1801

De insigni usu calculi imaginariorum in calculo integrali

Leonhard Euler

DE INSIGNI VSV

CALCVLI IMAGINARIORVM

IN CALCVLO INTEGRALI.

Auctore

L. EVLERO.

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§. 1.

Cum nuper integrale formulae differentialis

$$\frac{\partial x(1+xx)^2}{(1-xx)\sqrt[4]{(1-6xx+x^4)}}$$

eruissem, quod, posito brevitatis gratia $\sqrt[4]{(1-6xx+x^4)}=v$, inveneram

$$= \tfrac{1}{2} l \frac{1+xx+vv-2vx}{1+xx+vv+2vx} - \text{Arc. tang.} \frac{2vx}{1+xx-vv},$$

affirmare non dubitavi hoc ipsum integrale non nisi ope Calculi Imaginariorum obtineri posse. Tractaveram enim ante istam formulam differentialem:

$$\frac{\partial y(1-yy)^2}{(1+yy)\sqrt[4]{(1+6yy+y^4)^3}}$$

ex qua illa oritur, fi ftatuatur $y = x \sqrt{-1}$. Nunc ergo quoque, poftquam in integrali pofterioris loco y fcripfiffem $x \sqrt{-1}$, integrale fuperioris prodire debebat. Ad hoc autem requirebatur, ut tam logarithmi, quam arcus quantitatum imaginariarum ita evolverentur, ut ad formam generalem $A + B \sqrt{-1}$ reducerentur.

§. 2. Hoc autem phaenomenon in innumeris alii cafibus occurrere poteft, qui ex hac confideratione originen trahunt. Sit Z eiusmodi functio ipfius z, ut formulae diffe rentialis $Z \partial z$ integrale utcunque, five algebraice, five pe logarithmos, five arcus circulares exprimi queat, quod in tegrale per litteram V defignemus, ut fit $\int Z \partial z = V$. Ia loco z fubftituamus quantitatem imaginariam quamcunque quam uti conftat femper tali forma repraefentare lice $z = y (\cos. \theta + \sqrt{-1} \sin. \theta)$, ubi angulum θ ut conftante fpectabimus, ita ut fola y fit variabilis; hoc modo e $\partial z = \partial y (\cos. \theta + \sqrt{-1} \sin. \theta)$; functio autem Z recipi fimilem formam $Z = M + N \sqrt{-1}$, ita ut iam formula tegranda fit

$$\int Z \partial z = \int \partial y (M \cos. \theta - N \sin. \theta) + \sqrt{-1} \int \partial y (M \sin. \theta + N \cos. \theta)$$

cuius prior pars eft realis, pofterior vero imaginaria.

§. 3. Fiat nunc eadem fubftitutio. nempe Z = (cof. $\theta + \sqrt{-1}$ fin. θ) in integrali invento V, unde pari forma imaginaria $P + Q \sqrt{-1}$ prodeat neceffe eft; et q niam partes reales et imaginariae feorfim inter fe compa debent, hinc orientur duae fequentes aequalitates:

$$P = \cos. \theta \int M \partial y - \sin. \theta \int N \partial y;$$

$$Q = \sin. \theta \int M \partial y + \cos. \theta \int N \partial y;$$

unde colligimus

$$\int M\partial y = P\cos.\theta + Q\sin.\theta \text{ et}$$
$$\int N\partial y = Q\cos.\theta - P\sin.\theta$$

hocque modo si inventae fuerint binae quantitates P et Q, ambo integralia tam $\int M\partial y$ quam $\int N\partial y$ exhiberi poterunt.

§. 4. Nisi autem functio proposita Z fuerit admodum simplex, plerumque litterae M et N hinc proveniunt functiones tam complicatae novae variabilis y, ut vix alia via pateat, harum formularum $\int M\partial y$ et $\int N\partial y$ integralia investigandi, praeter hanc ipsam, quam modo indicavimus, et quae per imaginaria procedit; totum ergo negotium huc redit, ut ex invento integrali V ambae quantitates P et Q inde oriundae definiantur. Quatenus igitur istud integrale V partes continet algebraicas, ista operatio nulla laborat difficultate; quando autem logarithmos et arcus circulares involvit, haud exigua sagacitate opus est, ut eius valor in formam $P + Q\sqrt{-1}$ transmutetur, quam ob rem subsidia hic sum traditurus, quibus omnes huiusmodi transformationes perfici queant.

§. 5. Cuncta autem haec subsidia commodissime repeti possunt ex sola formula Arc. tang. $t\sqrt{-1}$: Cum enim eius differentiale fit $= \frac{\partial t\sqrt{-1}}{1-tt}$, huius integrale vicissim erit $= \frac{\sqrt{-1}}{2} l \frac{1+t}{1-t}$, siquidem ita definiatur, ut evanescat posito $t = 0$, quandoquidem hoc casu etiam arcus evanescit. Hinc igitur iam nacti sumus hanc primam reductionem:

$$\text{Arc. tang. } t\sqrt{-1} = \frac{\sqrt{-1}}{2} l \frac{1+t}{1-t}$$

ubi in generali forma $A + B\sqrt{-1}$ est $A = 0$.

