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Investigatio quarundam serierum, quae ad rationem peripheriae circuli ad diametrum vero proxime definiendam maxime sunt accommodatae

Leonhard Euler

INVESTIGATIO QVARVNDAM SERIERVM, QVAE AD RATIONEM PERIPHERIAE CIRCVLI AD DIAMETRVM VERO PROXIME DEFINIENDAM MAXIME SVNT ACCOMMODATAE.

Auctore

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§. 1.

Qui post *Ludolphum a Ceulen* veram rationem peripheriae ad diametrum proxime assignare susceperunt, usi sunt serie *Leibnitiana*, qua pro circulo, cuius radius $= 1$, arcus quicunque s per suam tangentem t ita exprimi solet, ut sit

$$s = t - \tfrac{1}{3}t^3 + \tfrac{1}{5}t^5 - \tfrac{1}{7}t^7 + \text{etc.},$$

quae eo magis convergit, quo minor tangens t accipiatur. Sed quia arcus s ad totam peripheriam, vel ad arcum quadrantis cognitam rationem tenere debet, pro arcu s vix minorem valorem assumere licet, quam 30 graduum quippe cuius tangens est $\frac{1}{\sqrt{3}}$, quo valore in serie substituto, si semiperipheria circuli per π designetur, erit $\pi = 6s$, unde deducitur haec series:

$$\pi = \sqrt{12} \times \left(1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \frac{1}{9.3^4} - \text{etc.}\right)$$

Hinc

Hinc patet, calculum huius ſeriei ante inſtitui non poſſe, quam radix quadrata ex numero 12 ad tot figuras decimales fuerit extracta, ad quot valor ipſius π deſideratur, quem ſtupendum laborem olim *Abrahamus Sharp* usque ad 72 figuras decimales; tum vero Profeſſor Greshamienſis *Machin* ad 100 figuras eſt exſecutus. Multo maiorem autem laborem ſollertiſſimus calculator Gallus *de Lagny* eſt exantlare coactus, qui ex eadem ſerie valorem ipſius π adeo usque ad 128 figuras decimales determinavit, qui labor certe plus quam Herculeus eſt cenſendus, cum tamen extractio radicis ex numero 12 tantum tanquam opus praeliminare ſit ſpectandum, iſtam enim immenſam fractionem decimalem demum opus erat continuo per 3 dividere, quo facto inſuper ſinguli termini per numeros impares 3, 5, 7, 9, 11, etc. ordine dividi debebant. Cum igitur iſtius ſeriei quilibet terminus in hac forma contineatur: $\frac{+\sqrt{12}}{(2n+1)\,3^n}$, ubi n denotat numerum terminorum, tot terminos computari oportet, donec fiat $\frac{(2n+1)\,3^n}{\sqrt{12}} = 10^{128}$, ſive, logarithmis vulgaribus ſumendis, donec fiat $l(2n+1) + n\,l\,3 - \frac{1}{2}l\,12 = 128$; unde primam partem $l(2n+1)$ negligendo colligitur $n = \frac{128 + \frac{1}{2}l\,12}{l\,3}$, hincque prodit terminorum numerus aliquanto minor quam 269; ex quo utique maxime eſt mirandum, quemquam fuiſſe repertum, qui hunc ſtupendum laborem exſequi ſit auſus.

§. 2. Iam dudum autem propoſui methodum iſtum laborem plurimum ſublevandi. Poſtquam ſcilicet oſtendi, duos

duos arcus ſatis exiguos in hunc uſum adhiberi poſſe, quorum quidem neuter ad peripheriam teneat rationem rationalem, quorum tamen ſumma talem rationem teneat. Tales arcus ſunt: A tang. $\frac{1}{2}$ + A tang. $\frac{1}{3}$ = A tang. 1 = $\frac{\pi}{4}$, ita ut $\pi = 4$ A tang. $\frac{1}{2}$ + 4 A tang. $\frac{1}{3}$, quorum uterque per noſtram ſeriem facile evolvitur, cum ſit:

$$\text{A tang.}\ \tfrac{1}{2} = \tfrac{1}{2} - \tfrac{1}{3\cdot 2^3} + \tfrac{1}{5\cdot 2^5} - \tfrac{1}{7\cdot 2^7} + \text{etc. et}$$

$$\text{A tang.}\ \tfrac{1}{3} = \tfrac{1}{3} - \tfrac{1}{3\cdot 3^3} + \tfrac{1}{5\cdot 3^5} - \tfrac{1}{7\cdot 3^7} + \text{etc.}$$

ubi termini illius ſeriei fere in ratione quadrupla decreſcunt, huius vero in ratione fere noncupla, ideoque multo magis convergunt, quam ſeries ab Auctoribus memoratis uſurpata. Praecipue vero notandum eſt hoc modo nullam extractionem radicis requiri, ſicque fere maximam partem illius laboris evitari; praeterea etiam ſinguli termini harum novarum ſeriérum facillime in fractiones decimales convertuntur, quae, quia figurae certum ordinem, imprimis ab initio, ſervant, computus ad quotcunque figuras ſine magno labore extenditur.

