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De evolutione potestatis polynomialis cuiuscunque $(1 + x + x^2 + x^3 + x^4 + \text{etc.})^n$

Leonhard Euler

DE EVOLVTIONE POTESTATIS POLYNOMIALIS CVIVSCVNQVE

$(1 + x + x^2 + x^3 + x^4 + \text{etc.})^n$

Auctore

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§. 1.

Incipiamus a poteſtate binomiali $(1+x)^n$, quae more ſolito evoluta deſignemus coëfficientem poteſtatis cuiusvis x^λ hoc charactere $(\frac{n}{\lambda})$, ita ut ſit

$$(1+x)^n = 1 + (\tfrac{n}{1})x + (\tfrac{n}{2})xx + (\tfrac{n}{3})x^3 + (\tfrac{n}{4})x^4 + \text{etc.} \ldots (\tfrac{n}{n})x^n,$$

ubi ergo erit

$$(\tfrac{n}{1}) = n; \quad (\tfrac{n}{2}) = \frac{n(n-1)}{1 \cdot 2}; \quad (\tfrac{n}{3}) = \frac{n \cdot (n-1)(n-2)}{1 \cdot 2 \cdot 3};$$

$$(\tfrac{n}{4}) = \frac{n \cdot (n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} \text{ etc.}$$

et in genere

$$(\tfrac{n}{\lambda}) = \frac{n \cdot (n-1)(n-2) \cdot \cdot \cdot \cdot (n-\lambda+1)}{1 \cdot 2 \cdot 3 \cdot \cdot \cdot \cdot \lambda},$$

unde patet caſu $\lambda = 0$ et $\lambda = n$ fore $(\frac{n}{0}) = (\frac{n}{n}) = 1$, atque adeo in genere $(\frac{n}{\lambda}) = (\frac{n}{n-\lambda})$. Praeterea vero notaſſe iuvabit, tam caſibus quibus λ eſt numerus negativus, quam qui-

quibus est numerus maior quam n, significatum formulae $\left(\frac{n}{\lambda}\right)$ semper esse nihilo aequalem.

§. 2. Quoniam per hos characteres calculus non mediocriter sublevatur et contrahitur, similibus characteribus utamur etiam in evolutione potestatum trinomialium, quadrinomialium, et generatim polynomialium quarumcunque. Hunc in finem superioribus characteribus pro binomio adhibitis adiungamus quasi exponentem 2, quandoquidem hinc nulla ambiguitas est metuenda, quoniam in huiusmodi calculis nullae potestates horum characterum occurrere solent; hoc modo pro evolutione potestatis binomialis habebimus:

$$(1+x)^n = \left(\tfrac{n}{0}\right)^2 + \left(\tfrac{n}{1}\right)^2 x + \left(\tfrac{n}{2}\right)^2 xx + \left(\tfrac{n}{3}\right)^2 x^3 + \text{etc.}$$

ubi ergo meminisse oportet esse in genere $\left(\frac{n}{\lambda}\right)^2 = \left(\frac{n}{n-\lambda}\right)^2$, tum vero perpetuo $\left(\frac{n}{0}\right)^2 = \left(\frac{n}{n}\right)^2 = 1$, atque has formulas in nihilum abire casibus, quibus est λ vel numerus integer negativus, vel positivus maior quam n.

§. 3. Iisdem igitur characteribus utemur pro evolutione potestatum polynomialium quarumcunque, dummodo pro trinomialibus adiungamus quasi exponentem ternarium, pro quadrinomialibus quaternarium, pro quinomialibus quinarium, et ita porro, hoc scilicet modo:

Pro trinomialibus $(1+x+xx)^n$ evolutio praebeat

$$\left(\tfrac{n}{0}\right)^3 + \left(\tfrac{n}{1}\right)^3 x + \left(\tfrac{n}{2}\right)^3 x^2 + \left(\tfrac{n}{3}\right)^3 x^3 + \left(\tfrac{n}{4}\right)^3 x^4 + \text{etc.}$$

Pro quadrinomialibus $(1+x+xx+x^3)^n$ evolutio praebeat

$$\left(\tfrac{n}{0}\right)^4 + \left(\tfrac{n}{1}\right)^4 x + \left(\tfrac{n}{2}\right)^4 x^2 + \left(\tfrac{n}{3}\right)^4 x^3 + \left(\tfrac{n}{4}\right)^4 x^4 + \text{etc.}$$