§. 6. Ponamus nunc $t = u\sqrt{-1}$, eritque $t\sqrt{-1} = -u$ et Arc. tang. $t\sqrt{-1} = -$ Arc. tang. u, ex quo habebimus

$$-\text{Arc. tang. } u = \frac{\sqrt{-1}}{2}\, l\, \frac{1 + u\sqrt{-1}}{1 - u\sqrt{-1}},$$

unde viciſſim colligitur

$$l\, \frac{1+u\sqrt{-1}}{1-u\sqrt{-1}} = -\frac{2}{\sqrt{-1}}\, \text{Arc. tang. } u = +2\sqrt{-1}\, \text{Arc. tang. } u.$$

Cum porro fit

$$\frac{1 + u\sqrt{-1}}{1 - u\sqrt{-1}} = \frac{(1 + u\sqrt{-1})^2}{1 + uu},\ \text{erit}$$

$$l\, \frac{1+u\sqrt{-1}}{1-u\sqrt{-1}} = 2\, l\, (1 + u\sqrt{-1}) - 2\, l\sqrt{(1 + uu)}$$

$$= 2\sqrt{-1}\, \text{Arc. tang. } u,$$

unde colligitur haec nova reductio:

$$l(1 + u\sqrt{-1}) = l\sqrt{(1 + uu)} + \sqrt{-1}\, \text{Arc. tang. } u.$$

§. 7. Cum igitur omnes formulae imaginariae ad formam $p(\text{coſ. } \alpha + \sqrt{-1}\, \text{ſin. } \alpha)$ reduci queant, erit

$$l\, p(\text{coſ. } \alpha + \sqrt{-1}\, \text{ſin. } \alpha) = l\, p\, \text{coſ. } \alpha + l(1 + \sqrt{-1}\, \text{tang. } \alpha)$$

et poſito $u = \text{tang. } \alpha$, fiet

$$l\,(1 + \text{tang. } \alpha\sqrt{-1}) = -l\, \text{coſ. } \alpha + \alpha\sqrt{-1}.$$

Hinc deducimus iſtam reductionem non minus memorabilem:

$$l\, p\, (\text{coſ. } \alpha + \sqrt{-1}\, \text{ſin. } \alpha) = l\, p + \alpha\sqrt{-1}$$

ideoque

$$l\,(\text{coſ. } \alpha + \sqrt{-1}\, \text{ſin. } \alpha) = \alpha\sqrt{-1}.$$

§. 8. Hinc igitur iam facilem modum impetravimus omnium quantitatum imaginariarum logarithmos ad formam $A + B\sqrt{-1}$ revocandi. At vero pro arcubus imaginariis hanc

hanc solam reductionem adhuc sumus nacti, qua erat Arc. tang. $t\sqrt{-1} = \frac{\sqrt{-1}}{2} l \frac{1+t}{1-t}$. Desideratur ergo adhuc regula huiusmodi arcum imaginarium Arc. tang. $(p + q\sqrt{-1})$ ad formam $A + B\sqrt{-1}$ reducendi. Talis quidem regula iam passim reperitur, quia autem plerumque nimis operose est eruta, sequenti modo eam immediate ex solo principio hic stabilito deducemus.

§. 9. Quaeramus scilicet primo summam huiusmodi binorum arcuum, quae sit

$$\text{Arc. tang. } (p + q\sqrt{-1}) + \text{Arc. tang. } (p - q\sqrt{-1})$$

quam designemus littera R, et cum in genere sit

$$\text{A tang. } a + \text{A tang. } b = \text{Arc. tang. } \frac{a+b}{1-ab},$$

ob $a = p + q\sqrt{-1}$ et $b = p - q\sqrt{-1}$, erit

$$R = \text{Arc. tang. } \frac{2p}{1-pp-qq}.$$

Simili modo ponatur eorundem arcuum differentia

$$\text{Arc. tang. } (p + q\sqrt{-1}) - \text{Arc. tang. } (p - q\sqrt{-1}) = S,$$

et quia

$$\text{Arc. tang. } a - \text{Arc. tang. } b = \text{Arc. tang. } \frac{a-b}{1+ab}, \text{ erit}$$

$$S = \text{Arc. tang. } \frac{2q\sqrt{-1}}{1+pp+qq}.$$

Initio autem vidimus esse

$$\text{Arc. tang. } t\sqrt{-1} = \frac{\sqrt{-1}}{2} l \frac{1+t}{1-t},$$

unde sumto $t = \frac{2q}{1+pp+qq}$, erit

$$S = \frac{\sqrt{-1}}{2} l \frac{(1+q)^2+pp}{(1-q)^2+pp}.$$

§. 10. Inventis igitur binarum illarum formularum tam summa R quam differentia S, utramque seorsim exhibere

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licet; erit enim

$$\text{Arc. tang.}\,(p+q\sqrt{-1}) = \frac{R+S}{2}, \text{ ideoque}$$

$$\text{Arc. tang.}\,(p+q\sqrt{-1}) = \tfrac{1}{2}\,\text{Arc. tang.}\,\frac{2p}{1-pp-qq} + \frac{\sqrt{-1}}{4}\, l\, \frac{(1+q)^2+p}{(1-q)^2+p}$$

similique modo erit

$$\text{Arc. tang.}\,(p-q\sqrt{-1}) = \tfrac{1}{2}\,\text{Arc. tang.}\,\frac{2p}{1-pp-qq} - \frac{\sqrt{-1}}{4}\, l\, \frac{(1+q)^2+p}{(1-q)^2+p}$$

quae quidem ex priore sponte deducitur, loco q scriber $-q$. Hic commode Arc. tang. $\frac{2p}{1-pp-qq}$ in duos resolvere lic quo facto erit

$$\text{Arc. tang.}\,(p+q\sqrt{-1}) = \tfrac{1}{2}\,\text{Arc. tang.}\,\frac{p}{1-q} + \tfrac{1}{2}\,\text{Arc. tang.}\,\frac{p}{1+q} + \frac{\sqrt{-1}}{4}\, l\, \frac{(1+q)^2+pp}{(1-q)^2+pp}.$$