§. 3. Multo magis autem labor diminuetur, ſi adhuc minores arcus in ſubſidium vocentur. Cum enim ſit

$$\text{A tang.}\ \tfrac{1}{2} = \text{A tang.}\ \tfrac{1}{3} + \text{A tang.}\ \tfrac{1}{7},$$

erit nunc

$$\pi = 8\ \text{A tang.}\ \tfrac{1}{3} + 4\ \text{A tang.}\ \tfrac{1}{7},$$

ſicque in ſerie priore termini ſtatim in ratione noncupla decreſcunt, in poſteriore vero adeo 49 vicibus evadunt minores. Unicum autem, quod hic deſiderari poſſet, in hoc conſiſtit, quod non tam facile per 49 continua diviſio inſtituatur, optandumque fuiſſet, ut iſta diviſio vel per poteſtatem denarii vel alius numeri ſimplicem ad 10 rationem tenentis, expediri poſſet.

§. 4.

§. 4. Incidi autem nuper in modum prorſus ſingularem, quo huic incommodo feliciſſimo ſucceſſu occurritur atque adeo ſeries praecedentes magis convergentes redduntur. Conſtat autem iſte modus in idonea transformatione ſeriei *Leibnitianae*, quae per ſequentes operationes procedit:

$$s = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9} + \text{etc.}$$

$$stt = t^3 - \frac{t^5}{3} + \frac{t^7}{5} - \frac{t^9}{7} + \text{etc.}$$

$$\text{ergo } s + stt = t + \tfrac{2}{3}t^3 - \tfrac{2}{3.5}t^5 + \tfrac{2}{5.7}t^7 - \text{etc.} = t + s'tt$$

$$\text{ergo } s' = \tfrac{2}{3}t - \tfrac{2}{3.5}t^3 + \tfrac{2}{5.7}t^5 - \tfrac{2}{7.9}t^7 + \text{etc.}$$

$$\text{hinc } s'tt = \tfrac{2}{1.3}t^3 - \tfrac{2}{3.5}t^5 + \tfrac{2}{5.7}t^7 - \text{etc.}$$

$$s'(1+tt) = \tfrac{2}{3}t + \tfrac{2.4}{3.5}t^3 - \tfrac{2.4}{3.5.7}t^5 + \tfrac{2.4}{5.7.9}t^7 - \text{etc.} = \tfrac{2}{3}t + s''tt$$

$$\text{ergo } s'' = \tfrac{2.4}{1.3.5}t - \tfrac{2.4}{3.5.7}t^3 + \tfrac{2.4}{5.7.9}t^5 - \text{etc.}$$

$$s''tt = + \tfrac{2.4}{1.3.5}t^3 - \tfrac{2.4}{3.5.7}t^5 + \text{etc.}$$

$$s''(1+tt) = \tfrac{2.4}{3.5}t + \tfrac{2.4.6}{1.3.5.7}t^3 - \tfrac{2.4.6}{3.5.7.9}t^5 + \text{etc.} = \tfrac{2.4}{3.5}t + s'''t$$

$$s''' = \tfrac{2.4.6}{3.5.7}t - \tfrac{2.4.6}{3.5.7.9}t^3 + \tfrac{2.4.6}{5.7.9.11}t^5 - \text{etc.}$$

$$s'''tt = + \tfrac{2.4.6}{1.3.5.7}t^3 - \tfrac{2.4.6}{3.5.7.9}t^5 + \text{etc.}$$

$$s'''(1+tt) = \tfrac{2.4.6}{3.5.7}t + \tfrac{2.4.6.8}{1.3.5.7.9}t^3 - \tfrac{2.4.6.8}{3.5.7.9.11}t^5 + \text{etc.}$$

etc. etc.

§. 5. Colligamus iam ſingulas ſubſtitutiones factas, quae ſunt:

$$s = \frac{t}{1+tt} + \frac{s'tt}{1+tt},$$

$$s' = \frac{2t}{3(1+tt)} + \frac{s''tt}{1+tt},$$

$$s'' = \frac{2.4t}{3.5(1+tt)} + \frac{s'''tt}{1+tt},$$
$$s''' = \frac{2.4.6t}{3.5.7(1+tt)} + \frac{s''''tt}{1+tt},$$
etc.