Pro quinomialibus $(1+x+xx+x^3+x^4)^n$ evolutio praebeat

$$\left(\tfrac{n}{0}\right)^5 + \left(\tfrac{n}{1}\right)^5 x + \left(\tfrac{n}{2}\right)^5 xx + \left(\tfrac{n}{3}\right)^5 x^3 + \left(\tfrac{n}{4}\right)^5 x^4 + \left(\tfrac{n}{5}\right)^5 x^5 + \text{etc.}$$

etc. etc.

§. 4. His explicatis inquiramus in veros valores horum characterum exponentibus 3, 4, 5, 6 etc. insignitorum, et videamus quomodo illi per characteres binario notatos, quippe quorum significatus est notissimus, determinari queant. Singulos igitur casus harum potestatum polynomialium ordine percurramus.

Evolutio potestatis trinomialis
$(1 + x + xx)^n$.

§. 5. Seriem hinc oriundam hoc modo repraesentemus:

$$\left(\frac{n}{0}\right)^3 + \left(\frac{n}{1}\right)^3 x + \left(\frac{n}{2}\right)^3 xx + \left(\frac{n}{3}\right)^3 x^3 + \left(\frac{n}{4}\right)^3 x^4 + \text{etc.}$$

cuius terminus ultimus erit $= \left(\frac{n}{2n}\right)^3 x^{2n}$, ubi coëfficientem $\left(\frac{n}{2n}\right)^3$ iam novimus esse unitati aequalem, perinde ac terminum primum $\left(\frac{n}{0}\right)^3$; tum vero quia coëfficientes isti retro eodem ordine progrediuntur, hinc sequitur fore:

$$\left(\frac{n}{1}\right)^3 = \left(\frac{n}{2n-1}\right)^3;\ \left(\frac{n}{2}\right)^3 = \left(\frac{n}{2n-2}\right)^3;$$

atque adeo in genere $\left(\frac{n}{\lambda}\right)^3 = \left(\frac{n}{2n-\lambda}\right)^3$. Porro hic evidens est valorem formulae $\left(\frac{n}{\lambda}\right)^3$ in nihilum abire tam casibus quibus λ est numerus integer negativus, quam casibus quibus est positivus maior quam $2n$.

§. 6. Ante autem quam determinationem horum characterum suscipiamus, haud incongruum erit evolutionem casuum simpliciorum ante oculos posuisse:

n	$(1+x+xx)^n$
0	1
1	$1+x+xx$
2	$1+2x+3xx+2x^3+x^4$
3	$1+3x+6xx+7x^3+6x^4+3x^5+x^6$
4	$1+4x+10xx+16x^3+19x^4+16x^5+10x^6+4x^7+x^8$
5	$1+5x+15xx+30x^3+45x^4+51x^5+45x^4+30x^3+$ etc.
6	$1+6x+21xx+50x^3+90x^4+126x^5+141x^6+126x^7+$ etc
etc.	etc.

Ex ultimo casu, quo $n=6$, patet igitur esse

$$(\tfrac{6}{0})^3=1;\ (\tfrac{6}{1})^3=6;\ (\tfrac{6}{2})^3=21;\ (\tfrac{6}{3})^3=50;\ (\tfrac{6}{4})^3=90;$$
$$(\tfrac{6}{5})^3=126;\ (\tfrac{6}{6})^3=141;\ (\tfrac{6}{7})^3=126;\ (\tfrac{6}{8})^3=90;$$
$$(\tfrac{6}{9})^3=50;\ (\tfrac{6}{10})^3=21;\ (\tfrac{6}{11})^3=6;\ (\tfrac{6}{12})^3=1.$$

§. 7. Vt nunc investigemus quomodo hi charactere ex trinomio orti per similes characteres ex binomio ortos ex primi queant; potestatem propositam sub forma binomial repraesentemus hoc modo: $[1+x(1+x)]^n$, cuius evoluti ergo praebebit hanc progressionem:

$$1+(\tfrac{n}{1})^2x(1+x)+(\tfrac{n}{2})^2xx(1+x)^2+(\tfrac{n}{3})^2x^3(1+x)^3$$
$$+(\tfrac{n}{4})^2x^4(1+x)^4+\text{etc.}$$

cuius terminus generalis hanc habebit formam: $(\frac{n}{\alpha})x$ $(n-x)^{\alpha}$.

