1801

Methodus nova ac facilis omnium aequationum algebraicarum radices non solum ipsas sed etiam quascumque earum potestates per series concinnas exprimendi

Leonhard Euler

E711

METHODVS NOVA AC FACILIS OMNIVM AEQVATIONVM ALGEBRAICARVM RADICES NON SOLVM IPSAS SED ETIAM QVASCVNQVE EARVM POTESTATES PER SERIES CONCINNAS EXPRIMENDI.

Auctore

L. EVLERO.

Conventui exhibit. die 21 Septemb. 1778.

§. 1.

Primus, qui hoc argumentum satis felici successu tractavit, erat ingeniosissimus *Lambert* non ita pridem beate defunctus, qui huiusmodi series pro aequationibus trinomialibus methodo prorsus singulari per approximationes procedente elicuit. Verum ista methodus calculos maxime operosos ac taediosos requirebat, ita ut inventis aliquot terminis initialibus eundem calculum ulterius prosequi non potuerit, sed tantum ex egregio ordine, qui in prioribus terminis observabatur, per inductionem sequentes concludere fuerit coactus. Quamobrem iam ex illo tempore plurimum studii collocavi in methodum directam et planiorem inquirendi, quae ad easdem series perduceret.

§. 2.

§. 2. In huiusmodi autem methodum non multo post incidi, quam in *Commentar. Novor. Tomo XV.* fusius exposui; ubi in genere aequationem sub hac forma contentam:

$$1 = \frac{A}{x} - \frac{B}{x^2} + \frac{C}{x^3} - \frac{D}{x^4} + \frac{E}{x^5} - \text{etc.}$$

sum contemplatus, cuius radices si fuerint α, β, γ, etc. notum est earum summam esse $= A$, summam quadratorum $= A^2 - 2B$, summam cuborum $= A^3 - 3AB + 3C$, et ita porro. Hinc igitur pro summa potestatum quarumcunque $\alpha^n + \beta^n + \gamma^n + \delta^n +$ etc. similem expressionem investigavi, cuius legem progressionis in infinitum extendi observavi; cum tamen pro quovis casu eos tantum terminos accipi oporteat, qui a fractionibus sint liberati; unde mihi in mentem venit in valores istarum serierum, si in infinitum continuentur, inquirere. Mox autem facili ratiocinio intellexi, earum summam eandem potestatem solius maximae radicis, puta α^n, definire.

§. 3. Cum igitur seriem infinitam essem adeptus, quae potestatem exponentis n maximae radicis, scilicet α, exprimeret, mox perspexi istam seriem egregie cum Lambertina convenire; tum vero haud amplius difficile erat similes series pro omnibus aequationibus sub hac forma multo generaliori contentas:

$$1 = \frac{A}{x^\alpha} + \frac{B}{x^\beta} + \frac{C}{x^\gamma} + \frac{D}{x^\delta} + \text{etc.}$$

exhibere, ubi exponentibus α, β, γ, δ, etc. adeo omnes plane valores sive positivos, sive negativos, sive integros, sive fractos tribuere licet. Nihilo vero minus singuli huius seriei termini egregio ordine procedunt, hosque adeo, quousque libue-

libuerit, facile continuare licet, ita ut hic nihil plane inductioni vel coniecturae concedere necesse sit.

§. 4. Cum autem haec methodus ex principio prorsus alieno, et per ambages non parum molestas, sit deducta, plurimum laboravi, ut methodum magis directam, et faciliori negotio ad scopum perducentem, perscrutarer, quin etiam labores meos in aliquot dissertationibus cum Academia communicatis accuratius exposui. Nunc autem idem argumentum retractans in methodum longe faciliorem, ac per nullas ambages procedentem, incidi, quam hoc loco clarius explicare constitui.

§. 5. Hic igitur consideraturus sum aequationem algebraicam sub hac forma generalissima contentam:

$$1 = \frac{A}{x^{\alpha}} + \frac{B}{x^{\beta}} + \frac{C}{x^{\gamma}} + \frac{D}{x^{\delta}} + \text{etc.}$$

ubi ante omnia facile patet sine ulla restrictione loco litterae A unitatem scribi posse, ita ut aequatio, quam hic tractare suscipio, sit:

$$1 = \frac{1}{x^{\alpha}} + \frac{B}{x^{\beta}} + \frac{C}{x^{\gamma}} + \frac{D}{x^{\delta}} + \text{etc.}$$

ex qua statim patet, si litterae B, C, D, etc. evanescerent, fore $x = 1$; unde sequitur in genere radicem x certe seriei infinitae aequalem statui posse, cuius primus terminus sit unitas, sequentes vero litteras B, C, D, utcunque inter se compositas, complectantur, quandoquidem in eam praeter ipsas has litteras singulas tam omnia producta ex binis quam ex ternis et pluribus ingredi debent.

§. 6. Quoniam autem hic mihi propositum est non solum in ipsam radicem x, sed in genere in eius potestatem quamcunque x^n inquirere, ipsam aequationem hac forma repraesentabo:

$$x^n - x^{n-\alpha} = B\,x^{n-\beta} + C\,x^{n-\gamma} + D\,x^{n-\delta} + \text{etc.}$$

ubi primum terminum a dextra in sinistram transtuli, ut ad dextram tantum litterae B, C, D, etc. cum suis potestatibus coniunctae occurrant, ex quarum permistione verum valorem potestatis x^n investigari oportet; ubi facile perspicitur eiusmodi seriem pro x^n prodire debere, in qua post terminum primum 1 non solum singulae litterae B, C, D, etc. sed etiam omnia producta tam ex binis quam ternis pluribusque occurrere debeant, ita ut totum negotium iam huc redeat, ut singulis his productis debiti coëfficientes assignentur, qui utique potissimum pendebunt ab exponente n, praeter reliquos exponentes datos α, β, γ, δ, ε, etc.

