \frac{\sqrt{2\alpha}}{\lambda}$, qui valor vtique satisfacit, sed tantum particulariter. Pro integrali autem completo inveniendo statuatur $x - y = v$, sive $x = y + v$, vnde aequatio differentialis evadet $\partial y = \frac{\alpha \partial v}{\lambda v v - 2\alpha}$, cujus ergo integrale completum sive a logarithmis, sive ab arcubus circularibus pendet.

§. 5. Ponamus nunc esse $\alpha = \gamma = \delta = \varepsilon = 0$, et aequatio nostra differentialis erit

$$2\beta y \partial x + \beta (x + y) \partial y = \lambda \partial y (x - y)^2,$$

cui ergo satisfacit hoc integrale ex I. forma

$$\frac{\sqrt{2\beta x} + \sqrt{2\beta y}}{x - y} = \sqrt{2\lambda}, \text{ vel ex II. forma}$$

$$2\beta\sqrt{xy} = \lambda (x - y)^2 - \beta (x + y).$$

Illa autem forma praebet $\sqrt{x} + \sqrt{y} = (x - y)\sqrt{\frac{\lambda}{\beta}}$, quae divisa per $\sqrt{x} + \sqrt{y}$ dat $1 = (\sqrt{x} - \sqrt{y})\sqrt{\frac{\lambda}{\beta}}$, sive $\sqrt{x} = \sqrt{y} + \sqrt{\frac{\beta}{\lambda}}$, hincque $x = y + 2\sqrt{\frac{\beta}{\lambda}y} + \frac{\beta}{\lambda}$, ideoque $\partial x = \partial y + \frac{\partial y \sqrt{\beta}}{\sqrt{\lambda y}}$, qui valores substituti aequationem identicam producunt.

§. 6. Cum igitur isti casus simplicissimi jam profundiorem indagationem requirant, hinc evidentissime elucet, si omnes sex litterae in calculo relinquantur, tum neminem

certe

certe vnquam ejus integrale saltem particulare esse erut
rum; unde haec ipsa aequatio generalis:

$$\left.\begin{matrix}\partial x(\alpha + 2\beta y + \gamma yy + 2\delta y^3 + \varepsilon y^4) + \\ +\partial x(\alpha + \beta(x+y) + \gamma xy + \partial xy(x+y) + \varepsilon xxyy)\end{matrix}\right\} = \lambda \partial y (x - y$$

omni attentione maxime digna videtur, cum ejus integral licet particulare, sit ipsa aequatio supra §. 2. assignata triplici forma. In sequentibus autem exemplis hujusmo aequationes differentiales proferemus, quarum adeo integ lia completa algebraice exhiberi queant.

Exemplum II.

§. 7. Cum sit $\partial x : \partial y = \sqrt{X} : \sqrt{Y}$, erit $\frac{\partial x + \partial}{\partial x +}$ $= \frac{\sqrt{X} + \sqrt{Y}}{\sqrt{X} - \sqrt{Y}}$. Jam haec fractio supra et infra multiplicet per $\sqrt{X} + \sqrt{Y}$ fietque $\frac{\partial x + \partial y}{\partial x - \partial y} = \frac{(\sqrt{X} + \sqrt{Y})^2}{X - Y}$, cujus numer tor ex prima forma integralis est

$$(x - y)^2 \, (2\lambda + \gamma + 2\delta(x + yy) + \varepsilon(x + y)^2$$

denominator vero erit

$$2\beta(x - y) + \gamma(xx - yy) + 2\delta(x^3 - y^3) + \varepsilon(x^4 - y^4)$$

sicque haec fractio per $x - y$ deprimi potest, ita vt habeam

$$\frac{\partial x + \partial y}{\partial y - \partial y} = \frac{(x - y)(2\lambda + \gamma + 2\delta(x + y) + \varepsilon(x + y)^2)}{2\beta + \gamma(x + y) + 2\delta(xx + xy + yy) + \varepsilon(x + y)(xx +}$$

cujus ergo integrale pariter erit ipsa aequatio finita su assignata, quae cum praeter quantitates constantes, in ip aequationem differentialem ingredientes, quae sunt β, δ, ε et λ, insuper litteram α contineat, vtique pro integ completo est habenda.