§. 11. Nunc igitur loco $p+q\sqrt{-1}$ substitua formam $r(\text{cos}.\,\alpha+\sqrt{-1}\,\text{sin}.\,\alpha)$, ut fit $p=r\,\text{cos}.\,\alpha$ et $q=r\,\text{si}$ ac reperietur

$$\text{Arc. tang.}\,r(\text{cos}.\,\alpha+\sqrt{-1}\,\text{sin}.\,\alpha) = \tfrac{1}{2}\,\text{Arc. tang.}\,\frac{2r\,\text{cos}.\,\alpha}{1-rr} + \frac{\sqrt{-1}}{4}\, l\, \frac{1+2r\,\text{sin}.\,\alpha+rr}{1-2r\,\text{sin}.\,\alpha+rr}.$$

Per posteriorem autem formam erit quoque

$$\text{Arc. tang.}\,r(\text{cos}.\,\alpha+\sqrt{-1}\,\text{sin}.\,\alpha) = \tfrac{1}{2}\,\text{Arc. tang.}\,\frac{r\,\text{cos}.\,\alpha}{1-r\,\text{sin}.\,\alpha} + \tfrac{1}{2}\,\text{Arc. tang.}\,\frac{r\,\text{cos}.\,\alpha}{1+r\,\text{sin}.\,\alpha} + \frac{\sqrt{-1}}{4}\, l\, \frac{1+2r\,\text{sin}.\,\alpha+rr}{1-2r\,\text{sin}.\,\alpha+rr},$$

§. 12. Hae iam formulae hactenus inventae c subsidia complectuntur, quibus indigebimus ad omnes rithmos et arcus circulares imaginarios resolvendos. F las autem inventas hic simul aspectui exponamus:

$$\text{I.}\quad l(a+b\sqrt{-1}) = l\,a + l\left(1+\frac{b\sqrt{-1}}{a}\right) = l\sqrt{(aa+bb)} + \sqrt{-1}\,\text{Arc. tang.}\,\frac{b}{a},$$

unde deducitur iſta ſaepiſſime occurrens:

$$l\frac{a+b\sqrt{-1}}{a-b\sqrt{-1}} = 2\sqrt{-1}\ \text{Arc. tang.}\ \frac{b}{a}.$$

Porro etiam notetur haec formula:

$$l\,a(\text{coſ.}\ \alpha + \sqrt{-1}\ \text{ſin.}\ \alpha) = l\,a + \alpha\sqrt{-1}.$$

Pro arcubus autem has adepti ſumus formulas:

$$\text{Arc. tang.}\ (a + b\sqrt{-1}) = \tfrac{1}{2}\ \text{Arc. tang.}\ \frac{2a}{1-aa-bb} + \frac{\sqrt{-1}}{4}\, l\frac{(1+b)^2+aa}{(1-b)^2+aa},$$

vel etiam

$$\text{Arc. tang.}\ a(\text{coſ.}\ \alpha + \sqrt{-1}\ \text{ſin.}\ \alpha) = \tfrac{1}{2}\ \text{Arc. tang.}\ \frac{2a\ \text{coſ.}\ \alpha}{1-aa} + \frac{\sqrt{-1}}{4}\, l\frac{1+2a\ \text{ſin.}\ \alpha+aa}{1-2a\ \text{ſin.}\ \alpha+aa}.$$

§. 13. His fundamentis conſtitutis conſideremus caſus, quibus integrale $\int Z\,\partial z$ per logarithmos et arcus circulares exprimi poteſt, id quod ſemper evenit, quando Z eſt functio rationalis ipſius z, tum autem integrale componitur ex huiusmodi partibus:

I. $l(1 \pm z)$;

II. $l(1 - 2z\ \text{coſ.}\ \alpha + zz)$;

III. Arc. tang. $\frac{z\ \text{ſin.}\ \alpha}{1 - z\ \text{coſ.}\ \alpha}$;

vel ſaltem integralia, quae reperiuntur, facile ad tales formas redigi poſſunt. Harum ergo reſolutionem, quando ſtatuitur $z = y(\text{coſ.}\ \theta + \sqrt{-1}\ \text{ſin.}\ \theta)$, nonnullas in ſequentibus problematibus expediemus.

Problema 1.

§. 14. *Hanc formulam logarithmicam* $l(1 \pm z)$, *poſito*

$$z = y\ \text{coſ.}\ \theta + \sqrt{-1}\ \text{ſin.}\ \theta)$$

ad formam generalem $A + B\sqrt{-1}$ *reducere.*

Solutio.

Evolvamus primo formulam

$$l(1+z) = l(1 + y\cos.\theta + y\sqrt{-1}\sin.\theta)$$

et comparatione cum superiore forma generali facta erit

$$a = 1 + y\cos.\theta \text{ et } b = y\sin.\theta,$$

unde colligitur

$$l(1+z) = l\sqrt{(1 + 2y\cos.\theta + yy)} + \sqrt{-1}\,\text{Arc. tang.}\,\frac{y\sin.\theta}{1 + y\cos.\theta}.$$

Hinc autem alter casus $l(1-z)$ sponte derivatur, sumendo y negative, eritque ergo $l(1-z) =$

$$l\sqrt{(1 - 2y\cos.\theta + yy)} - \sqrt{-1}\,\text{Arc. tang.}\,\frac{y\sin.\theta}{1 - y\cos.\theta}.$$

Saepenumero autem in integralibus occurrere solet formula $l\frac{1+z}{1-z}$, cuius ergo valor, posito

$$z = y(\cos.\theta + \sqrt{-1}\sin.\theta),$$

sequenti modo exprimetur:

$$l\frac{1+z}{1-z} = \frac{1}{2}l\frac{1 + 2y\cos.\theta + yy}{1 - 2y\cos.\theta + yy} + \sqrt{-1}\,\text{Arc. tang.}\,\frac{y\sin.\theta}{1 - y\cos.\theta}$$
$$+ \sqrt{-1}\,\text{Arc. tang.}\,\frac{y\sin.\theta}{1 + y\cos.\theta}$$

quare si ambo arcus in unum contrahantur, prodibit

$$l\frac{1+z}{1-z} = \frac{1}{2}l\frac{1 + 2y\cos.\theta + yy}{1 - 2y\cos.\theta + yy} + \sqrt{-1}\,\text{Arc. tang.}\,\frac{2y\sin.\theta}{1 - yy}.$$

Problema 2.