§. 7. Hic autem plurimum iuvabit istos coëfficientes per idoneos characteres repraesentare, quibus scilicet omnis confusio ex infinita multitudine terminorum oriunda evitari queat. Ita coëfficientes ipsarum litterarum B, C, etc. quatenus ad potestatem exponentis n referuntur, his signis denotabo: $(\overset{n}{B})$, $(\overset{n}{C})$, $(\overset{n}{D})$, etc. Vnde si alius quicunque exponens, puta m, proponeretur, hi coëfficientes ita forent designandi: $(\overset{m}{B})$, $(\overset{m}{C})$, $(\overset{m}{D})$, etc., quod idem tenendum est de omnibus productis ex binis pluribusve harum litterarum compositis. Veluti si in genere occurrat hoc productum $B^b. C^c. D^d.$ etc., eius coëfficientem pro potestate exponentis m ita sum repraesentaturus: $(B^b. \overset{m}{C^c}. D^d. \text{etc.})$.

§. 8. Hoc igitur fignandi modo conftituto valor poteftatis quaefitae x^n per huiusmodi feriem exprimetur:

$$x^n = 1 + (\overset{n}{B})B + (\overset{n}{C})C + (\overset{n}{D})D + (\overset{n}{E})E + \text{etc.}$$
$$+ (\overset{n}{B^2})B^2 + (\overset{n}{C^2})C^2 + (\overset{n}{D^2})D^2 + (\overset{n}{E^2})E^2 + \text{etc.}$$
$$+ (\overset{n}{BC})BC + (\overset{n}{BD})BD + (\overset{n}{BE})BE + \text{etc.}$$

cuius ergo feriei terminus generalis omnes plane in fe complectens erit

$$(\overset{n}{B^b . C^c . D^d . E^e . \text{etc.}})\, B^b . C^c . D^d . E^e . \text{etc.}$$

§. 9. Ne autem hic opus fit calculum ad plures horum terminorum fimul applicare, praecipuum momentum huc redit, ut fingulos coëfficientes ex paucioribus terminis iam cognitis inveftigare doceamus. Ac primo quidem fi quaeratur coëfficiens $(\overset{n}{B})$, five terminus $(\overset{n}{B})B$, pro poteftate x^n, evidens eft pro $x^{n-\alpha}$ hunc terminum fore $(\overset{n-\alpha}{B})B$, unde ex ipfa aequatione, quatenus hic tantum de terminis formae B agitur, erit

$$(\overset{n}{B})B - (\overset{n-\alpha}{B})B = B\,x^{n-\beta} = B;$$

propterea quod poteftas $x^{n-\beta}$ nullas harum litterarum involvere debet, ideoque pro $x^{n-\beta}$ fcribi debet unitas, utpote prima pars valoris veri. Quoniam nunc hic per B dividi poteft, habebimus pro coëfficiente quaefito $(\overset{n}{B})$ hanc aequationem: $(\overset{n}{B}) - (\overset{n-\alpha}{B}) = 1$; fimili modo pro reliquis habebimus has aequationes:

$$(\overset{n}{C}) - (\overset{n-\gamma}{C}) = 1;\ (\overset{n}{D}) - (\overset{n-\alpha}{D}) = 1;\ (\overset{n}{E}) - (\overset{n-\alpha}{E}) = 1;\ \text{etc.}$$

§. 10. Sin autem quaeramus coëfficientem $(\overset{n}{B^2})$, evidens eſt ex parte ſiniſtra noſtrae aequationis ſolum terminum $B x^{n-\beta}$ in computum venire, quia nullae aliae litterae hic occurrunt. Erit igitur

$(\overset{n}{B^2})\, B^2 - (\overset{n-\alpha}{B^2})\, B^2 = (\overset{n-\beta}{B})\, B^2$; ergo per B^2 dividendo

$(\overset{n}{B^2}) - (\overset{n-\alpha}{B^2}) = (\overset{n-\beta}{B})$. Simili modo erit

$(\overset{n}{C^2}) - (\overset{n-\alpha}{C^2}) = (\overset{n-\gamma}{C})$, tum vero

$(\overset{n}{D^2}) - (\overset{n-\alpha}{D^2}) = (\overset{n-\delta}{D})$,

etc.

§. 11. Sin autem porro quaeratur coëfficiens $(\overset{n}{BC})$ ſive terminus $(\overset{n}{BC})\, B\, C$, manifeſtum eſt ex parte aequationis dextra binos terminos $B x^{n-\beta}$ et $C x^{n-\gamma}$ hic in ſubſidium vocari debere. Vt enim forma $B\, C$ reſultet, pro priore parte pro $x^{n-\beta}$ ſumi debet $(\overset{n-\beta}{C})\, C$; pro poſteriore autem loco $x^{n-\gamma}$ ſcribi debet $(\overset{n-\gamma}{B})\, B$, ſicque noſtra aequatio per $B\, C$ diviſa erit:

$(\overset{n}{BC}) - (\overset{n-\alpha}{BC}) = (\overset{n-\beta}{C}) + (\overset{n-\gamma}{B})$. Eodem modo erit

$(\overset{n}{BD}) - (\overset{n-\alpha}{BD}) = (\overset{n-\beta}{D}) + (\overset{n-\delta}{B})$

$(\overset{n}{CD}) - (\overset{n-\alpha}{CD}) = (\overset{n-\gamma}{D}) + (\overset{n-\delta}{C})$

et ita porro.