§. 8. Quo hanc aequationem in ordinem redigam primo eam in hanc formam convertamus:

$$\frac{\partial x}{\partial y} = \frac{\beta + \lambda(x - y) + \gamma x + \delta x(2x + y) + \varepsilon xx(x + y)}{\lambda(x - y) - \beta - \gamma y - \delta y(2y + x) - \varepsilon yy(x + y)}.$$

Nu

Hinc igitur fractionibus sublatis prodibit haec aequatio:

$$\left.\begin{array}{l}\lambda\partial x(x-y)-\beta\partial x-\gamma y\partial x-\delta y\partial x(2y+x)-\varepsilon yy\partial x(x+y)\\-\lambda\partial y(x-y)-\beta\partial y-\gamma x\partial y-\delta x\partial y(2x+y)-\varepsilon xx\partial y(x+y)\end{array}\right\}=0.$$

Hujus ergo aequationis integrale completum est ipsa illa aequatio finita, quam supra sub triplici forma repraesentavimus, in qua littera α est constans arbitraria per integrationem ingressa, vnde ex tertia forma integrale ita referri poterit

$$\alpha(2\lambda+\gamma+2\delta(x+y)+\varepsilon(x+y)^2)=\lambda\lambda(x-y)^2-2\lambda\beta(x+y)$$
$$-2\lambda\gamma xy-2\lambda\delta xy(x+y)-2\lambda\varepsilon xxyy+\beta\beta-2\beta\delta xy$$
$$-2\beta\varepsilon xy(x+y)+(\delta\delta-\gamma\varepsilon)xxyy.$$

sive

$$\alpha=\frac{\lambda\lambda(x-y)^2-2\lambda\beta(x+y)-2\lambda\gamma xy-2\lambda\delta xy(x+y)-2\lambda\varepsilon xxyy+\beta\beta-2\beta\delta xy-2\beta\varepsilon xy(x+y)+(\delta\delta-\gamma\varepsilon)xxyy}{2\lambda+\gamma+2\delta(x+y)+\varepsilon(x+y)^2}$$

§. 9. Quia in hac aequatione plures occurrunt litterae, scilicet λ, β, γ, δ, ε, contemplemur primo casus speciales, quibus duae tantum litterae occurrunt, reliquis ad nihilum redactis.

Casus I.

quo $\gamma=\delta=\varepsilon=0$.

§. 10. Aequatio ergo differentialis erit

$$\lambda\partial x(x-y)-\lambda\partial y(x-y)-\beta\partial x-\beta\partial y=0$$

sive $\lambda(x-y)(\partial x-\partial y)-\beta(\partial x+\partial y)=0$

cujus integrale sponte se prodit

$$\lambda(x-y)^2-2\beta(x+y)=\text{const.}$$

Generalis vero integralis forma hoc casu praebet

$$\alpha=\frac{\lambda\lambda(x-y)^2-2\lambda\beta(x+y)+\beta\beta}{2\lambda}$$

Casus

Casus II.

quo $\beta = \delta = \varepsilon = 0$.

Hoc casu aequatio differentialis erit $\lambda \partial x (x - y) - \lambda \partial y (x - y) - \gamma (y \partial x + x \partial y) =$ cujus integrale pariter sponte se offert, quandoquidem

$$\lambda (x - y)^2 - 2 \gamma x y = \text{const.}$$

Ex forma generali integrale fit $\alpha = \frac{\lambda\lambda (x - y)^2 - 2 \lambda \gamma x y}{2 \lambda + \gamma}$. Quin etiam si fuerit tantum $\delta = \varepsilon = 0$, qui fit

Casus III.

Aequatio differentialis erit $\lambda (x - y)(\partial x - \partial y) - \beta (\partial x + \partial y) - \gamma (y \partial x + x \partial y) =$ cujus integrale est manifesto

$$\lambda (x - y)^2 - 2 \beta (x + y) - 2 \gamma x y = \text{const.}$$

Forma generalis autem praebet

$$\alpha = \frac{\lambda\lambda (x - y)^2 - 2 \lambda \beta (x + y) - 2 \lambda \gamma x y + \beta\beta}{2 \lambda + \gamma},$$

vbi consensus est manifestus, sicque quoties ambae litt δ et ε evanescunt, res nihil plane habet in recessu; ve si litterarum δ et ε, vel altera tantum, vel ambae affue ejusmodi oriuntur aequationes differentiales, quarum gratio per methodos usitatas non parum difficultatis i vit; hujusmodi igitur casus hic data opera evolvamus

Casus IV.

quo $\beta = \gamma = \varepsilon = 0$.