§. 8. Consideremus nunc pro evolutione proposit potestatem ipsius x quamcunque x^λ, eiusque coëfficien est $(\frac{n}{\lambda})^3$, cuius valorem investigemus. Hunc in finem ex singulis membris binomialibus modo inventis depron debet potestas x^λ, quatenus quidem in iis continetur. Fo

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ma autem generalis eſt $(\frac{n}{\alpha})\, x^{\alpha}\,(1+x)^{\alpha}$; unde ob

$$(1+x)^{\alpha} = 1 + (\tfrac{\alpha}{1})^2 x + (\tfrac{\alpha}{2})^2 x^2 + (\tfrac{\alpha}{3})^2 x^3 + (\tfrac{\alpha}{4})^2 x^4 + \text{etc.}$$

quia hic occurrit generatim terminus $(\frac{\alpha}{\beta})^2 x^{\beta}$, is ductus in $(\frac{n}{\alpha})^2 x^{\alpha}$ praebet $(\frac{\alpha}{\beta})^2 (\frac{n}{\alpha})^2 x^{\alpha+\beta}$. Quod ſi ergo fuerit $\alpha + \beta = \lambda$, coëfficiens $(\frac{\alpha}{\beta})^2 (\frac{n}{\alpha})^2$ pars erit coëfficientis quaeſiti $(\frac{n}{\lambda})^3$.

§. 9. Quamobrem ad valorem coëfficientis $(\frac{n}{\lambda})^3$, eruendum tantum opus eſt litteris α et β omnes valores in integris tribuere, quibus prodire poteſt $\alpha + \beta = \lambda$. Evidens autem eſt ambos hos numeros α et β neque negativos neque maiores quam n capi debere, quia alioquin iſta forma evaneſceret; tum vero etiam ſi eſſet $\beta > \alpha$, formula $(\frac{\alpha}{\beta})^2$ pariter eſſet nulla. Hinc igitur maximus valor pro α aſſumendus erit $= \lambda$, tum vero $\beta = 0$; unde ſequitur

ſi α	λ	$\lambda - 1$	$\lambda - 2$	$\lambda - 3$	$\lambda - 4$	etc.
fore β	0	1	2	3	4	

§. 10. Producta igitur ex ſingulis his caſibus orta et in unam ſummam collecta dabunt valorem quaeſitum characteris $(\frac{n}{\lambda})^3$, ita ut nacti ſimus hanc determinationem:

$$(\tfrac{n}{\lambda})^3 = (\tfrac{\lambda}{0})^2 (\tfrac{n}{\lambda})^2 + (\tfrac{\lambda-1}{1})^2 (\tfrac{n}{\lambda-1})^2 + (\tfrac{\lambda-2}{2})^2 (\tfrac{n}{\lambda-2})^2$$
$$+ (\tfrac{\lambda-3}{3})^2 (\tfrac{n}{\lambda-3})^2 + \text{etc.}$$

ſicque iſte valor per partes cognitas exprimitur, quarum numerus quovis caſu eſt finitus.

§. 11. Quo haec melius intelligantur, evolvamus caſus ſimpliciores, tribuendo ipſi λ valores 0, 1, 2, 3, 4 etc.

eritque ut sequitur $\left(\frac{n}{0}\right)^3 = 1$; $\left(\frac{n}{1}\right)^3 = \left(\frac{1}{0}\right)^2\left(\frac{n}{1}\right)^2 = n$;

$$\left(\frac{n}{2}\right)^3 = \left(\frac{2}{0}\right)^2\left(\frac{n}{2}\right)^2 + \left(\frac{1}{1}\right)^2\left(\frac{n}{1}\right)^2 = \left(\frac{n}{2}\right)^2 + \left(\frac{n}{1}\right)^2 = \frac{n(n-1)}{1.\ 2} + n = \frac{n(n+1)}{1.\ 2};$$