ſimi-

ſimilique modo evidens eſt fore

$$(B\overset{n}{C}D) - (B\overset{n-\alpha}{C}D) = (\overset{n-\beta}{CD}) + (\overset{n-\gamma}{BC}) + (\overset{n-\delta}{BC})$$

atque porro

$$(BC\overset{n}{D}E) - (BC\overset{n-\alpha}{D}E) = (\overset{n-\varepsilon}{BCD}) + (\overset{n-\beta}{CDE}) + (\overset{n-\gamma}{BDE}) + (\overset{n-\delta}{BCE}).$$

§. 12. Quod ſi eadem littera ſaepius occurrat, ipſa quidem quadrata iam evolvimus, pro cubis vero habebimus:

$$(\overset{n}{B^3}) - (\overset{n-\alpha}{B^3}) = (\overset{n-\beta}{B^2})$$

$$(\overset{n}{C^3}) - (\overset{n-\alpha}{C^3}) = (\overset{n-\gamma}{C^2})$$

$$(\overset{n}{D^3}) - (\overset{n-\alpha}{D^3}) = (\overset{n-\delta}{D^2})$$

etc.

Eodemque modo erit pro ſuperioribus poteſtatibus

$$(\overset{n}{B^4}) - (\overset{n-\alpha}{B^4}) = (\overset{n-\beta}{B^3})$$

$$(\overset{n}{B^5}) - (\overset{n-\alpha}{B^5}) = (\overset{n-\beta}{B^4})$$

$$(\overset{n}{B^6}) - (\overset{n-\alpha}{B^6}) = (\overset{n-\beta}{B^5})$$

etc.

§. 13. Sin autem plures litterae ingrediantur, ex parte aequationis dextra etiam plures termini in ſubſidium vocari debent, veluti ex ſequentibus formulis pateſcit:

$$(\overset{n}{B^2C}) - (\overset{n-\alpha}{B^2C}) = (\overset{n-\gamma}{B^2}) + (\overset{n-\beta}{BC})$$

$$(\overset{n}{B^2C^2}) - (\overset{n-\alpha}{B^2C^2}) = (\overset{n-\gamma}{B^2C}) + (\overset{n-\beta}{BC^2})$$

(B³

$$(\overset{n}{B^3C}) - (\overset{n-\alpha}{B^3C}) = (\overset{n-\gamma}{B^3}) + (\overset{n-\beta}{B^2C})$$

$$(\overset{n}{B^3C^2}) - (\overset{n-\alpha}{B^3C^2}) = (\overset{n-\gamma}{B^3C}) + (\overset{n-\beta}{B^2C^2})$$

$$(\overset{n}{B^3C^3}) - (\overset{n-\alpha}{B^3C^3}) = (\overset{n-\gamma}{B^3C^2}) + (\overset{n-\beta}{B^2C^3})$$

Simili modo perſpicuum eſt fore

$$(\overset{n}{B^3C^2D}) - (\overset{n-\alpha}{B^3C^2D}) = (\overset{n-\beta}{B^2C^2D}) + (\overset{n-\gamma}{B^3CD}) + (\overset{n-\delta}{B^3C^2}).$$

Haecque exempla abunde ſufficiunt ad coëfficientes omnium plane productorum per huiusmodi aequationes deſignandos.

§. 14. Per tales autem aequationes inveſtigatio coëfficientium utcunque complexorum ad coëfficientes ſimpliciorum productorum reducitur, quos tanquam iam cognitos ſpectare licet, quandoquidem a determinatione ſimpliciorum operationes inchoamus. Scilicet ſi coëfficiens quaeſitus quicunque deſignetur per $\Phi : n$, ſiquidem tanquam functio ipſius n ſpectari poteſt: reſolutio omnium harum aequationum revocatur ad hanc formam:

$$\Phi : n - \Phi : (n - \alpha) = \Pi,$$

ubi Π eſt functio iam cognita litterae n. At vero mox videbimus, huius aequationis reſolutionem pro noſtro inſtituto ſatis commode expediri poſſe.

§. 15. Reſolutio huius aequationis ad calculum differentiarum finitarum eſt referenda, & perinde ac differentialium quantitatem conſtantem arbitrariam recipiet. Quare ne hinc ulla incertitudo relinquatur, ante omnia probe eſt notandum, omnes coëfficientes, quos quaerimus, ita comparaparatos esse debere, ut evanescant posito $n = 0$. Cum enim hoc casu fiat $x^n = 1$, ideoque ipsi primo termino nostrae seriei aequalis, sequentes termini omnes litteras B, C, D, etc. involventes hoc casu evanescere debebunt; unde necesse est ut eorum coëfficientes factorem n involvant.

§. 16. Resolutio autem generalis huius aequationis $\Phi : n - \Phi : (n - \alpha) = \Pi$, parum adiumenti in hoc negotio esset allatura. At vero datur solutio particularis ad nostrum institutum imprimis accommodata, quam hic evolvi conveniet. Denotante scilicet n' ipsam quantitatem variabilem n, constante quapiam c sive auctam sive minutam, ita ut sit $n' = n \pm c$, si fuerit

$$\Phi : n = \Delta\, n\, (n' + \alpha)\,(n' + 2\,\alpha) \,.\,.\,.\,.\,.\, (n' + i\,\alpha)$$

ubi Δ itidem significat quantitatem constantem, erit

$$\Phi : (n - \alpha) = \Delta\,(n - \alpha)\, n'\,(n' + \alpha)\,(n' + 2\,\alpha) \,.\,.\,.\,.\, [n' + (i - 1)\,\alpha],$$

unde ob factores communes

$$(n' + \alpha)\,(n' + 2\,\alpha) \,.\,.\,.\,.\,.\, [n' + (i - 1)\,\alpha], \text{ erit}$$

$$\Phi : n - \Phi : (n - \alpha) = \Delta\,(n'\,n + i\,n\,\alpha - n'\,n + \alpha\,n') \,.\,.\,.\,.\,.\,.$$

$$= \Delta\,\alpha\,(n' + i\,n)\,(n' + \alpha)\,(n' + 2\,\alpha) \,.\,.\,.\,.\,.\, n' + (i - 1)\,\alpha$$

quae expressio ergo aequabitur quantitati illi Π, unde sequens Lemma fundamenti loco hic constituamus.