§. 11. Hoc ergo casu aequatio differentialis er $\lambda (x - y)(\partial x - \partial y) - \delta y \partial x (2 y + x) - \delta x \partial y (2 x + y$ cujus integrale ex forma generali resultat

$$\alpha = \frac{\lambda\lambda (x - y)^2 - 2 \lambda \delta x y (x + y) + \delta\delta x x y y}{2 \lambda + 2 \delta (x + y)},$$

veritas neutiquam tam clare perſpicitur, quam cum in praecedentibus; namque poſito brevitatis gratia $\lambda = n\delta$, vt habeatur haec aequatio:

$n(x-y)(\partial x - \partial y) = y\partial x(2y+x) + x\partial y(2x+y)$,

ejus prius membrum ſponte eſt integrabile, hincque etiam ſi multiplicetur per functionem quamcunque $x - y$. Verum nulla hujusmodi functio datur, qua etiam poſterius membrum integrabile reddatur. Vt autem more ſolito in ejus integrale inquiramus, ponamus $x + y = p$ et $x - y = q$, vt ſit $x = \frac{p+q}{2}$ et $y = \frac{p-q}{2}$, atque aequatio noſtra induet hanc formam: $nq\partial q = \frac{1}{4}\partial p(3pp + qq) - pq\partial q$. Ponamus hic $qq = v$, vt ſit $2q\partial q = \partial v$, et aequatio noſtra erit: $2n\partial v + 2p\partial v - v\partial p = 3pp\partial p$. In qua aequatione quia v vnicam tantum habet dimenſionem, ea methodo conſueta reſolvi poterit: diviſa enim per $2n + 2p$, praebet $\partial v - \frac{v\partial p}{2n+2p} = \frac{3pp\partial p}{2n+2p}$.

§. 12. Conſtat autem hanc aequationem generalem: $\partial v + Pv\partial p = Q\partial p$, vbi P et Q ſint functiones quaecunque ipſius p, integrabilem reddi, ſi ducatur in $e^{\int P\partial p}$; tum enim integrale fit $e^{\int P\partial p}v = \int e^{\int P\partial p}Q\partial p$. Hinc autem pro noſtro caſu habebimus $P = \frac{-1}{2n+2p}$ et $Q = \frac{3pp}{2n+2p}$; quamobrem hinc $\int P\partial p = -\frac{1}{2}l(2n+2p) + \frac{1}{2}l\,2 = -\frac{1}{2}l(n+p)$, ideoque $e^{\int P\partial p} = \frac{1}{\sqrt{(n+p)}}$, ergo aequatio integralis erit $\frac{v}{\sqrt{(n+p)}} = \frac{3}{2}\int\frac{pp\partial p}{(n+p)^{\frac{3}{2}}}$. Pro poſtremo membro ponatur $n + p = zz$, ſive $p = zz - n$ eritque $(n+p)^{\frac{3}{2}} = z^3$, tum vero fiet $\frac{pp\partial p}{(n+p)^{\frac{3}{2}}} = \frac{2\partial z(z^4 - 2nzz + nn)}{zz} = 2zz\partial z - 4n\partial z + \frac{2nn\partial z}{zz}$, cujus integrale eſt $\frac{2}{3}z^3 - 4nz - \frac{2nn}{z}$, conſequenter noſtra aequatio integralis erit: $\frac{v}{\sqrt{(n+p)}} = z^3 - 6nz - \frac{3nn}{z} + \text{conſt.}$

ſive

five $\frac{v}{\sqrt{(n+p)}} = (n+p)^{\frac{3}{2}} - 6n\sqrt{(n+p)} - \frac{3nn}{\sqrt{(n+p)}} + C$, qu[...]
aequatio reducitur ad hanc formam:

$$v = (n+p)^2 - 6n(n+p) - 3nn + C\sqrt{(n+p)},$$

five $v = pp - 4np - 8nn + C\sqrt{(n+p)}$.

§. 13. Erat autem $v = qq$, ficque integrale noftru[...] erit $qq = pp - 4np - 8nn + C\sqrt{(n+p)}$. At ve[...] integrale fupra datum, fi pariter ad quantitates p et q [...] ducatur, in hanc formam transmutatur:

$$\frac{2\alpha}{\delta} = \frac{nnqq - \frac{np(pp-qq)}{2} + \frac{(pp-qq)^2}{16}}{n+p}$$

$$= \frac{16nnqq - 8np(pp-qq) + (pp-qq)^2}{16(n+p)}.$$

Ex forma autem inventa conftans arbitraria C hoc mo[...] definitur: $-C = \frac{pp-qq-4np-8nn}{\sqrt{(n+p)}}$,
cujus quadratum praebet

$$CC = \frac{(pp-qq)^2 - 8np(pp-qq) - 16nn(pp-qq) + 16nnpp + 64n^3p + 64[...]}{n+p}$$

hincque jam elicitur $\frac{32\alpha}{\delta} - CC = 64n^3$. Unde patet am[...] haec integralia perfecte inter fe convenire, fiquidem tant[...] quantitate conftante a fe invicem discrepant.