$$\left(\frac{n}{3}\right)^3 = \left(\frac{3}{0}\right)^2\left(\frac{n}{3}\right)^2 + \left(\frac{2}{1}\right)^2\left(\frac{n}{2}\right)^2, \text{ sive}$$

$$\left(\frac{n}{3}\right)^3 = \left(\frac{n}{3}\right)^2 + 2\left(\frac{n}{2}\right)^2,$$

$$\left(\frac{n}{4}\right)^3 = \left(\frac{4}{0}\right)^2\left(\frac{n}{4}\right)^2 + \left(\frac{3}{1}\right)^2\left(\frac{n}{3}\right)^2 + \left(\frac{2}{2}\right)^2\left(\frac{n}{2}\right)^2, \text{ sive}$$

$$\left(\frac{n}{4}\right)^3 = \left(\frac{n}{4}\right)^2 + 3\left(\frac{n}{3}\right)^2 + \left(\frac{n}{2}\right)^2,$$

$$\left(\frac{n}{5}\right)^3 = \left(\frac{5}{0}\right)^2\left(\frac{n}{5}\right)^2 + \left(\frac{4}{1}\right)^2\left(\frac{n}{4}\right)^2 + \left(\frac{3}{2}\right)^2\left(\frac{n}{3}\right)^2, \text{ sive}$$

$$\left(\frac{n}{5}\right)^3 = \left(\frac{n}{5}\right)^2 + 4\left(\frac{n}{4}\right)^2 + 3\left(\frac{n}{3}\right)^2,$$

$$\left(\frac{n}{6}\right)^3 = \left(\frac{6}{0}\right)^2\left(\frac{n}{6}\right)^2 + \left(\frac{5}{1}\right)^2\left(\frac{n}{5}\right)^2 + \left(\frac{4}{2}\right)^2\left(\frac{n}{4}\right)^2 + \left(\frac{3}{3}\right)^2\left(\frac{n}{3}\right)^2, \text{ sive}$$

$$\left(\frac{n}{6}\right)^3 = \left(\frac{n}{6}\right)^2 + 5\left(\frac{n}{5}\right)^2 + 6\left(\frac{n}{4}\right)^2 + \left(\frac{n}{3}\right)^2,$$

$$\left(\frac{n}{7}\right)^3 = \left(\frac{7}{0}\right)^2\left(\frac{n}{7}\right)^2 + \left(\frac{6}{1}\right)^2\left(\frac{n}{6}\right)^2 + \left(\frac{5}{2}\right)^2\left(\frac{n}{5}\right)^2 + \left(\frac{4}{3}\right)^2\left(\frac{n}{4}\right)^2, \text{ sive}$$

$$\left(\frac{n}{7}\right)^3 = \left(\frac{n}{7}\right)^2 + 6\left(\frac{n}{6}\right)^2 + 10\left(\frac{n}{5}\right)^2 + 4\left(\frac{n}{4}\right)^2,$$

$$\left(\frac{n}{8}\right)^3 = \left(\frac{8}{0}\right)^2\left(\frac{n}{8}\right)^2 + \left(\frac{7}{1}\right)^2\left(\frac{n}{7}\right)^2 + \left(\frac{6}{2}\right)^2\left(\frac{n}{6}\right)^2 + \left(\frac{5}{3}\right)^2\left(\frac{n}{5}\right)^2 + \left(\frac{4}{4}\right)^2\left(\frac{n}{4}\right)^2, \text{ sive}$$

$$\left(\frac{n}{8}\right)^3 = \left(\frac{n}{8}\right)^2 + 7\left(\frac{n}{7}\right)^2 + 15\left(\frac{n}{6}\right)^2 + 10\left(\frac{n}{5}\right)^2 + \left(\frac{n}{4}\right)^2,$$

$$\left(\frac{n}{9}\right)^3 = \left(\frac{9}{0}\right)^2\left(\frac{n}{9}\right)^2 + \left(\frac{8}{1}\right)^2\left(\frac{n}{8}\right)^2 + \left(\frac{7}{2}\right)^2\left(\frac{n}{7}\right)^2 + \left(\frac{6}{3}\right)^2\left(\frac{n}{6}\right)^2 + \left(\frac{5}{4}\right)^2\left(\frac{n}{5}\right)^2, \text{ sive}$$