Lemma.

§. 17. *Proposita aequatione resolvenda*

$$\Phi : n - \Phi : (n - \alpha) = \Pi,$$

qualis ista quantitas Π *in hac forma continebitur:*

$\Pi =$

$\Pi = \Delta\alpha(n'+in)[(n'+\alpha)(n'+2\alpha)(n'+3\alpha)\ldots[n'+(i-1)]]$ tum semper erit $\Phi:n = \Delta n(n'+\alpha)(n'+2\alpha)\ldots(n'+i\alpha)$ existente $n' = n \pm c$. Haec forma iam ita est comparata, ut evanescat posito $n = 0$, sicque ad coëfficientes quaesitos definiendos apprime est accommodata.

§. 18. Iam huius Lemmatis beneficio omnes nostros coëfficientes satis expedite determinare licebit; et quia magis compositos perpetuo ex simplicioribus derivari oportet, omnes terminos seriei generalis, quam quaerimus, pro potestate indefinita x^i in certos ordines distinguamus, quorum primus comprehendat terminos ipsas litteras B, C, D, etc. simpliciter continentes; ad ordinem secundum referamus producta ex binis harum litterarum, cuiusmodi sunt B^2, BC, C^2, etc.; tertius ordo contineat productum ex ternis, cuiusmodi sunt B^3, B^2C, BCD, etc. quartus ordo producta ex quaternis, et ita porro. Pro singulis ergo his ordinibus coëfficientes investigabimus.

Investigatio

Terminorum primi ordinis.

§. 19. Omnium horum terminorum unica est forma B, pro cuius coëfficiente supra habuimus hanc aequationem: $(\overset{n}{B})B - (\overset{n-\alpha}{B}) = 1$; unde posito $(\overset{n}{B}) = \Phi:n$, erit hic $\Pi = 1$. Sumatur ergo in Lemmate praemisso $\Phi:n = \Delta n$, ita ut hic sit $i = 0$, et quoniam hinc fit $\Phi:(n-\alpha) = \Delta(n-\alpha)$, erit $\Pi = \Delta\alpha = 1$, unde fit $\Delta = \frac{1}{\alpha}$; quamobrem coëfficiens no-

ster

ſter erit $(\overset{n}{B}) = \frac{n}{\alpha}$, ſimilique modo pro caeteris huius ordinis erit $(\overset{n}{C}) = \frac{n}{\alpha}$, $(\overset{n}{D}) = \frac{n}{\alpha}$, etc. ita ut ipſi termini huius ordinis futuri ſint $\frac{n}{\alpha} B + \frac{n}{\alpha} C + \frac{n}{\alpha} D + \frac{n}{\alpha} E +$ etc.

Inveſtigatio
Terminorum ſecundi ordinis.

§. 20. Horum igitur terminorum dabitur duplex forma, vel B^2, vel BC, quorum ergo coëfficientes ſunt $(\overset{n}{B^2})$, vel $(\overset{n}{BC})$, quos indagari oportet. Pro priore autem ſupra iam dedimus hanc aequationem: $(\overset{n}{B^2}) - (\overset{n'}{B^2}) = (\overset{n\ \beta}{B})$; ita ut poſito $(\overset{n}{B^2}) = \Phi : n$ fit $\Pi = (\overset{n\ \beta}{B})$. Cum igitur modo invenerimus eſſe $(\overset{n}{B}) = \frac{n}{\alpha}$, erit $\Pi = \frac{n-\beta}{\alpha}$. In Lemmate igitur fiat $i = 1$, ita ut ſit $\Phi : n = \Delta n (n' + \alpha)$, unde oritur $\Pi = \Delta \alpha (n' + n) = \frac{n-\beta}{\alpha}$. Hinc ergo reſtituto $n' = n + c$, erit $\frac{n-\beta}{\alpha} = \Delta \alpha (2n + c)$, unde ſequitur fore $2 \Delta \alpha n = \frac{n}{\alpha}$ et $\Delta \alpha c = -\frac{\beta}{\alpha}$, unde fit $\Delta = \frac{1}{2\alpha\alpha}$ et $c = -\frac{\beta}{\Delta\alpha\alpha} = -2\beta$, ita ut ſit $n' = n - 2\beta$.

§. 21. Cum igitur ſit $\Delta = \frac{1}{2\alpha\alpha}$ et $n' = n - 2\beta$, erit coëfficiens ipſius B^2 quaeſitus, ſcilicet

$$(\overset{n}{B^2}) = \frac{n(n+\alpha-2\beta)}{2\alpha\alpha} = \frac{n}{\alpha} \cdot \frac{n+\alpha-2\beta}{2\alpha}.$$

Quare termini ſecundi ordinis formae B^2 erunt ſequentes:

$$\frac{n}{\alpha} \cdot \frac{n+\alpha-2\beta}{2\alpha} \cdot B^2 + \frac{n}{\alpha} \cdot \frac{n+\alpha-2\gamma}{2\alpha} C^2 + \frac{n}{\alpha} \cdot \frac{n+\alpha-2\delta}{2\alpha} \cdot D + \text{etc.}$$

§. 22. Pro altera forma B C supra attulimus hanc aequationem:

$$(\overset{n}{B}C) - (\overset{n-\gamma}{B}C) = (\overset{n-\beta}{C}) + (\overset{n-\gamma}{B}).$$

Cum igitur sit $(\overset{n}{B}) = (\overset{n}{C}) = \frac{n}{\alpha}$, si ponamus $(\overset{n}{B}C) = \Phi : n$, erit

$$\Pi = \frac{n-\beta}{\alpha} + \frac{n-\gamma}{\alpha} = \frac{\alpha n - \beta - \gamma}{\alpha}.$$

In Lemmate igitur nostro sumamus $i = 1$, ut sit

$\Phi : n = \Delta n (n' + \alpha)$, fierique debet

$\Delta \alpha (n' + n) = \frac{2n - \beta - \gamma}{\alpha}$.

