1798

De novo genere serierum rationalium et valde convergentium, quibus ratio peripheriae ad diametrum exprimi potest

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DE NOVO GENERE SERIERVM RATIONALIVM ET VALDE CONVERGENTIVM QVIBVS RATIO PERIPHERIAE AD DIAMETRVM EXPRIMI POTEST.

Auctore
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Conventui exhibita die 17 Iunii 1779.

§. 1.

Principium, unde hae series sunt deductae, situm est in hac formula binomiali: $4 + x^4$, quam constat involvere hos duos factores rationales: $2 + 2x + xx$ et $2 - 2x + xx$. Hinc enim statim sequitur hanc formulam integralem: $\int \frac{\partial x (2 + 2x + xx)}{4 + x^4}$, quam signo $\odot$ indicemus, reduci ad hanc: $\odot = \int \frac{\partial x}{2 - 2x + xx}$, cuius integrale, ita sumtum, ut evanescat posito $x = 0$, est A tang. $\frac{x}{2 - x}$. Vbi observetur, casu $x = 1$ esse $\odot = \frac{\pi}{4}$; at vero casu $x = \frac{1}{2}$ erit $\odot =$ A tang. $\frac{1}{3}$, tum vero casu $x = \frac{1}{4}$ erit $\odot =$ A tang. $\frac{1}{7}$. Notum autem est esse

$$2 \text{ A tang. } \tfrac{1}{3} + \text{A tang. } \tfrac{1}{7} = \text{A tang. } 1 = \tfrac{\pi}{4}.$$

§. 2.

§. 2. Cum igitur formula integralis illa signo ⊙ indicata tribus constet partibus, singulas seorsim evolvamus, quas brev. gr. sequentibus characteribus insigniamus:

I. $\int \frac{\partial x}{4+x^4} =$ ♄; II. $\int \frac{x\,\partial x}{4+x^4} =$ ♃; III. $\int \frac{xx\,\partial x}{4+x^4} =$ ♂;

ita ut sit

⊙ = 2 ♄ + 2 ♃ + ♂ = A tang. $\frac{x}{2-x}$.

Nunc igitur istas tres formulas integrales more solito in series infinitas evolvamus, inde formandas, quod sit

$$\frac{1}{4+x^4} = \frac{1}{4}\left(1 - \frac{x^4}{4} + \frac{x^8}{4^2} - \frac{x^{12}}{4^3} + \frac{x^{16}}{4^4} - \text{etc.}\right)$$

§. 3. Quod si iam primo istam seriem ducamus in ∂x et integremus, prima formula ♄ per sequentem seriem exprimetur:

♄ $= \frac{x}{4}\left[1 - \frac{1}{5}\cdot\frac{x^4}{4} + \frac{1}{9}\left(\frac{x^4}{4}\right)^2 - \frac{1}{13}\left(\frac{x^4}{4}\right)^3 + \text{etc.}\right].$

At vero illa series ducta in $x\,\partial x$ et integrata dabit

♃ $= \frac{xx}{8}\left[1 - \frac{1}{3}\cdot\frac{x^4}{4} + \frac{1}{5}\left(\frac{x^4}{4}\right)^2 - \frac{1}{7}\left(\frac{x^4}{4}\right)^3 + \text{etc.}\right].$

Denique eadem series ducta in $xx\,\partial x$ et integrata praebet

♂ $= \frac{x^3}{4}\left[\frac{1}{3} - \frac{1}{7}\frac{x^4}{4} + \frac{1}{11}\left(\frac{x^4}{4}\right)^2 - \frac{1}{15}\left(\frac{x^4}{4}\right)^3 + \text{etc.}\right].$

§. 4. Cum igitur sit ⊙ = 2 ♄ + 2 ♃ + ♂, evolvamus seorsim casus initio memoratos, quibus est vel $x = 1$, vel $x = \frac{1}{2}$, vel $x = \frac{1}{4}$, quorum primo est $\frac{x^4}{4} = \frac{1}{4}$; secundo vero est $\frac{x^4}{4} = \frac{1}{64}$; tertio vero $\frac{x^4}{4} = \frac{1}{1024}$; unde patet binos casus posteriores maxime convergere; quin etiam ipsa prima, cuius termini in ratione quadrupla decrescunt, iam magis convergit quam series Leibnitiana, sumto arcu cuius tangens est

est $\frac{1}{\sqrt{3}}$, praeterquam quod hic calculus nulla irrationalitate perturbatur.

Evolutio casus primi.

quo $x = 1$ et ⊙ $=$ A tang. $1 = \frac{\pi}{4}$.