$$\left(\frac{n}{9}\right)^3 = \left(\frac{n}{9}\right)^2 + 8\left(\frac{n}{8}\right)^2 + 21\left(\frac{n}{7}\right)^2 + 20\left(\frac{n}{6}\right)^2 + 5\left(\frac{n}{5}\right)^2,$$

$$\left(\frac{n}{10}\right)^3 = \left(\frac{10}{0}\right)^2\left(\frac{n}{10}\right)^2 + \left(\frac{9}{1}\right)^2\left(\frac{n}{9}\right)^2 + \left(\frac{8}{2}\right)^2\left(\frac{n}{8}\right)^2 + \left(\frac{7}{3}\right)^2\left(\frac{n}{7}\right)^2 + \left(\frac{6}{4}\right)^2\left(\frac{n}{6}\right)^2 + \left(\frac{5}{5}\right)^2\left(\frac{n}{5}\right)^2, \text{ sive}$$

$$\left(\frac{n}{10}\right)^3 = \left(\frac{n}{10}\right)^2 + 9\left(\frac{n}{9}\right)^2 + 28\left(\frac{n}{8}\right)^2 + 35\left(\frac{n}{7}\right)^2 + 15\left(\frac{n}{6}\right)^2 + \left(\frac{n}{5}\right)^2,$$

etc. etc.

§. 12. Applicemus haec exempli loco ad casum $n = 6$, quippe quem supra §. 6. iam evolvimus ac reperiemus:

$$\left(\frac{6}{0}\right)^3 = 1,$$

$$\left(\frac{6}{1}\right)^3 = 6,$$

$$\left(\frac{6}{2}\right)^3 = 21,$$

$$\left(\frac{6}{3}\right)^3 = \left(\frac{6}{3}\right)^2 + 2\left(\frac{6}{2}\right)^2 = 50,$$

$\left(\frac{6}{4}\right.$

$(\frac{6}{4})^3 = (\frac{6}{4})^2 + 3(\frac{6}{3})^2 + (\frac{6}{2})^2 = 15 + 3.20 + 15 = 90,$

$(\frac{6}{5})^3 = (\frac{6}{5})^2 + 4(\frac{6}{4})^2 + 3(\frac{6}{3})^2 = 6 + 4.15 + 3.20 = 126,$

$(\frac{6}{6})^3 = (\frac{6}{6})^2 + 5(\frac{6}{5})^2 + 6(\frac{6}{4})^2 + (\frac{6}{3})^2 = 1 + 5.6 + 6.15 + 20 = 141,$

$(\frac{6}{7})^3 = 6(\frac{6}{6})^2 + 10(\frac{6}{5})^2 + 4(\frac{6}{4})^2 + (\frac{6}{3})^2$, five

$(\frac{6}{7})^3 = 6 + 10.6 + 4.15 = 126,$

fcilicet cum fit $(\frac{6}{\alpha})^3 = (\frac{6}{12-\alpha})^3$, erit utique $(\frac{6}{7})^3 = (\frac{6}{5})^3 = 126$; fimili modo erit $(\frac{6}{8})^3 = (\frac{6}{4})^3 = 90$; $(\frac{6}{9})^3 = (\frac{6}{3})^3 = 50$; $(\frac{6}{10})^3 = (\frac{6}{2})^3 = 21$; $(\frac{6}{11})^3 = (\frac{6}{1})^3 = 6$; ac denique $(\frac{6}{12})^3 = (\frac{6}{0})^3 = 1$, qui valores cum fupra datis egregie conveniunt.

Evolutio poteftatis quadrinomialis
$(1 + x + xx + x^3)^n$.

§. 13. Valorem igitur hunc evolutum ita repraefentabimus:

$$1 + (\tfrac{n}{1})^4 x + (\tfrac{n}{2})^4 xx + (\tfrac{n}{3})^4 x^3 + (\tfrac{n}{4})^4 x^4 + (\tfrac{n}{5})^4 x^5 + \text{etc.}$$

ubi fcilicet eft $(\frac{n}{0})^4 = 1$. Deinde quia ultimus terminus eft x^{3n}, erit $(\frac{n}{3n})^4 = 1$; et quia coëfficientes retro fcripti eundem ordinem fervant, erit $(\frac{n}{3n-1})^4 = (\frac{n}{1})^4$, atque in genere $(\frac{n}{3n-\lambda})^4 = (\frac{n}{\lambda})^4$; ubi obfervetur, tam cafibus quibus λ eft numerus integer negativus, quam pofitivus maior quam $3n$, valores huius formulae in nihilum abire. Quibus notatis hic mihi eft propofitum indagare quomodo hi characteres quaternario notati per characteres five binario five ternario notatos, utpote iam cognitos, definiri queant.