Sumatur ergo primo $n' = n - \beta - \gamma$, ut fiat $\Delta \alpha = \frac{1}{\alpha}$, ideoque $\Delta = \frac{1}{\alpha\alpha}$, sicque erit coëfficiens quaesitus

$$(\overset{n}{B}C) = \frac{n(2n-\beta-\gamma)}{\alpha\alpha} = \frac{2n}{\alpha} \cdot \frac{n+\alpha-\beta-\gamma}{2\alpha}.$$

§. 23. Hinc ergo pro secundo ordine termini formae BC erunt $(\overset{n}{B}C) = \frac{2n}{\alpha} \cdot \frac{n+\alpha-\beta-\gamma}{2\alpha}$, hincque

$$\frac{n}{\alpha} \cdot \frac{n+\alpha-\beta-\gamma}{2\alpha} \cdot 2\,BC + \frac{n}{\alpha} \cdot \frac{n+\alpha-\beta-\delta}{2\alpha} \cdot 2\,BD$$
$$+ \frac{n}{\alpha} \cdot \frac{n+\alpha-\gamma-\delta}{2\alpha} \cdot 2\,CD + \text{etc.}$$

quibus si adiungantur termini formae B^2 modo ante inventi, totus ordo secundus iam est absolutus.

Investigatio

Terminorum tertii ordinis.

§. 24. Prima forma in hoc ordine occurrens est B^3, pro cuius coëfficiente supra nacti sumus hanc aequationem:

(B)

$$(\overset{n}{B^3}) - (\overset{n-\alpha}{B^3}) = (\overset{n-\beta}{B^2}).$$

Cum igitur modo invenerimus

$$(\overset{n}{B^2}) = \frac{n(n+\alpha-2\beta)}{\alpha.2\alpha}, \text{ erit hic}$$

$$(\overset{n-\beta}{B^2}) = \Pi = \frac{n-\beta}{\alpha} \cdot \frac{n+\alpha-3\beta}{2\alpha}.$$

Hinc in Lemmate praemiſſo ſumamus $i = 2$, ut inde fiat

$$\Pi = \Delta\,\alpha\,(n' + 2n)(n' + \alpha).$$

Vt igitur poſteriores factores evadant aequales, ſtatui debet $n' = n - 3\beta$, quo factore communi ſublato relinquetur haec aequatio:

$$\Delta\,\alpha\,(n' + 2n) = \Delta\,\alpha\,(3n - 3\beta) = \frac{n-\beta}{2\alpha^2}.$$

Hic igitur commode diviſio per $n - \beta$ ſuccedit, ita ut hinc fiat $\Delta = \frac{1}{6\alpha^3}$, ſicque coëfficiens quaeſitus erit

$$(\overset{n}{B^3}) = \frac{n(n+\alpha-3\beta)(n+2\alpha-3\beta)}{\alpha.2\alpha.3\alpha};$$

unde per ſe patet termini C^3 coëfficientem fore

$$(\overset{n}{C^3}) = \frac{n(n+\alpha-3\gamma)(n+2\alpha-3\gamma)}{\alpha.2\alpha.3\alpha}.$$

§. 25. Secunda forma huius ordinis erit B^2C, pro cuius coëfficiente ſupra reperimus hanc aequationem:

$$(\overset{n}{B^2C}) - (\overset{n-\alpha}{B^2C}) = (\overset{n-\gamma}{B^2}) + (\overset{n-\beta}{BC}) = \Pi,$$

ergo ex valoribus iam inventis derivamus iſtas duas partes:

$$(\overset{n-\gamma}{B^2}) = \frac{(n-\gamma)(n+\alpha-2\beta-\gamma)}{\alpha.2\alpha};$$

deinde

$$\overset{n-\beta}{(B\,C)} = \frac{2(n-\beta)(n+\alpha-2\beta-\gamma)}{\alpha.2\alpha},$$

ubi evidens eſt poſteriores factores aequales inter ſe prodire debuiſſe; unde ex additione orietur

$$\Pi = \frac{(3n-2\beta-\gamma)(n+\alpha-2\beta-\gamma)}{\alpha.2\alpha}.$$

In Lemmate igitur noſtro ſumi debet $i = 2$, indeque fiet

$$\Pi = \Delta\,\alpha\,(n'+2n)(n'+\alpha);$$

quare, ut poſteriores factores congruant, ſumi debet $n' = n$ $-2\beta-\gamma$, quibus ſublatis remanebit haec aequatio:

$$\frac{3n-2\beta-\gamma}{2.2} = \Delta\,\alpha\,(3n-2\beta-\gamma),$$

ubi iterum diviſio per $3n-2\beta-\gamma$ ſuccedit, ita u hinc $\Delta = \frac{1}{2\alpha^3}$, conſequenter coëfficiens quaeſitus erit

$$\overset{n}{(B^2\,C)} = \frac{n(n+\alpha-2\beta-\gamma)(n+2\alpha-2\beta-\gamma)}{\alpha.2\alpha.\alpha},$$

ſive hoc modo

$$\tfrac{1}{3}\overset{n}{(B^2\,C)} = \frac{n(n+\alpha-2\beta-\gamma)(n+2\alpha-2\beta-\gamma)}{\alpha.2\alpha.3\alpha}.$$