§. 5. Cum igitur hic sit $\frac{x^4}{4} = \frac{1}{4}$, tres nostrae series principales ♄, ♃, ⊙ sequenti modo procedent:

♄ $= \frac{1}{4}\left[1 - \frac{1}{5}\cdot\frac{1}{4} + \frac{1}{9}\left(\frac{1}{4}\right)^2 - \frac{1}{13}\left(\frac{1}{4}\right)^3 + \frac{1}{17}\left(\frac{1}{4}\right)^4 - \text{etc.}\right]$

♃ $= \frac{1}{8}\left[1 - \frac{1}{3}\cdot\frac{1}{4} + \frac{1}{5}\left(\frac{1}{4}\right)^2 - \frac{1}{7}\left(\frac{1}{4}\right)^3 + \frac{1}{9}\left(\frac{1}{4}\right)^4 - \text{etc.}\right]$

♂ $= \frac{1}{4}\left[\frac{1}{3} - \frac{1}{7}\cdot\frac{1}{4} + \frac{1}{11}\left(\frac{1}{4}\right)^2 - \frac{1}{15}\left(\frac{1}{4}\right)^3 + \frac{1}{19}\left(\frac{1}{4}\right)^4 - \text{etc.}\right]$

§. 6. Cum igitur sit ⊙ $=$ 2 ♄ $+$ 2 ♃ $+$ ♂ $= \frac{\pi}{4}$, per 4 multiplicando valor ipsius π per sequentes tres series exprimitur

$$\pi = \begin{cases} 2\left(1 - \frac{1}{5}\cdot\frac{1}{4} + \frac{1}{9}\cdot\frac{1}{4^2} - \frac{1}{13}\cdot\frac{1}{4^3} + \frac{1}{17}\cdot\frac{1}{4^4} - \text{etc.}\right) \\ 1\left(1 - \frac{1}{3}\cdot\frac{1}{4} + \frac{1}{5}\cdot\frac{1}{4^2} - \frac{1}{7}\cdot\frac{1}{4^3} + \frac{1}{9}\cdot\frac{1}{4^4} - \text{etc.}\right) \\ 1\left(\frac{1}{3} - \frac{1}{7}\cdot\frac{1}{4} + \frac{1}{11}\cdot\frac{1}{4^2} - \frac{1}{15}\cdot\frac{1}{4^3} + \frac{1}{19}\cdot\frac{1}{4^4} - \text{etc.}\right) \end{cases}$$

§. 7. Ex his certe ternis seriebus ratio peripheriae ad diametrum multo minore labore computari potuisset quam ex serie *Leibnitiana*, qua Auctores illi meritissimi *Sharp*; *Machin* & *Lagny* sunt usi, quorum primus valorem ipsius π in fractione decimali usque ad 72 figuras, secundus ad 100, ac postremus adeo usque ad 128 determinavit. At vero sequentes casus multo magis istum laborem sublevare possent.

Evo-

Evolutio casus secundi, quo $x = \frac{1}{2}$.

§. 8. Hoc igitur casu erit $\frac{x^4}{4} = \frac{1}{64}$, unde tres illae series sequenti modo referentur:

$$♄ = \tfrac{1}{8}\left(1 - \tfrac{1}{5}\cdot\tfrac{1}{64} + \tfrac{1}{9}\cdot\tfrac{1}{64^2} - \tfrac{1}{13}\cdot\tfrac{1}{64^3} + \text{etc.}\right)$$

$$♃ = \tfrac{1}{32}\left(1 - \tfrac{1}{3}\cdot\tfrac{1}{64} + \tfrac{1}{5}\cdot\tfrac{1}{64^2} - \tfrac{1}{7}\cdot\tfrac{1}{64^3} + \text{etc.}\right)$$

$$♂ = \tfrac{1}{32}\left(\tfrac{1}{3} - \tfrac{1}{7}\cdot\tfrac{1}{64} + \tfrac{1}{11}\cdot\tfrac{1}{64^2} - \tfrac{1}{15}\cdot\tfrac{1}{64^3} + \text{etc.}\right)$$

§. 9. Cum igitur $2♄ + 2♃ + ♂ = \text{A tang.}\,\frac{1}{3}$, erit

$$\text{A tang.}\,\tfrac{1}{3} = \left\{\begin{array}{l} \tfrac{1}{4}\left(1 - \tfrac{1}{5}\cdot\tfrac{1}{64} + \tfrac{1}{9}\cdot\tfrac{1}{64^2} - \tfrac{1}{13}\cdot\tfrac{1}{64^3} + \text{etc.}\right) \\ \tfrac{1}{16}\left(1 - \tfrac{1}{3}\cdot\tfrac{1}{64} + \tfrac{1}{5}\cdot\tfrac{1}{64^2} - \tfrac{1}{7}\cdot\tfrac{1}{64^3} + \text{etc.}\right) \\ \tfrac{1}{32}\left(\tfrac{1}{3} - \tfrac{1}{7}\cdot\tfrac{1}{64} + \tfrac{1}{11}\cdot\tfrac{1}{64^2} - \tfrac{1}{15}\cdot\tfrac{1}{64^3} + \text{etc.}\right) \end{array}\right\}$$

Etsi autem hic tres computandae sunt series, tamen, quia singulae secundum eandem rationem 1 : 64 decrescunt, laborem mirum in modum contrahere licebit.