§. 15. *Proposita formula logarithmica*

$$l(1 - 2z\cos.\alpha + zz),$$

si in ea ponatur $z = y(\cos.\theta + \sqrt{-1}\sin.\theta)$, *eius valorem ad formulam postulatam* $A + B\sqrt{-1}$ *reducere.*

Solu

Solutio.

Si hic immediate substitutionem facere vellemus, in calculos satis molestos delaberemur, quos ut evitemus, observasse iuvabit, formulam $1 - 2z\cos.\alpha + zz$ esse productum ex his factoribus:

$$[1 - z(\cos.\alpha + \sqrt{-1}\sin.\alpha)\,(1 - z(\cos.\alpha - \sqrt{-1}\sin.\alpha)],$$

quorum ergo logarithmos invicem addi oportet.

Tractemus ergo primo formulam

$$l[1 - z(\cos.\alpha + \sqrt{-1}\sin.\alpha)],$$

et cum sit

$$y(\cos.\theta + \sqrt{-1}\sin.\theta)(\cos.\alpha + \sqrt{-1}\sin.\alpha) = z(\cos.\alpha + \sqrt{-1}\sin.\alpha)$$

quoniam in genere est

$$(\cos.\beta + \sqrt{-1}\sin.\beta)(\cos.\gamma + \sqrt{-1}\sin.\gamma) = \cos.(\beta+\gamma) + \sqrt{-1}\sin.(\beta+\gamma),$$ erit

$$l[1 - z(\cos.\alpha + \sqrt{-1}\sin.\alpha)] = l[1 - y\cos.(\alpha+\theta) + \sqrt{-1}\sin.(\alpha+\theta)].$$

Hic ergo facta comparatione erit

$$a = 1 - y\cos.(\alpha+\theta) \text{ et } b = -y\sin.(\alpha+\theta);$$

unde eius valor resolutus erit

$$l[1 - z(\cos.\alpha + \sqrt{-1}\sin.\alpha)] = \tfrac{1}{2}l[1 - 2y\cos.(\alpha+\theta) + yy] - \sqrt{-1}\,\text{Arc.tang.}\frac{y\sin.(\alpha+\theta)}{1 - y\cos.(\alpha+\theta)}.$$

Hinc altera formula facile deducitur, sumendo angulum negative, eritque

$$l[1 - z(\cos.\alpha - \sqrt{-1}\sin.\alpha)] = \tfrac{1}{2}l[1 - 2y\cos.(\theta-\alpha) + yy] - \sqrt{-1}\,\text{Arc.tang.}\frac{y\sin.(\theta-\alpha)}{1 - y\cos.(\theta-\alpha)}.$$

 Nunc

Nunc igitur tantum opus eſt, ambos valores, quos modo invenimus, invicem addere, ſicque prodibit haec reductio:

$$l\,(1-2z\cos.\,\alpha+zz)$$
$$=\tfrac{1}{2}\,l\,[1-2y\cos.(\alpha+\theta)+yy]-\sqrt{-1}\,\text{Arc. tg.}\,\frac{y\sin.\,\alpha+\theta}{1-y\cos.(\alpha+\theta)}$$
$$+\tfrac{1}{2}\,l\,[1-2y\cos.(\theta-\alpha)+yy]-\sqrt{-1}\,\text{Arc. tg.}\,\frac{y\sin.(\theta-\alpha)}{1-y\cos.(\theta-\alpha)}$$

Problema 3.

§. 16. *Propoſita formula pro arcu circulari*

$$T=\text{Arc. tang.}\,\frac{z\sin.\,\alpha}{1-z\cos.\,\alpha},$$

ſi in ea ponatur $z=y(\cos.\,\theta+\sqrt{-1}\sin.\,\theta)$, *eius valorem inde reſultantem ad formam* $A+B\sqrt{-1}$ *revocare.*

Solutio.

Quia hic in numeratore et denominatore imaginaria occurrunt, ad ſimpliciorem formam perveniemus, ſi utrinque addamus Arc. α, ſive Arc. tang. $\frac{\sin.\,\alpha}{\cos.\,\alpha}$; ſic enim erit

$$T+\alpha=\text{Arc. tang.}\,\frac{\sin.\,\alpha}{\cos.\,\alpha-z}=90^\circ-\text{Arc. tang.}\,\frac{\cos.(\alpha-z)}{\sin.\,\alpha}$$

ideoque

$$T=90^\circ-\alpha-\text{Arc. tang.}\,\frac{\cos.(\alpha-z)}{\sin.\,\alpha}.$$

Iam in hac poſtrema formula ponamus

$$z=y(\cos.\,\theta+\sqrt{-1}\,\sin.\,\theta),\text{ fietque}$$

$$\text{Arc. tang.}\,\frac{\cos.(\alpha-z)}{\sin.\,\alpha}=\text{Arc. tang.}\,\frac{\cos.\,\alpha-y\cos.\,\theta-y\sqrt{-1}\sin}{\sin.\,\alpha}$$

quae expreſſio comparata cum formula generali

Ar

Arc. tang. $(a + b\sqrt{-1})$ dat

$$a = \frac{\cos. \quad - y \cos.\theta}{\sin.\alpha} \text{ et } b = -\frac{y \sin.\theta}{\sin.\alpha}.$$