§. 26. Tertia denique forma huius ordinis eſt BCD pro cuius coëfficiente ſupra data eſt haec aequatio:

$$\overset{n}{(B\,C\,D)} - \overset{n-\alpha}{(B\,C\,D)} = \overset{n-\beta}{(C\,D)} + \overset{n-\gamma}{(B\,D)} + \overset{n-\delta}{(B\,C)} = \Pi,$$

ita ut hic Π componatur ex tribus partibus, quae ad prae ſentes indices reductae erunt

$$1^\circ.\ \overset{n-\beta}{(C\,D)} = \frac{2(n-\beta)(n+\alpha-\beta-\gamma-\delta)}{\alpha.2\alpha}$$

$$2^\circ.\ \overset{n-\gamma}{(B\,D)} = \frac{2(n-\gamma)(n+\alpha-\beta-\gamma-\delta)}{\alpha.2\alpha}$$

$$3^\circ.\ \overset{n-\delta}{(B\,C)} = \frac{2(n-\delta)(n+\alpha-\beta-\gamma-\delta)}{\alpha.2\alpha}$$

ub

ubi evidens eſt poſteriores factores neceſſario inter ſe aequales prodire debuiſſe, ſicque his iunctis erit

$$\Pi = \frac{2(3n - \beta - \gamma - \delta)(n + \alpha - \beta - \gamma - \delta)}{\alpha . 2\alpha}.$$

In noſtro igitur Lemmate ſumi oportet $i = 2$, ut inde prodeat $\Pi = \Delta\,\alpha\,(n' + 2n\,(n' + \alpha)$; ubi manifeſto ſumi debet $n' = n - \beta - \gamma - \delta$; ſicque etiam priores factores tolli poterunt, hincque concludetur fore $\Delta = \frac{1}{\alpha^3}$, conſequenter coëfficiens quaeſitus producti B C D erit

$$(B \overset{n}{C} D) = \frac{6n(n - \alpha - \beta - \gamma - \delta)(n + 2\alpha - \beta - \gamma - \delta)}{\alpha . 2\alpha . 3\alpha}.$$

Inveſtigatio
Terminorum quarti ordinis.

§. 27. Prima forma in hoc ordine occurrens, quando ſcilicet omnes quatuor factores ſunt inter ſe aequales, eſt B^4, pro cuius coëfficiente ſupra haec aequatio eſt data:

$$(\overset{n}{B^4}) - (\overset{n-\alpha}{B^4}) = (\overset{n-\beta}{B^3}) = \Pi.$$

Cum igitur modo invenerimus

$$(\overset{n}{B^3}) = \frac{n}{\alpha} . \frac{n + \alpha - 3\beta}{2\alpha} . \frac{n + 2\alpha - 3\beta}{3\alpha}, \text{ erit}$$

$$(\overset{n-\beta}{B^3}) = \frac{n - \beta}{\alpha} . \frac{n + \alpha - 4\beta}{2\alpha} . \frac{n + 2\alpha - 4\beta}{3\alpha} = \Pi.$$

Quia hic habentur tres factores, in Lemmate praemiſſo ſumi debet $i = 3$, indeque orietur

$$\Pi = \Delta\,\alpha\,(n' + 3n)(n' + \alpha)(n' + 2\alpha);$$

ubi bini poſteriores factores ſponte ſe tollunt, ponendo $n' = n - 4\beta$; tum autem relinquetur haec aequatio: $\frac{n - \beta}{6\alpha^3} =$

$4\Delta\alpha(n-\beta)$, unde fit $\Delta = \frac{1}{24\alpha^4}$, ficque coëfficiens qua-
tus pro forma B^4 erit

$$\overset{n}{(B^4)} = \frac{n}{\alpha} \cdot \frac{n+\alpha-4\beta}{2\alpha} \cdot \frac{n+2\alpha-4\beta}{3\alpha} \cdot \frac{n+3\alpha-4\beta}{4\alpha}.$$

§. 28. Secunda forma hic occurrens eft B^3C, p
cuius coëfficiente fupra dedimus hanc aequationem:

$$\overset{n}{(B^3C)} - \overset{n-\alpha}{(B^3C)} = \overset{n-\gamma}{(B^3)} + \overset{n-\beta}{(B^2C)} = \Pi.$$

Colligantur ergo ex formis fupra inventis hae duae part-
ac reperietur

$$\overset{n-\gamma}{(B^3)} = \frac{n-\gamma}{\alpha} \cdot \frac{n+\alpha-3\beta-\gamma}{2\alpha} \cdot \frac{n+2\alpha-3\beta-\gamma}{3\alpha}$$

$$\overset{n-\beta}{(B^2C)} = \frac{3(n-\beta)}{\alpha} \frac{n+\alpha-3\beta-\gamma}{2\alpha} \cdot \frac{n+2\alpha-3\beta-\gamma}{3\alpha};$$

vnde oritur

$$\Pi = \frac{4n-3\beta-\gamma}{\alpha} \cdot \frac{n+\alpha-3\beta-\gamma}{2\alpha} \cdot \frac{n+2\alpha-3\beta-\gamma}{3\alpha}.$$

In Lemmate ergo pro hoc cafu fumi debet $i=3$, ut pr-
deat inde

$$\Pi = \Delta\alpha(n'+3n)(n'+\alpha)(n'+2\alpha),$$

ubi ftatim patet fumi debere $n' = n-3\beta-\gamma$, hocque m-
do omnes factores litteram n involventes fe tolli patiuntu-
quo facto reperietur $\Delta = \frac{1}{6\alpha^4}$, confequenter formae B^3C co-
ciens quaefitus erit

$$\overset{n}{(B^3C)} = \frac{4n}{\alpha} \cdot \frac{n+\alpha-3\beta-\gamma}{2\alpha} \cdot \frac{n+2\alpha-3\beta-\gamma}{3\alpha} \cdot \frac{n+3\alpha-3\beta-\gamma}{4\alpha}.$$