Evolutio casus tertii, quo $x = \frac{1}{4}$.

§. 10. Cum igitur hic fit $\frac{x^4}{4} = \frac{1}{1024}$, series nostrae tres principales ita se habebunt:

$$♄ = \tfrac{1}{16}\left(1 - \tfrac{1}{5}\cdot\tfrac{1}{1024} + \tfrac{1}{9}\cdot\tfrac{1}{1024^2} - \tfrac{1}{13}\cdot\tfrac{1}{1024^3} + \text{etc.}\right)$$

$$♃ = \tfrac{1}{128}\left(1 - \tfrac{1}{3}\cdot\tfrac{1}{1024} + \tfrac{1}{5}\cdot\tfrac{1}{1024^2} - \tfrac{1}{7}\cdot\tfrac{1}{1024^3} + \text{etc.}\right)$$

$$♂ = \tfrac{1}{256}\left(\tfrac{1}{3} - \tfrac{1}{7}\cdot\tfrac{1}{1024} + \tfrac{1}{11}\cdot\tfrac{1}{1024^2} - \tfrac{1}{15}\cdot\tfrac{1}{1024^3} + \text{etc.}\right)$$

§. 11. Cum igitur 2 ♄ + 2 ♃ + ♂ = A tang. $\frac{1}{7}$, erit his feriebus debite iunctis:

$$\text{A tang.}\ \tfrac{1}{7} = \left\{ \begin{array}{l} \frac{1}{8}\left(1 - \frac{1}{5}\cdot\frac{1}{1024} + \frac{1}{9}\cdot\frac{1}{1024^2} - \frac{1}{13}\cdot\frac{1}{1024^3} + \text{etc.}\right) \\ \frac{1}{64}\left(1 - \frac{1}{3}\cdot\frac{1}{1024} + \frac{1}{5}\cdot\frac{1}{1024^2} - \frac{1}{7}\cdot\frac{1}{1024^2} + \text{etc.}\right) \\ \frac{1}{256}\left(\frac{1}{3} - \frac{1}{7}\cdot\frac{1}{1024} + \frac{1}{11}\cdot\frac{1}{1024^2} - \frac{1}{15}\cdot\frac{1}{1024^3} + \text{etc.}\right) \end{array} \right\}$$

Applicatio binorum cafuum pofteriorum ad peripheriam circuli per feries maxime convergentes exprimendam.

§. 12. Cum fit, uti iam obfervavimus,

$\frac{\pi}{4} = 2\,\text{A tang.}\,\frac{1}{3} + \text{A tang.}\,\frac{1}{7}$ erit $\pi = 8\,\text{A tang.}\,\frac{1}{3} + 4\,\text{A tang.}\,\frac{1}{7}$ feriebus fupra inventis fubftitutis valor ipfius π per fex fequentes feries coniunctim exprimetur:

$$\pi = \left\{ \begin{array}{l} 2\left(1 - \frac{1}{5}\cdot\frac{1}{64} + \frac{1}{9}\cdot\frac{1}{64^2} - \frac{1}{13}\cdot\frac{1}{64^3} + \text{etc.}\right) \\ \frac{1}{2}\left(1 - \frac{1}{3}\cdot\frac{1}{64} + \frac{1}{5}\cdot\frac{1}{64^2} - \frac{1}{7}\cdot\frac{1}{64^3} + \text{etc.}\right) \\ \frac{1}{4}\left(\frac{1}{3} - \frac{1}{7}\cdot\frac{1}{64} + \frac{1}{11}\cdot\frac{1}{64^2} - \frac{1}{15}\cdot\frac{1}{64^3} + \text{etc.}\right) \\ \frac{1}{2}\left(1 - \frac{1}{5}\cdot\frac{1}{1024} + \frac{1}{9}\cdot\frac{1}{1024^2} - \frac{1}{13}\cdot\frac{1}{1024^3} + \text{etc.}\right) \\ \frac{1}{16}\left(1 - \frac{1}{3}\cdot\frac{1}{1024} + \frac{1}{5}\cdot\frac{1}{1024^2} - \frac{1}{7}\cdot\frac{1}{1024^3} + \text{etc.}\right) \\ \frac{1}{64}\left(1 - \frac{1}{7}\cdot\frac{1}{1024} + \frac{1}{11}\cdot\frac{1}{1024^2} - \frac{1}{15}\cdot\frac{1}{1024^3} + \text{etc.}\right) \end{array} \right.$$

Hic ergo maxime notatu dignum occurrit, quod omnes iftae feries per folas poteftates binarii procedant.

VI.