Hinc igitur erit

$$1 - aa - bb = -\frac{\cos.2\alpha + 2y\cos.\alpha\cos.\theta - yy}{\sin.\alpha^2}$$

ideoque

$$\frac{2a}{1 - aa - bb} = \frac{\sin.2\alpha - 2y\sin.\alpha\cos.\theta}{-\cos.2\alpha + 2y\cos.\alpha\cos.\theta - yy}, \text{ ergo}$$

$$\text{Arc. tang.} \frac{2a}{1 - aa - bb} = -\text{Arc. tang.} \frac{\sin.2\alpha - 2y\sin.\alpha\cos.\theta}{\cos.2\alpha - 2y\cos.\alpha\cos.\theta + yy}.$$

Iam pro parte imaginaria erit

$$1 + aa + bb = \frac{1 - 2y\cos.\alpha\cos.\theta + yy}{\sin.\alpha^2},$$

unde colligitur numerator

$$(1 + b)^2 + aa = \frac{1 - 2y\cos.(\theta - \alpha) + yy}{\sin.\alpha^2}$$

et denominator

$$(1 - b)^2 + aa = \frac{1 - 2y\cos.(\alpha + \theta) + yy}{\sin.\alpha^2},$$

sicque pars imaginaria erit

$$\frac{\sqrt{-1}}{4} l \frac{(1 + b)^2 + aa}{(1 - b)^2 + aa} = \frac{\sqrt{-1}}{4} l \frac{1 - 2y\cos.(\theta - \alpha) + yy}{1 - 2y\cos.(\alpha + \theta) + yy},$$

quamobrem hinc colligimus

$$\text{Arc. tang.} \frac{\cos.(\; - z)}{\sin.\;} = -\tfrac{1}{2}\text{Arc. tang.} \frac{\sin.2\alpha - 2y\sin.\alpha\cos.\theta}{\cos.2\alpha - 2y\cos.\alpha\cos.\theta + yy}$$
$$+ \frac{\sqrt{-1}}{4} l \frac{1 - 2y\cos.(\theta - \;) + yy}{1 - 2y\cos.(\alpha + \theta) + yy}.$$

His iam formulis inventis reductio ipsius formulae propositae ita se habebit:

$$\text{Arc. tang.} \frac{z\sin.\alpha}{1 - z\cos.\alpha} = 90^0 - \alpha + \tfrac{1}{2}\text{Arc. tg.} \frac{\sin.2\alpha - 2y\sin.\alpha\cos.\theta}{\cos.2\alpha - 2y\cos.\alpha\cos.\theta + yy}$$
$$- \frac{\sqrt{-1}}{4} l \frac{1 - 2y\cos.(\theta - \alpha) + yy}{1 - 2y\cos.(\alpha + \theta) + yy}.$$

Hae

Hae iam reductiones haud difficulter ad omnes formulas accommodari poterunt, quod quo clarius appareat, sequens exemplum adiungamus.

Integratio Formulae differentialis

$$\frac{\partial x}{(3 - xx)\sqrt[3]{(1 - 3xx)}} = \partial V.$$

§. 17. Quoniam nondum apparet, quomodo hanc ipsam formulam tractari conveniat, eam ad sequentem formam imaginariam, ponendo $x = z\sqrt{-1}$, reducamus, ut fi

$$\partial V = \frac{\partial z\sqrt{-1}}{(3 + zz)\sqrt[3]{(1 + 3zz)}},$$

quae forma iam ita comparata deprehenditur, ut per praecepta non ita pridem tradita ad integrationem perduci posfit, eius ergo resolutionem sequenti modo expedire poterimus.

§. 18. Ponamus igitur

$$\frac{\partial z}{(3 + zz)\sqrt[3]{(1 + 3zz)}} = \partial T,$$

ut fit $V = T\sqrt{-1}$. Hanc autem formam sequenti mod repraesentemus: $\partial T = \frac{z\,\partial z}{(3z + z^3)\sqrt[3]{(1 + 3zz)}}$, ubi brevitatis gratia statuamus $\sqrt[3]{(1 + 3zz)} = v$, ut fit

∂

$$\partial T = \frac{z \partial z}{v(3z + z^3)},$$

hicque secundum nostra praecepta statuamus $p = \frac{1+z}{v}$ et $q = \frac{1-z}{v}$, unde fit $p + q = \frac{2}{v}$ et $p - q = \frac{2z}{v}$, hincque $z = \frac{p-q}{p+q}$, ideoque differentiando

$$\partial z = \frac{2q\partial p - 2p\partial q}{(p+q)^2} = \tfrac{1}{2} v v (q \partial p - p \partial q),$$

quo valore substituto impetramus

$$\partial T = \frac{v z (q \partial p - p \partial q)}{2(3z + z^3)}.$$

§. 19. Cum iam sit $1 + z = pv$ et $1 - z = qv$, erit primo $2z = v(p - q)$, tum vero summa cuborum dabit

$$(1+z)^3 + (1-z)^3 = v^3(p^3 + q^3) = 2 + 6zz.$$

Quoniam igitur posuimus $\sqrt[3]{(1 + 3zz)} = v$, erit $v^3 = 1 + 3zz$; quam ob rem habebimus $p^3 + q^3(1 + 3zz) = 2 + 6zz$, consequenter $p^3 + q^3 = 2$. Denique vero differentia cuborum praebet $(p^3 - q^3)v^3 = 6z + 2z^3$; unde patet esse $3z + z^3 = \frac{1}{2}(p^3 - q^3)v^3$; at vero differentia quadratorum dat $(pp - qq)vv = 4z$, unde fit $z = \frac{1}{4}vv(pp - qq)$.