§. 29. Tertia forma in hoc ordine eft B^2C^2, pro c-
ius coëfficiente fupra data eft haec aequatio:

(I

$$\overset{n}{(B^2 C^2)} - \overset{n-\alpha}{(B^2 C^2)} = \overset{n-\gamma}{(B^2 C)} + \overset{n-\beta}{(B\, C^2)} = \Pi.$$

Hic vero erit primo

$$\overset{n-\gamma}{(B^2 C)} = \frac{3(n-\gamma)}{\alpha} \cdot \frac{n+\alpha-2\beta-2\gamma}{2\alpha} \cdot \frac{n+2\alpha-2\beta-2\gamma}{3\alpha}$$

$$\overset{n-\beta}{(B\, C^2)} = \frac{3(n-\beta)}{\alpha} \cdot \frac{n+\alpha-2\beta-2\gamma}{2\alpha} \cdot \frac{n+2\alpha-2\beta-2\gamma}{3\alpha},$$

unde fit

$$\Pi = \frac{2n-\beta-\gamma}{2\alpha^3} \cdot (n+\alpha-2\beta-2\gamma)(n+2\alpha-2\beta-2\gamma).$$

Quare fi in Lemmate fumamus $i = 3$, quoque ut ante effe debet

$$\Pi = \Delta\,\alpha\,(n'+3\,n)(n'+\alpha)(n'+2\,\alpha),$$

ubi manifefto fumi debet $n' = n - 2\beta - 2\gamma$, quo facto reperitur $\Delta = \frac{1}{4\alpha^4}$, ficque huius formae $B^2 C^2$ coëfficiens quaefitus erit

$$\overset{n}{(B^2 C^2)} = \frac{6n}{\alpha} \cdot \frac{n+\alpha-2\beta-2\gamma}{2\alpha} \cdot \frac{n+2\alpha-2\beta-2\gamma}{3\alpha} \cdot \frac{n+3\alpha-2\beta-2\gamma}{4\alpha}.$$

§. 30. Quarta forma ad hunc ordinem referenda eft $B^2 C\, D$, pro cuius coëfficiente ex principiis fupra ftabilitis haec aequatio ftatui debet:

$$\overset{n}{(B^2 C\, D)} - \overset{n-\alpha}{(B^2 C\, D)} = \overset{n-\delta}{(B^2 C)} + \overset{n-\gamma}{(B^2 D)} + \overset{n-\beta}{(B\, C\, D)} = \Pi,$$

ficque Π conftat ex tribus partibus

$$\overset{n-\delta}{(B^2 C)} = \frac{3(n-\delta)}{\alpha} \cdot \frac{n+\alpha-2\beta-\gamma-\delta}{2\alpha} \cdot \frac{n+2\alpha-2\beta-\gamma-\delta}{3\alpha}$$

$$\overset{n-\gamma}{(B^2 D)} = \frac{3(n-\gamma)}{\alpha} \cdot \frac{n+\alpha-2\beta-\gamma-\delta}{2\alpha} \cdot \frac{n+2\alpha-2\beta-\gamma-\delta}{3\alpha}$$

$$\overset{n-\beta}{(B\, C\, D)} = \frac{6(n-\beta)}{\alpha} \cdot \frac{n+\alpha-2\beta-\gamma-\delta}{2\alpha} \cdot \frac{n+2\alpha-2\beta-\gamma-\delta}{3\alpha}$$

qui-

quibus ergo collectis fit

$$\Pi = \frac{4n - 2\beta - \gamma - \delta}{2 \cdot 3} (n + \alpha - 2\beta - \gamma - \delta)(n + 2\alpha - 2\beta - \gamma - \delta).$$

cui ex Lemmate haec expressio:

$$\Delta\, \alpha\, (n' + 3n)(n' + \alpha)(n' + 2\alpha)$$

reddi debet aequalis, quod egregie succedet sumendo $n' = n - 2\beta - \gamma - \delta$; hinc enim colligetur $\Delta = \frac{1}{2\alpha^4}$, consequenter huius formae B^2CD coëfficiens debitus ita exprimetur:

$$\overset{n}{(B^2CD)} = \frac{12n}{\alpha} \cdot \frac{n + \alpha - 2\beta - \gamma - \delta}{2\alpha} \cdot \frac{n + 2\alpha - 2\beta - \gamma - \delta}{3\alpha} \cdot \frac{n + 3\alpha - 2\beta - \gamma - \delta}{4\alpha}.$$

§. 31. Postrema forma huius ordinis est BCDE, pro cuius coëfficiente habetur haec aequatio:

$$\overset{n}{(BCDE)} - \overset{n-\alpha}{(BCDE)} = \overset{n-\varepsilon}{(BCD)} + \overset{n-\delta}{(BCE)} + \overset{n-\gamma}{(BDE)} + \overset{n-\beta}{(CDE)}.$$

Has igitur quatuor partes hic evolvamus

$$\overset{n-\varepsilon}{(BCD)} = \frac{6(n-\varepsilon)}{\alpha} \cdot \frac{n + \alpha - \beta - \gamma - \delta - \varepsilon}{2\alpha} \cdot \frac{n + 2\alpha - \beta - \gamma - \delta - \varepsilon}{3\alpha}$$

$$\overset{n-\delta}{(BCE)} = \frac{6(n-\delta)}{\alpha} \cdot \frac{n + \alpha - \beta - \gamma - \delta - \varepsilon}{2\alpha} \cdot \frac{n + 2\alpha - \beta - \gamma - \delta - \varepsilon}{3\alpha}$$

$$\overset{n-\gamma}{(BDE)} = \frac{6(n-\gamma)}{\alpha} \cdot \frac{n + \alpha - \beta - \gamma - \delta - \varepsilon}{2\alpha} \cdot \frac{n + 2\alpha - \beta - \gamma - \delta - \varepsilon}{3\alpha}$$

$$\overset{n-\beta}{(CDE)} = \frac{6(n-\beta)}{\alpha} \cdot \frac{n + \alpha - \beta - \gamma - \delta - \varepsilon}{2\alpha} \cdot \frac{n + 2\alpha - \beta - \gamma - \delta - \varepsilon}{3\alpha}$$