§. 14. Antequam hunc laborem fufcipiamus, cafus fimpliciores formulae propofitae in tabula fubiuncta ob oculos ponamus:

n

n	$(1 + x + xx + x^3)^n$
0	1
1	$1 + x + xx + x^3$
2	$1 + 2x + 3xx + 4x^3 + 3x^4 + 2x^5 + x^6$
3	$1 + 3x + 6xx + 10x^3 + 12x^4 + 12x^5 + 10x^6 + 6x^7 + 3x^8 + x^9$
4	$1 + 4x + 10xx + 20x^3 + 31x^4 + 40x^5 + 44x^6 + 40x^7 + 31x^8 +$ etc.
5	$1 + 5x + 15xx + 35x^3 + 65x^4 + 101x^5 + 135x^6 + 155x^7 + 155x^8 +$ etc.
6	$1 + 6x + 21xx + 56x^3 + 120x^4 + 216x^5 +$ etc.
	etc. etc.

§. 15. Nunc formulam propofitam fub hac binomiali: $[1 + x(1 + x + xx)]^n$ referamus, eiusque evolutio nobis praebet hanc feriem:

$$1 + (\tfrac{n}{1})^2 [x(1 + x + xx)] + (\tfrac{n}{2})^2 x^2 (1 + x + xx)^2 + \text{etc.}$$

cuius terminus generalis eft $(\frac{n}{\alpha})^2 x^\alpha (1 + x + xx)^\alpha$. Nunc vero, quia $(1 + x + xx)^\alpha$ eft poteftas trinomialis, erit

$$(1 + x + xx)^\alpha = 1 + (\tfrac{\alpha}{1})^3 x + (\tfrac{\alpha}{2})^3 xx + (\tfrac{\alpha}{3})^3 x^3 + \text{etc}$$

cuius iterum terminus generalis eft $(\frac{\alpha}{\beta})^3 x^\beta$; unde fi propo natur poteftas x^λ, exiftente $\lambda = \alpha + \beta$, ex hoc membro orie tur pro hac poteftate $(\frac{\alpha}{\beta})^3 (\frac{n}{\alpha})^2 x^\lambda$.

§. 16. Cum igitur in evolutione quaefita poteftati x^λ coëfficiens fit $(\frac{n}{\lambda})^4$, eius valor reperietur, fi, ob $\lambda = \alpha + \beta$ omnes valores formulae $(\frac{n}{\alpha})^2 (\frac{\alpha}{\beta})^3$ in unam fummam coll gantur; quo facto erit

$$(\tfrac{n}{\lambda})^4 = (\tfrac{n}{\lambda})^2 (\tfrac{\lambda}{0})^3 + (\tfrac{n}{\lambda - 1})^2 (\tfrac{\lambda - 1}{1})^3 + (\tfrac{n}{\lambda - 2})^2 (\tfrac{\lambda - 2}{2})^3 + (\tfrac{n}{\lambda - 3})^2 (\tfrac{\lambda - 3}{3})^3 \text{ etc.}$$

Si

Sicque patet quomodo omnes characteres quaternario notati per iam cognitos, ſive binario, ſive ternario notatos, determinentur; quod quo clarius appareat loco λ ſucceſſive ſcribamus numeros 0, 1, 2, 3, 4, etc. ac reperiemus:

$$(\frac{n}{0})^4 = (\frac{n}{0})^2 (\frac{0}{0})^3 = 1,$$
$$(\frac{n}{1})^4 = (\frac{n}{1})^2 (\frac{1}{0})^3 = n,$$
$$(\frac{n}{2})^4 = (\frac{n}{2})^2 (\frac{2}{0})^3 + (\frac{n}{1})^2 (\frac{1}{1})^3 = (\frac{n}{2})^2 + n,$$
$$(\frac{n}{3})^4 = (\frac{n}{3})^2 (\frac{3}{0})^3 + (\frac{n}{2})^2 (\frac{2}{1})^3 + (\frac{n}{1})^2 (\frac{1}{2})^3, \text{ ſive}$$
$$(\frac{n}{3})^4 = (\frac{n}{3})^2 + 2 (\frac{n}{2})^2 + (\frac{n}{1})^2,$$
$$(\frac{n}{4})^4 = (\frac{n}{4})^2 (\frac{4}{0})^3 + (\frac{n}{3})^2 (\frac{3}{1})^3 + (\frac{n}{2})^2 (\frac{2}{2})^3 + (\frac{n}{1})^2 (\frac{1}{3})^3, \text{ ſive}$$
$$(\frac{n}{4})^4 = (\frac{n}{4})^2 + 3 (\frac{n}{3})^2 + 3 (\frac{n}{2})^2,$$
$$(\frac{n}{5})^4 = (\frac{n}{5})^2 (\frac{5}{0})^3 + (\frac{n}{4})^2 (\frac{4}{1})^3 + (\frac{n}{3})^2 (\frac{3}{2})^3 + (\frac{n}{2})^2 (\frac{2}{3})^3,$$
$$(\frac{n}{6})^4 = (\frac{n}{6})^2 (\frac{6}{0})^3 + (\frac{n}{5})^2 (\frac{5}{1})^3 + (\frac{n}{4})^2 (\frac{4}{2})^3 + (\frac{n}{3})^2 (\frac{3}{3})^3 + (\frac{n}{2})^2 (\frac{2}{4})^3 + \text{etc.}$$
$$(\frac{n}{7})^4 = (\frac{n}{7})^2 (\frac{7}{0})^3 + (\frac{n}{6})^2 (\frac{6}{1})^3 + (\frac{n}{5})^2 (\frac{5}{2})^3 + (\frac{n}{4})^2 (\frac{4}{3})^3 + (\frac{n}{3})^2 (\frac{3}{4})^3 + \text{etc.}$$
$$(\frac{n}{8})^4 = (\frac{n}{8})^2 (\frac{8}{0})^3 + (\frac{n}{7})^2 (\frac{7}{1})^3 + (\frac{n}{6})^2 (\frac{6}{2})^3 + (\frac{n}{5})^2 (\frac{5}{3})^3 + (\frac{n}{4})^2 (\frac{4}{4})^3 + (\frac{n}{3})^2 (\frac{3}{5})^3 + \text{etc.}$$

etc. etc.

Evolutio poteſtatis quinomialis
$$(1 + x + xx + x^3 + x^4)^n.$$

§. 17. Eius ergo valorem evolutum ita exhibemus:

$$1 + (\frac{n}{1})^5 x + (\frac{n}{2})^5 x^2 + (\frac{n}{3})^5 x^3 + (\frac{n}{4})^5 x^4 + (\frac{n}{5})^5 x^5 + \text{etc.}$$

ubi eſt $(\frac{n}{0})^5 = (\frac{n}{4n})^5 = 1$, atque in genere $(\frac{n}{\lambda})^5 = (\frac{n}{4n-\lambda})^5$; tum vero patet hos valores evaneſcere tam caſibus, quibus eſt λ numerus integer negativus, quam quibus eſt poſitivus maior quam $4n$.

§. 18.

§. 18. Nunc eadem forma tanquam binomium repraesentata erit $[1+x(1+x+xx+x^3)]^n$, cuius evolutio in genere praebet membrum $(\frac{n}{\alpha})^2 x^\alpha (1+x+xx+x^3)^\alpha$, ubi factor $(1+x+xx+x^3)^\alpha$ continet terminum $(\frac{\alpha}{\beta})^4 x^\beta$, ita ut iunctim habeatur iste terminus $(\frac{n}{\alpha})^2 (\frac{\alpha}{\beta})^4 x^{\alpha+\beta}$. Quare si fuerit $\alpha+\beta=\lambda$, potestatis x^λ ex hoc membro coëfficiens erit $(\frac{n}{\alpha})^2(\frac{\alpha}{\beta})^4$. Iam litteris α et β tribuantur omnes valores, quos recipere possunt, incipiendo ab $\alpha=\lambda$, atque coëfficiens quaesitus erit:

$$(\frac{n}{\lambda})^5 = (\frac{n}{\lambda})^2(\frac{\lambda}{0})^4 + (\frac{n}{\lambda-1})^2(\frac{\lambda-1}{1})^4 + (\frac{n}{\lambda-2})^2(\frac{\lambda-2}{2})^4 + (\frac{n}{\lambda-3})^2(\frac{\lambda-3}{3})^4 + \text{etc.}$$

sicque omnes characteres numero 5 notati per characteres ordinis praecedentis numero 4 notati, una cum characteribus numero 2 notatis definientur.