Quod ſi iam valores poſteriores in praecedentibus ſubſtituantur, pro arcu s ſequens obtinebitur nova ſeries:

$$s = \frac{t}{1+tt} + \frac{2}{3}\cdot\frac{t^3}{(1+tt)^2} + \frac{2.4}{3.5}\cdot\frac{t^5}{(1+tt)^3} + \frac{2.4.6}{3.5.7}\cdot\frac{t^7}{(1+tt)^4} + \text{etc.}$$

quae ad ſequentem formam commodiorem reducitur:

$$s = \frac{t}{1+tt}\left[1 + \frac{2}{3}\left(\frac{tt}{1+tt}\right) + \frac{2.4}{3.5}\left(\frac{tt}{1+tt}\right)^2 + \frac{2.4.6}{3.5.7}\left(\frac{tt}{1+tt}\right)^3 + \text{etc.}\right]$$

ubi ſinguli termini adhuc facilius evolvuntur quam in ſerie praecedente, propterea quod ex quolibet termino ſequens immediate determinari poteſt. Ita ex primo termino reperitur ſecundus, ſi ille per $\frac{2}{3}$ et per $\frac{tt}{1+tt}$ multiplicetur (Multiplicatio autem per $\frac{2}{3}$ fit, dum pars tertia ſubtrahitur). Secundus per $\frac{4}{5}\left(\frac{tt}{1+tt}\right)$ multiplicatus dat tertium; hic vero, per $\frac{6}{7}\left(\frac{tt}{1+tt}\right)$ multiplicatus, dat quartum, et ita porro. Facillime autem per fractiones $\frac{4}{5}, \frac{6}{7}, \frac{8}{9}$, etc. multiplicatur. Praeterea vero haud exiguum eſt lucrum, quod omnes termini ſunt poſitivi, eorumque ergo ſola additio arcum quaeſitum s ſuppeditat.

§. 6. Ad hanc autem novam ſeriem primum methodo longe alia ſum perductus, quam hic appoſuiſſe operae erit pretium. Cum ſit $s = \int \frac{\partial t}{1+tt}$, quaeſtionem hoc modo determinate ſum contemplatus, ut ſcilicet quaereretur valor huius formulae integralis, ſi a termino $t = 0$ usque ad terminum $t = a$ extendatur, ita ut futurum ſit $s =$ A tang. a.

§. 7. Tum vero huius formulae denominatorem $1 + tt$ ſub hac forma repraeſento: $1 + aa - (aa - tt)$, hinc-

hincque porro ſub hac: $1 + aa(1 - \frac{aa - tt}{1 + aa})$, quo facto fractio $\frac{1}{1 + tt}$ evolvetur in hanc ſeriem:

$$\frac{1}{1 + aa}[1 + \frac{aa - tt}{1 + aa} + (\frac{aa - tt}{1 + aa})^2 + (\frac{aa - tt}{1 + aa})^3 + \text{etc.}]$$

ſicque erit

$$s = \frac{1}{1 + aa}\int \partial t [1 + \frac{aa - tt}{1 + aa} + (\frac{aa - tt}{1 + aa})^2 + \text{etc.}],$$

poſtquam ſcilicet integratio a $t = 0$ usque ad $t = a$ fuerit extenſa; unde ſtatim patet, pro primo termino fore $\int \partial t = a$; pro ſecundo autem $\int \partial t (aa - tt) = \frac{2}{3} a^3$.

§. 8. At vero, quo facilius omnes termini ſequentes integrentur, ſequentem aequationem evolvi conveniet:

$$\int \partial t (aa - tt)^{n+1} = A \int \partial t (aa - tt)^n + Bt(aa - tt)^{n+1},$$

quae differentiata ac per $\partial t (aa - tt)^n$ diviſa praebet:

$$aa - tt = A + B(aa - tt) - 2(n + 1) Btt,$$

ubi duplicis generis termini occurrunt, ſcilicet vel mere conſtantes, vel quadrato tt affecti, qui ſeorſim ſe mutuo tollere debent.