His igitur in unam summam collectis erit

$$\Pi = \frac{4n - \beta - \gamma - \delta - \varepsilon}{3} (n + \alpha - \beta - \gamma - \delta - \varepsilon)(n + 2\alpha - \beta - \gamma - \delta - \varepsilon).$$

Quare ut Lemmatis forma pro Π data huic evadat aequalis, manifesto sumi debet $n = n - \beta - \gamma - \delta - \varepsilon$, unde fit $\Delta = \frac{1}{\alpha^4}$, sicque istius formae BCDE coëfficiens erit

$$\overset{n}{(BCDE)} = \frac{24n}{\alpha} \cdot \frac{n + \alpha - \beta - \gamma - \delta - \varepsilon}{2\alpha} \cdot \frac{n + 2\alpha - \beta - \gamma - \delta - \varepsilon}{3\alpha} \cdot \frac{n + 3\alpha - \beta - \gamma - \delta - \varepsilon}{4\alpha}$$

Con-

Conclufio generalis.

§. 32. Lex qua iftae expreffiones ulterius progrediuntur, iam ita eft manifefta, ut fuperfluum foret has operationes ulterius continuare, id quod unico exemplo illuftraffe fufficiet. Sit igitur propofita haec forma ordinis noni $B^4 . C^3 . D^2$, cuius coëfficiens naturalis ex lege combinationis ortus eft, ut conftat,

$$N = \frac{1.2.3 \ldots\ldots 9}{1.2.3.4. \times 1.2.3. \times 1.2.}$$

Iam fi brevitatis gratia ponamus $4\beta + 3\gamma + 2\delta = \lambda$, coëfficiens huius formae pro noftro inftituto erit

$$(B^4 \overset{n}{C^3} D^2 = N . \frac{n}{\alpha} . \frac{n+\alpha-\lambda}{2\alpha} . \frac{n+2\alpha-\lambda}{3\alpha} . \frac{n+3\alpha-\lambda}{4\alpha} . \frac{n+4\alpha-\lambda}{5\alpha} \ldots\ldots \frac{n+8\alpha-\lambda}{9\alpha}$$

ubi fi loco N valorem modo datum fubftituamus, nancifcemur

$$(B^4 \overset{n}{C^3} D^2) = \frac{1}{\alpha^9} \frac{n(n+\alpha-\lambda)(n+2\alpha-\lambda) \ldots\ldots (n+8\alpha-\lambda)}{1.2.3.4. \times 1.2.3. \times 1.2.}.$$

§. 33. Hinc iam in genere pro producto $B^b . C^c . D^d . E^e$. etc. eundem coëfficientem reperimus, quem iam olim in Tomo Comentar. XV. ex longe aliis principiis elicueram; fcilicet, fi fumma omnium exponentium $b + c + d + e +$ etct. $= i$, quo numero ordo, ad quem hoc productum eft referendum, indicatur; tum vero ftatuatur

$$b\beta + c\gamma + d\delta + e\varepsilon + \text{etc.} = \lambda,$$

coëfficiens iftius producti ita exprimetur:

$$\frac{1}{\alpha^i} . \frac{n(n+\alpha-\lambda)(n+2\alpha-\lambda)(n+3\alpha-\lambda) \ldots\ldots [n+(i-1)\alpha-\lambda]}{1.2.3 \ldots b \times 1.2.3 \ldots c \times 1.2.3 \ldots d \times 1.2.3 \ldots e \times \text{etc.}}.$$

§. 34. Quod fi ergo omnia poffibilia producta litterarum B, C, D, E, etc. cum his coëfficientibus coniuncta in unam summam colligantur, eique unitas praefigatur, habetur valor potestatis indefinitae x^n, qui convenit huic aequationi algebraicae:

$$1 = \frac{1}{x^\alpha} + \frac{B}{x^\beta} + \frac{C}{x^\gamma} + \frac{D}{x^\delta} + \frac{E}{x^\varepsilon} + \text{etc.}$$

§. 35. In hac aequatione omnes numeratores litteris diversis B, C, D, E, etc. designavimus, propterea quod ad diversas potestates ipsius x referuntur. Vnde intelligitur, etiamsi forte esset C = D, tamen producti quod esset B^2 coefficientem neutiquam ex forma $(\overset{n}{B}B)$, sed perpetuo ex forma $(\overset{n}{B}C)$ repeti debere.

§. 36. Quoniam ope huius methodi valorem cuiuscunque potestatis indefinitae x^n formavimus, nil certe facilius est, quam hinc ipsam aequationis propositae radicem x determinare, ponendo scilicet $n = 1$. Vnde hoc insigne Paradoxon se offert, quod eadem methodus nullum plane usum praestatura fuisset, si eius ope ipsam radicem x elicere voluissemus, propterea quod vis istius methodi in eo ipso est constituenda, quod potestatem plane indefinitam x^n iam statim ab initio simus contemplati, unde omnium aliarum potestatum $x^{n-\alpha}$, $x^{n-\beta}$, $x^{n-\gamma}$, etc. valores exprimere licuit. Ceterum alia insignia Phaenomena, quae series hoc modo formatae offerunt, hic non commemoranda censeo, cum hoc argumentum iam alibi fusius sim prosecutus, hoc vero loco mihi potissimum fuerit propositum methodum directam in medium afferre, tales series satis expedite inveniendi.