Conclusio generalis.

§. 19. Ex his iam satis liquet, si proponatur potestas polynomialis in genere ex terminis numero $\theta+1$ constantibus, scilicet $(1+x+xx+x^3 \;.\;.\;.\;.\; x^\theta)^n$, tum termini potestatem x^λ continentis coëfficientem fore $(\frac{n}{\lambda})^{\theta+1}$, qui ita ex characteribus numero θ notatis componetur ut sit

$$(\frac{n}{\lambda})^{\theta+1} = (\frac{n}{\lambda})^2(\frac{\lambda}{0})^\theta + (\frac{n}{\lambda-1})^2(\frac{\lambda-1}{1})^\theta + (\frac{n}{\lambda-2})^2(\frac{\lambda-2}{2})^\theta + \text{etc.}$$

quae forma omnes praecedentes in se complectitur. Si enim incipiamus a valore $\theta=1$, hoc casu habetur potestas binomialis $(1+x)^n$, characteres autem unitate notati oriuntur ex potestate monomiali 1^n, unde oritur $(\frac{n}{0})=1$, reliqui vero omnes in nihilum abeunt. Hinc per casus procedendo habebimus ut sequitur:

$(\frac{n}{\lambda})^2$

$$\left(\frac{n}{\lambda}\right)^2 = \left(\frac{n}{\lambda}\right)^2 \left(\frac{\lambda}{0}\right) = \left(\frac{n}{\lambda}\right)^2,$$

$$\left(\frac{n}{\lambda}\right)^3 = \left(\frac{n}{\lambda}\right)^2 \left(\frac{\lambda}{0}\right)^2 + \left(\frac{n}{\lambda-1}\right)^2 \left(\frac{\lambda-1}{1}\right)^2 + \left(\frac{n}{\lambda-2}\right)^2 \left(\frac{\lambda-2}{2}\right)^2 + \text{etc.}$$

$$\left(\frac{n}{\lambda}\right)^4 = \left(\frac{n}{\lambda}\right)^2 \left(\frac{\lambda}{0}\right)^3 + \left(\frac{n}{\lambda-1}\right)^2 \left(\frac{\lambda-1}{1}\right)^3 + \left(\frac{n}{\lambda-2}\right)^2 \left(\frac{\lambda-2}{2}\right)^3 + \text{etc.}$$

$$\left(\frac{n}{\lambda}\right)^5 = \left(\frac{n}{\lambda}\right)^2 \left(\frac{\lambda}{0}\right)^4 + \left(\frac{n}{\lambda-1}\right)^2 \left(\frac{\lambda-1}{1}\right)^4 + \left(\frac{n}{\lambda-2}\right)^2 \left(\frac{\lambda-2}{2}\right)^4 + \text{etc.}$$

$$\left(\frac{n}{\lambda}\right)^6 = \left(\frac{n}{\lambda}\right)^2 \left(\frac{\lambda}{0}\right)^5 + \left(\frac{n}{\lambda-1}\right)^2 \left(\frac{\lambda-1}{1}\right)^5 + \left(\frac{n}{\lambda-2}\right)^2 \left(\frac{\lambda-2}{2}\right)^5 + \text{etc.}$$

$$\left(\frac{n}{\lambda}\right)^7 = \left(\frac{n}{\lambda}\right)^2 \left(\frac{\lambda}{0}\right)^6 + \left(\frac{n}{\lambda-1}\right)^2 \left(\frac{\lambda-1}{1}\right)^6 + \left(\frac{n}{\lambda-2}\right)^2 \left(\frac{\lambda-2}{2}\right)^6 + \text{etc.}$$

etc. etc.