§. 20. Substituantur nunc isti valores loco z et $3z + z^3$, atque nostra formula evadet

$$\partial T = \frac{(pp - qq)(q\partial p - p\partial q)}{4(p^3 - q^3)},$$

ubi ergo tantum binae litterae p et q occurrunt, quae ita a se invicem pendent, vt sit $p^3 + q^3 = 2$, ideoque differentiando $pp\partial p + qq\partial q = 0$, consequenter sive $\partial p = \frac{-qq\partial q}{pp}$, sive $\partial q = \frac{-pp\partial p}{qq}$.

§. 21.

§. 21. Dividatur nunc haec forma in duas partes, ponendo

$$\frac{pp(q\partial p - p\partial q)}{p^3 - q^3} = \partial P \text{ et } \frac{qq(q\partial p - p\partial q)}{p^3 - q^3} = \partial Q,$$

vt fit $\partial T = \frac{1}{4}\partial P - \frac{1}{4}\partial Q$, hicque ſtatim patet, ſi in priore formula loco $pp\partial p$ ſcribatur $-qq\partial q$, tum prodire

$$\partial P = -\frac{\partial q(p^3 + q^3)}{p^3 - q^3}.$$

Quia vero eſt $p^3 + q^3 = 2$, ideoque $p^3 = 2 - q^3$, elementum ∂P per ſolam litteram q ita exprimetur, ut fit

$$\partial P = \frac{\partial q}{1 - q^3}.$$

§. 22. Simili modo ſi in altera formula ∂Q loco $qq\partial q$ ſcribatur $-pp\partial p$, prodibit $\partial Q = \frac{\partial p(p^3 + q^3)}{p^3 - q^3}$, quae ergo ob relationem inter p et q ſuppeditat hanc formulam:

$$\partial Q = \frac{\partial p}{p^3 - 1} = \frac{-\partial p}{1 - p}.$$

His igitur coniunctis erit

$$4\partial T = \frac{\partial p}{1 - p^3} - \frac{\partial q}{1 - q^3},$$

ſicque totum negotium perductum eſt ad duas formulas differentiales rationales, quas ergo per logarithmos et arcus circulares integrare licet.

§. 23. Ad haec integralia invenienda ſtatuatur

$$\frac{1}{1 - p^3} = \frac{F}{1 - p} + \frac{G}{1 + p + pp},$$

ubi notetur fore

$$F = \frac{1 - p}{1 - p^3} = \frac{1}{1 - p + pp},$$

poſito $1 - p = 0$, ſive $p = 1$, unde fit $F = \frac{1}{3}$, tum vero erit

$$G = \frac{1 + p - pp}{1 - p^3} = \frac{1}{1 - p},$$

po-

posito $1+p+pp=0$. Hic iam ut littera p ex denominatore tolli queat, multiplicetur supra et infra per $2+p$, fiet $G=\frac{2+p}{2-p-pp}$. Quia igitur est $p+pp=-1$, erit $G=\frac{2+p}{3}$; quam ob rem habebimus $\frac{3\partial p}{1-p^3}=\frac{\partial p}{1-p}+\frac{(2+p)\partial p}{1+p+pp}$. Constat autem esse $\int\frac{\partial p}{1-p}=-l(1-p)$ et

$$\int\frac{p\partial p+2\partial p}{pp+p+1}=\frac{1}{2}l(1+p+pp)+\frac{3}{2}\int\frac{\partial p}{1+p+pp}.$$

Novimus autem in genere esse

$$\int\frac{\partial p}{1-2p\,cos.\,\alpha+pp}=\frac{1}{sin.\,\alpha}\text{ Arc. tang. }\frac{p\,sin.\,\alpha}{1-p\,cos.\,\alpha},$$

unde patet sumi debere $\alpha=120°$, et ob sin. $\alpha=\frac{\sqrt{3}}{2}$, erit

$$\int\frac{\partial p}{1+p+pp}=\frac{2}{\sqrt{3}}\text{ Arc. tang. }\frac{p\sqrt{3}}{2+p},$$

sicque totum integrale erit

$$3\int\frac{\partial p}{1-p^3}=-l(1-p)+\frac{1}{2}l(1+p+pp)$$
$$+\sqrt{3}\text{ Arc. tang. }\frac{p\sqrt{3}}{2+p},$$

similique modo erit

$$3\int\frac{\partial q}{1-q^3}=-l(1-q)+\frac{1}{2}l(1+q+qq)$$
$$+\sqrt{3}\text{ Arc. tang. }\frac{q\sqrt{3}}{3+q}.$$

§. 24. His igitur inventis erit

$$12\,T=l\frac{1-q}{1-p}+\frac{1}{2}l\frac{1+p+pp}{1+q+qq}$$
$$+\sqrt{3}\text{ Arc. tang. }\frac{p\sqrt{3}}{2+p}-\sqrt{3}\text{ Arc. tang. }\frac{q\sqrt{3}}{2+q}.$$