§. 9. Quoniam autem huius aequationis membrum primum et tertium continet factorem $aa - tt$, neceſſe eſt ut ſecundum cum quarto eundem factorem involvat, quod evenit, ſtatuendo $A = 2(n + 1) Baa$, quo facto, ſi aequatio inſuper per $aa - tt$ dividatur, prodibit $1 = B(2n + 3)$; unde colligitur: $B = \frac{1}{2n + 3}$ hincque $A = \frac{2(n + 1)}{2n + 3} aa$, ſicque aequatio noſtra aſſumta iam erit:

$$\int \partial t (aa - tt)^{n+1} = \frac{2(n + 1)}{2n + 3} aa \int \partial t (aa - tt)^n + \frac{t}{2n + 3}(aa - tt)^{n+1}.$$

Qua-

Quare si integralia a $t = o$ usque ad $t = a$ extendantur, postremum membrum sponte abit in nihilum, sicque habebimus hanc reductionem generalem:

$$\int \partial t\,(aa - tt)^{n+1} = \frac{2(n+1)aa}{2n+3}\int \partial t\,(aa - tt)^n.$$

§. 10. Iam ope huius reductionis ex quolibet termino nostrae seriei facillime terminus sequens assignari poterit. Quod si enim loco exponentis n successive omnes valores $0, 1, 2, 3, 4, 5$, etc. ponamus, sequentia integralia nanciscemur:

$$\int \partial t\,(aa - tt) = \tfrac{2}{3}a^3,$$
$$\int \partial t\,(aa - tt)^2 = \tfrac{2.4}{3.5}a^5,$$
$$\int \partial t\,(aa - tt)^3 = \tfrac{2.4.6}{3.5.7}a^7,$$
$$\int \partial t\,(aa - tt)^4 = \tfrac{2.4.6.8}{3.5.7.9}a^9,$$
etc.

§. 11. Quod si iam singuli hi valores in nostra serie substituantur, integrale, quod quaerimus, sequenti modo exprimetur:

$$s = \text{A tag.}\,a = \frac{1}{1+aa}\left(a + \frac{\frac{2}{3}a^3}{1+aa} + \frac{\frac{2.4}{3.5}a^5}{(1+aa)^2} + \frac{\frac{2.4.6}{3.5.7}a^7}{(1+aa)^3} + \text{etc.}\right)$$

unde, si loco a restituamus t, orietur ipsa series methodo praecedente inventa, scilicet:

$$s = \text{A tag.}\,t = \frac{t}{1+tt}\left[1 + \tfrac{2}{3}\left(\frac{tt}{1+tt}\right) + \tfrac{2.4}{3.5}\left(\frac{tt}{1+tt}\right)^2 + \tfrac{2.4.6}{3.5.7}\left(\frac{tt}{1+tt}\right)^3 + \text{etc.}\right].$$

§. 12. Nunc igitur hanc novam seriem ad nostrum institutum propius accommodemus, et quoniam supra primo hanc habuimus aequationem: $\pi = 4\,\text{A tang.}\,\frac{1}{2} + 4\,\text{A tang.}\,\frac{1}{3}$, pro

pro priore parte, ubi $t = \frac{1}{2}$, obtinebimus hanc feriem:

$$\text{A tang.}\, \tfrac{1}{2} = \tfrac{2}{5}\left(1 + \tfrac{2}{3}\cdot\tfrac{1}{5} + \tfrac{2.4}{3.5}\cdot\tfrac{1}{5^2} + \tfrac{2.4.6}{3.5.7}\cdot\tfrac{1}{5^3} + \text{etc.}\right)$$

pro altera autem parte, ubi $t = \frac{1}{3}$, erit

$$\text{A tang.}\, \tfrac{1}{3} = \tfrac{3}{10}\left(1 + \tfrac{2}{3}\cdot\tfrac{1}{10} + \tfrac{2.4}{3.5}\cdot\tfrac{1}{10^2} + \tfrac{2.4.6}{3.5.7}\cdot\tfrac{1}{10^3} + \text{etc.}\right)$$

confequenter valor ipfius π per binas fequentes feries exprimetur:

$$\pi = \left\{\begin{array}{l} + \tfrac{16}{10}\left(1 + \tfrac{2}{3}\left(\tfrac{2}{10}\right) + \tfrac{2.4}{3.5}\left(\tfrac{2}{10}\right)^2 + \tfrac{2.4.6}{3.5.7}\left(\tfrac{2}{10}\right)^3 + \text{etc.}\right. \\ + \tfrac{12}{10}\left(1 + \tfrac{2}{3}\left(\tfrac{1}{10}\right) + \tfrac{2.4}{3.5}\left(\tfrac{1}{10}\right)^2 + \tfrac{2.4.6}{3.5.7}\left(\tfrac{1}{10}\right)^3 + \text{etc.}\right. \end{array}\right\},$$

quae duae feries manifefto multo minore labore per numeros evolvuntur, quam eae, quas fupra dedimus, propterea quod hic in factoribus habemus ipfum denarium, atque hae feries adeo magis convergunt.