§. 14. Ob tantas ergo ambages, quibus vfi fum[...] ad integrale eliciendum, ifte cafus tanto majore attentio[...] dignus eft cenfendus. Interim tamen, quoniam integra[...] denominatorem habet $n+p$, atque ipfa fractio differentia[...] noftram aequationem differentialem reproducere debet, [...] ceffe eft vt ipfa noftra aequatio differentialis

$$4nq\partial q + 4pq\partial q - 3pp\partial p - qq\partial p = 0$$

integrabilis reddatur, fi per certam fractionem, quae reperi[...] $\frac{pp-qq+4np+8nn}{(n+p)^2}$, multiplicetur; id quod calculum i[...]

tue[...]

[illegible] per plures demum ambages patebit, si formulam pro $\frac{32\alpha}{\varepsilon}$ supra exhibitam differentiare voluerit, quem laborem autem hic suscipere non vacat, praesertim postquam consensum amborum integralium jam ostenderimus; quam ob causam iste casus maximam attentionem meretur.

Casus V.

quo $\beta = \gamma = \delta = 0$.

§. 15. Hoc ergo casu aequatio differentialis erit $\lambda(x-y)(\partial x - \partial y) - \varepsilon(x+y)(yy\,\partial x + xx\,\partial y) = 0$, cujus ergo integrale completum erit

$$\alpha = \frac{\lambda\lambda(x-y)^2 - 2\lambda\varepsilon xxyy}{2\lambda + \varepsilon(x+y)^2}.$$

Fiat nunc iterum $x + y = p$ et $x - y = q$, ponaturque $\lambda = n\varepsilon$, et aequatio differentialis prodibit

$$nq\partial q - \tfrac{1}{4}p\partial p\,(pp+qq) + \tfrac{1}{2}ppq\partial q = 0.$$

Integrale vero erit $\frac{\alpha}{\varepsilon} = \dfrac{nnqq - \frac{1}{8}n(pp-qq)^2}{2n+pp}$.

Ista autem aequatio pariter nulla laborat difficultate; posito enim $qq = v$, vt sit $2q\partial q = \partial v$, prodibit haec forma:

$$2n\partial v - pv\partial p + pp\partial v = p^3\partial p,$$

haecque divisa per $2n + pp$, erit $\partial v - \frac{vp\partial p}{2n+pp} = \frac{p^3\partial p}{2n+pp}$, quae cum aequatione generali §. 12. comparata dat $P = \frac{-p}{2n+pp}$ et $Q = \frac{p^3}{2n+pp}$. Fiet ergo $\int P\partial p = -\frac{1}{2}l(2n+pp)$, ideoque $e^{\int P\partial p} = \frac{1}{\sqrt{(2n+pp)}}$, ergo aequatio integralis erit

$$\frac{v}{\sqrt{(2n+pp)}} = \int \frac{p^3\partial p}{(2n+pp)^{\frac{3}{2}}} = \frac{4n+pp}{\sqrt{(2n+pp)}} + \text{Const.}$$

Sicque integrale completum erit $qq = 4n + pp + C\sqrt{(2n+pp)}$, sive habebimus $C = \frac{qq - pp - 4n}{\sqrt{(2n+pp)}}$, quae forma, vt cum supra assignata comparari possit, quadretur, fietque

 CC=

$CC = \frac{q^4 - 2ppqq - 8nqq + p^4 + 8npp + 16nn}{2n + pp}$. Erat autem =
$= + \frac{(pp-qq)^2 - 8nqq}{2n+pp}$, quarum expreſſionum differentia
$CC + \frac{8\alpha}{n\varepsilon} = 8n$; unde patet conſtantem C ita definiri,
ſit $CC = 8n - \frac{8\alpha}{n\varepsilon}$.

Caſus generalis,

vbi omnes litterae admittuntur.

§. 16. Poſito nunc in genere $x + y =$ et $x - y =$
aequatio noſtra differentialis erit

$$\lambda q\partial q - \beta\partial p - \tfrac{1}{2}\gamma(p\partial p - q\partial q) - \tfrac{1}{4}\delta\partial p(3pp + \ldots$$
$$+ \delta pq\partial q - \tfrac{1}{4}\varepsilon p\partial p(pp + qq) + \tfrac{1}{2}\varepsilon ppq\partial q = 0$$

cujus ergo integrale completum erit

$$\alpha = \frac{\left\{\begin{array}{l} + \lambda\lambda qq - 2\lambda\beta p - \tfrac{1}{2}\lambda\gamma(pp-qq) - \tfrac{1}{2}\lambda\partial p(pp-q\ldots \\ - \tfrac{1}{8}\lambda\varepsilon(pp-qq)^2 + \beta\beta - \tfrac{1}{2}\beta\varepsilon p(pp-qq) + \tfrac{1}{16}(\delta\delta - \gamma\varepsilon)(pp-q\ldots \end{array}\right.}{2\lambda + \gamma + 2\delta p + \varepsilon pp}.$$