Quare cum sit $p=\frac{1+z}{v}$ et $q=\frac{1-z}{v}$, habebimus

$$T=\frac{1}{12}l\frac{v-1+z}{v-1-z}+\frac{1}{24}l\frac{vv+v(1+z)+(1+z)^2}{vv+v(1-z)+(1-z)^2}$$
$$+\frac{1}{4\sqrt{3}}\text{ Arc. tang. }\frac{(1+z)\sqrt{3}}{2v+1+z}-\frac{1}{4\sqrt{3}}\text{ Arc. tang. }\frac{(1-z)\sqrt{3}}{2v+1-z}.$$

§. 25. Nunc secundum praecepta supra exposita, ubi sumsimus $z = y(\text{cos.}\,\theta + \sqrt{-1}\,\text{sin.}\,\theta)$, quia est $x = z\sqrt{-1}$, erit

$$z = -x\sqrt{-1} = y(\text{cos.}\,\theta + \sqrt{-1}\,\text{sin.}\,\theta);$$

unde patet statui debere $\theta = 90°$ et $y = -x$, hocque notato, ut superiores reductiones ad nostrum casum propius accommodemus, ibi ubique loco z et y scribamus $\frac{z}{s}$ et $\frac{y}{s}$, quo facto reductiones erunt

$$\text{I. } l(s+z) = +\tfrac{1}{2}l(ss+2sy\,\text{cos.}\,\theta+yy) + \sqrt{-1}\,\text{Arc. tg.}\,\frac{y\,\text{sin.}\,\theta}{s+y\,\text{cos.}\,\theta}.$$

$$\text{II. } l(ss-2sz\,\text{cos.}\,\alpha+zz) = \tfrac{1}{2}l[ss-2sy\,\text{cos.}(\alpha+\theta)+yy] - \sqrt{-1}\,\text{Arc. tang.}\,\frac{y\,\text{sin.}(\alpha+\theta)}{s-y\,\text{cos.}(\alpha+\theta)}$$
$$+\tfrac{1}{2}l(ss-2sy\,\text{cos.}(\theta-\alpha)+yy) - \sqrt{-1}\,\text{Arc. tang.}\,\frac{y\,\text{sin.}(\theta-\alpha)}{s-y\,\text{cos.}(\theta-\alpha)}.$$

$$\text{III. Arc. tang.}\,\frac{z\,\text{sin.}\,\alpha}{s-z\,\text{cos.}\,\alpha} = 90° - \alpha + \tfrac{1}{9}\,\text{Arc. tg.}\,\frac{ss\,\text{sin.}\,2\alpha - 2sy\,\text{sin.}\,\alpha\,\text{cos.}\,\theta}{ss\,\text{cos.}\,2\alpha - 2sy\,\text{cos.}\,\alpha\,\text{cos.}\,\theta + yy}$$
$$-\frac{\sqrt{-1}}{4}\,l\,\frac{ss-2sy\,\text{cos.}(\theta-\alpha)+yy}{ss-2sy\,\text{cos.}(\alpha+\theta)+yy}.$$

§. 26. Iam haec praecepta ad singulas partes integralis inventi applicemus, ac primo quidem pro formula $l(v-1+z)$ erit $s = v-1$, et ob $y = -x$ et $\theta = 90°$ colligitur

$$\text{I. } l(v-1+z) = \tfrac{1}{2}l[(v-1)^2+xx] - \sqrt{-1}\,\text{Arc. tang.}\,\frac{x}{v-1}.$$

$$\text{II. } l(v-1-z) = \tfrac{1}{2}l[(v-1)^2+xx] + \sqrt{-1}\,\text{Arc. tang.}\,\frac{x}{v-1}.$$

III. Pro formula $l[vv+v(1+z)+(1+z)^2]$ patet fore

$$ss = vv+v+1, \text{ seu } s = \sqrt{(vv+v+1)};$$
$$\text{cos.}\,\alpha = -\frac{v-2}{2\sqrt{(vv+v+1)}} \text{ et sin.}\,\alpha = \frac{v\sqrt{3}}{2\sqrt{(vv+v+1)}},$$

unde ob $\theta = 90°$ erit

$$\text{cos.}(\alpha+\theta) = -\text{sin.}\,\alpha;\ \text{cos.}(\theta-\alpha) = \text{sin.}\,\alpha;$$
$$\text{sin.}(\alpha+\theta) = \text{cos.}\,\alpha \text{ et sin.}(\theta-\alpha) = \text{cos.}\,\alpha;$$

quo

quo observato erit:

$$l[vv+v(1+z)+(1+z)^2]=\tfrac{1}{2}l[vv+v+1-vx\sqrt{3}+xx]$$
$$+\tfrac{1}{2}l[vv+v+1+vx\sqrt{3}+xx]$$
$$-\sqrt{-1}\,\text{Arc. tang.}\frac{x(v+2)}{2(vv+v+1)-vx\sqrt{3}}$$
$$-\sqrt{-1}\,\text{Arc. tang.}\frac{x(v+2)}{2(vv+v+1)+vx\sqrt{3}},$$

hinc simul mutato signo litterarum z et x erit

$$l[vv+v(1-z)+(1-z)^2]=\tfrac{1}{2}l[vv+v+1+vx\sqrt{3}+xx]$$
$$+\tfrac{1}{2}l[vv+v+1-vx\sqrt{3}+xx]$$
$$+\sqrt{-1}\,\text{Arc. tang.}\frac{x(v+2)}{2(vv+v+1)+vx\sqrt{3}}$$
$$+\sqrt{-1}\,\text{Arc. tang.}\frac{x(v+2)}{2(vv+v+1)-vx\sqrt{3}}.$$

§. 27. Nunc porro pro Arc. tang. $\frac{(1+z)\sqrt{3}}{2v+1+z}$, quae forma in regula nostra non continetur, notetur esse

$$\text{Arc. tg.}\frac{(1+z)\sqrt{3}}{2v+1+z}=\text{Arc. tg.}\frac{\sqrt{3}}{2v+1}+\text{Arc. tg.}\frac{vz\sqrt{3}}{2(vv+v+1)+(v+2)z},$$

qui postremus valor comparatus cum Arc. tang. $\frac{z\,\text{fin.}\,\alpha}{1-z\,\text{cof.}\,\alpha}$ iterum praebet fin. $\alpha=\frac{v\sqrt{3}}{2\sqrt{(vv+v+1)}}$, sicque anguli $\alpha+\theta$ et $\theta-\alpha$ manent iidem, ut ante; unde reductio praebet

$$\text{Arc. tang.}\frac{vz\sqrt{3}}{2(vv+v+1)+(v+2)z}=-\tfrac{1}{2}\text{Arc. tg.}\frac{v(v+2)\sqrt{3}}{-vv+2v+2+2xx}$$
$$-\frac{\sqrt{-1}}{4}l\frac{vv+v+1+vx\sqrt{3}+xx}{vv+v+1-vx\sqrt{3}+xx}.$$

Nacti ergo sumus has reductiones:

$$\text{Arc. tg.}\frac{(1+z)\sqrt{3}}{2v+1+z}=\text{Arc. tg.}\frac{\sqrt{3}}{2v+1}-\tfrac{1}{2}\text{Arc. tg.}\frac{v(v+2)\sqrt{3}}{-vv+2v+2+2xx}$$
$$-\frac{\sqrt{-1}}{4}l\frac{vv+v+1+vx\sqrt{3}+xx}{vv+v+1-vx\sqrt{3}+xx}\ \text{et}$$
$$\text{Arc. tg.}\frac{(1-z)\sqrt{3}}{2v+1-z}=\text{Arc. tg}\frac{\sqrt{3}}{2v+1}-\tfrac{1}{2}\text{Arc. tg.}\frac{v(v+2)\sqrt{3}}{-vv+2v+2+2xx}$$
$$-\frac{\sqrt{-1}}{4}l\frac{vv+v+1-vx\sqrt{3}+xx}{vv+v+1+vx\sqrt{3}+xx}.$$

§. 28. Quodfi iam omnes has partes rite colligamus, reperiemus

$$T = -\frac{\sqrt{-1}}{6}\,\text{Arc. tang.}\,\frac{x}{v-1} - \frac{\sqrt{-1}}{12}\,\text{Arc. tg.}\,\frac{x(v+2)}{2(vv+v+1)-vx\sqrt{3}}$$
$$-\frac{\sqrt{-1}}{12}\,\text{Arc. tg.}\,\frac{x(v+2)}{2(vv+v+1)+vx\sqrt{3}} - \frac{\sqrt{-1}}{8\sqrt{3}}\,l\,\frac{vv+v+1+vx\sqrt{3}+xx}{vv+v+1-vx\sqrt{3}+xx}.$$

§. 29. Hic igitur commode ufu venit, ut omnes partes reales fe mutuo deftruxerint, imaginariae vero duplicatae prodierint, quemadmodum natura rei manifefto poftulat. Cum igitur integrale quaefitum fit $V = T\sqrt{-1}$, nunc eius valor pulcherrime prodit realis, quocirca perducti fumus ad hanc integrationem:

$$\int \frac{\partial x}{(3-xx)\sqrt[3]{(1-3xx)}} = +\tfrac{1}{6}\,\text{Arc. tang.}\,\frac{x}{v-1}$$
$$+\tfrac{1}{12}\,\text{Arc. tang.}\,\frac{x(v+2)}{2(vv+v+1)-vx\sqrt{3}}$$
$$+\tfrac{1}{12}\,\text{Arc. tang.}\,\frac{x(v+2)}{2(vv+v+1)+vx\sqrt{3}}$$
$$+\frac{1}{8\sqrt{3}}\,l\,\frac{(vv+v+1+vx\sqrt{3}+xx)}{vv+v+1-vx\sqrt{3}+xx},$$

ubi eft $v = \sqrt[3]{(1-3xx)}$. Hae formulae aliquanto fir pliciores reddi poffunt, confiderando quod fit $1-v^3 = 3xx$ ideoque $1+v+vv = \frac{3xx}{1-v}$, unde cum plures fubftitution adhiberi queant, iis hic non immorandum cenfemus, f contenti effe poffumus, iftius formulae differentialis integra eruiffe, ad quod per nullam aliam methodum aditus p tere videtur.

§. 3

§. 30. Caeterum calculus facilior evadet, ſi in integrali primum invento ambo arcus per $\frac{1}{4\sqrt{3}}$ multiplicati in unum colligantur: inde enim prodit $\frac{1}{4\sqrt{3}}$ Arc. tg. $\frac{vz\sqrt{3}}{vv+v+1-zz}$. Hic iam ſtatim ponatur $z = -x\sqrt{-1}$, ut formula prodeat $-\frac{1}{4\sqrt{3}}$ Arc. tang. $\frac{vx\sqrt{3}\sqrt{-1}}{vv+v+1+xx}$, quae comparata cum canonica Arc. tang. $t\sqrt{-1} = \frac{\sqrt{-1}}{2}\,l\,\frac{1+t}{1-t}$, ob $t = \frac{vx\sqrt{3}}{vv+v+1+xx}$, ſtatim perducit ad hanc formulam reductam:

$$\frac{-\sqrt{-1}}{8\sqrt{3}}\,l\,\frac{vv+v+1+vx\sqrt{3}+xx}{vv+v+1-vx\sqrt{3}+xx}.$$

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