§. 17. Poſtquam autem noſtra aequatio ad ha
formam eſt reducta; ejus reſolutio nulla amplius difficult
laborat; poſito enim $qq = v$, et terminis ſive v, ſive
continentibus in vnam partem translatis, iſta forma provenit

$$(2\lambda + \gamma + 2\delta p + \varepsilon pp)\partial v - v(\delta + \varepsilon p)\partial p$$
$$= (4\beta + 2\gamma p + 3\delta pp + \varepsilon p^3)\partial p.$$
$$\partial v - \frac{v\partial p(\delta + \varepsilon p)}{2\lambda + \gamma + 2\delta p + \varepsilon pp} = \frac{\partial p(4\beta + 2\gamma p + 3\delta pp + \varepsilon p^3)}{2\lambda + \gamma + 2\delta p + \varepsilon pp},$$

haec forma cum generali §. 12. comparata dat

$$P = \frac{-\delta - \varepsilon p}{2\lambda + \gamma + 2\delta p + \varepsilon pp} \text{ et } Q = \frac{4\beta + 2\gamma p + 3\delta pp + \varepsilon p^3}{2\lambda + \gamma + 2\delta p + \varepsilon pp},$$

fiet ergo $\int P\partial p = -\tfrac{1}{2}l(2\lambda + \gamma + 2\delta p + \varepsilon pp)$,
ideoque $e^{\int P\partial p} = \frac{1}{\sqrt{(2\lambda + \gamma + 2\delta p + \varepsilon pp)}}$,

quocir

integratio dabit

$$\frac{v}{\sqrt{(2\lambda+\gamma+2\delta p+\varepsilon pp)}} = \int \frac{\partial p\,(4\beta+2\gamma p+3\delta pp+\varepsilon p^3)}{(2\lambda+\gamma+2\delta p+\varepsilon pp)^{\frac{3}{2}}}.$$

§. 18. Vt nunc poftremam formulam integralem facillime evolvamus, ponamus ejus integrale effe $\frac{A+Bp+Cpp}{\sqrt{(2\lambda+\gamma+2\delta p+\varepsilon pp)}}$, cujus formae differentiale debitum habebit denominatorem, at vero numerator ad hanc formam reducitur:

$$\partial p\,((2\lambda+\gamma)B - A\delta) + p\partial p\,(B\delta + 2C(2\lambda+\gamma) - A\varepsilon) + pp\partial p\,.\,3\delta C + p^3\partial p\,.\,\varepsilon C;$$

hinc ergo obtinemus quatuor fequentes aequationes:

1. $4\beta = (2\lambda+\gamma)B - A\delta$,
2. $2\gamma = B\delta + 2C(2\lambda+\gamma) - A\varepsilon$.
3. $3\delta = 3\delta C$,
4. $\varepsilon = \varepsilon C$,

vbi hae poftremae manifefto praebent $C = 1$, tum vero fecunda fit $B\delta + 4\lambda - A\varepsilon = 0$, ex qua cum prima conjuncta elicitur $B = \frac{4\beta\varepsilon + 4\lambda\delta}{(2\lambda+\gamma)\varepsilon - \delta\delta}$, ac denique $A = \frac{4\beta\delta + 4\lambda(2\lambda+\gamma)}{(2\lambda+\gamma)\varepsilon - \delta\delta}$, quibus valoribus inventis aequatio noftra integralis erit

$$\frac{qq}{\sqrt{(2\lambda+\gamma+2\delta p+\varepsilon pp)}} = \frac{A+Bp+Cpp}{\sqrt{(2\lambda+\gamma+2\delta p+\varepsilon pp)}} + \Delta,$$

five $\Delta = \frac{qq - A - Bp - Cpp}{\sqrt{(2\lambda+\gamma+2\delta p+\varepsilon pp)}}$, five $-\Delta = \frac{Cpp - qq + A + Bp}{\sqrt{(2\lambda+\gamma+2\delta p+\varepsilon pp)}}$, cujus quadratum a valore ipfius $\frac{16\alpha}{\delta\delta - \gamma\varepsilon - 2\lambda\varepsilon}$ fubtractum relinquit quantitatem conftantem.

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