§. 13. Lucrum autem adhuc multo erit maius, fi forma $\pi = 8\,\text{A tang.}\,\frac{1}{3} + 4\,\text{A tang.}\,\frac{1}{7}$ per novam feriem evolvatur, cuius pars prior iam eft evoluta; pro altera autem, ubi $t = \frac{1}{7}$, nunc habebimus:

$$\text{A tang.}\, \tfrac{1}{7} = \tfrac{7}{50}\left(1 + \tfrac{2}{3}\cdot\tfrac{1}{50} + \tfrac{2.4}{3.5}\cdot\tfrac{1}{50^2} + \tfrac{2.4.6}{3.5.7}\cdot\tfrac{1}{50^3} + \text{etc.}\right).$$

Hinc igitur nanciscemur fequentes feries pro valore femiperipheriae π indagando:

$$\pi = \left\{\begin{array}{l} + \tfrac{24}{10}\left(1 + \tfrac{2}{3}\left(\tfrac{1}{10}\right) + \tfrac{2.4}{3.5}\left(\tfrac{1}{10}\right)^2 + \tfrac{2.4.6}{3.5.7}\left(\tfrac{1}{10}\right)^3 + \text{etc.}\right. \\ + \tfrac{28}{50}\left(1 + \tfrac{2}{3}\left(\tfrac{2}{100}\right) + \tfrac{2.4}{3.5}\left(\tfrac{2}{100}\right)^2 + \tfrac{2.4.6}{3.5.7}\left(\tfrac{2}{100}\right)^3 + \text{etc.}\right. \end{array}\right\},$$

haeque duae feries funt aptiffimae ad valorem ipfius π ad quotcunque figuras decimales exprimendum, propterea quod finguli termini ex praecedentibus facillime formantur atque adeo prioris feriei termini iam in ratione decupla, pofterioris vero in quinquies decupla decrefcunt. Vnde fi quis hunc valorem

lorem ad 128 figures definire vellet, pro priori ferie computare deberet terminos centum viginti octo, posterioris vero septuaginta quinque tantum.

§. 14. Quo usus harum serierum clarius appareat, utriusque seriei octo terminos priores in fractiones decimales evolvamus, eritque

Pro parte priore.

term.	I.	= 2,4 etc.
—	II.	= 16 etc.
—	III.	= 128 etc.
—	IV.	= 109,714285,714285,714285,7142 etc.
—	V.	= 97,523809,523809,523809,523 etc.
—	VI.	= 8,865800,865800,865800,865 etc.
—	VII.	= 81,838161,838161,838161,8 etc.
—	VIII.	= 76,382284,382284,382284, etc.
Pars.	I.	= 2,574004427 | 231435 | 231435 | 231435 etc.

Pro parte posteriore.

term.	I.	= 0,560 etc.
—	II.	= 746666666666666666666666 etc.
—	III.	= 11946666666666666666666 etc.
—	IV.	= 2048000000000000000000 etc.
—	V.	= 3640888888888888888 etc.
—	VI.	= 661979797979797979 etc.
—	VII.	12221165501165501 etc.
—	VIII.	= 228128422688422 etc.
Pars	II.	= 0,56758821841665131 | 412587 | 4125 etc.

Hinc

Hinc patet iſtas ſummas octo priorum terminorum, ob revolutiones periodicas in figuris occurrentes, ſine ullo labore ad quotcunque figuras continuari poſſe.

§. 15. Ex hoc ſchemate iam ſtatim verus valor ipſius π ad octo figuras usque aſſignari poterit. Cum enim octo priorum terminorum ſumma ſit

Partis prioris $= 2,57400443$
Partis poſterioris $= 0,56758822$

erit valor ipſius $\pi = 3,14159265$

ubi ne in ultima quidem figura erratur. Facile autem iſte calculus ad plures figuras extendi poteſt, propterea quod termini octavum ſubſequentes ex eo ipſo ſine difficultate computantur. Eſt enim

Pro parte priore.

$$\text{terminus IX.} = \tfrac{1}{10}\left(1 - \tfrac{1}{17}\right)\text{ VIII.}$$
$$\text{— X.} = \tfrac{1}{10}\left(1 - \tfrac{1}{19}\right)\text{ IX.}$$
$$\text{— XI.} = \tfrac{1}{10}\left(1 - \tfrac{1}{21}\right)\text{ X.}$$
etc.

Pro parte poſteriore.

$$\text{terminus IX.} = \tfrac{2}{100}\left(1 - \tfrac{1}{17}\right)\text{ VIII.}$$
$$\text{— X.} = \tfrac{2}{100}\left(1 - \tfrac{1}{19}\right)\text{ IX.}$$
$$\text{— XI.} = \tfrac{2}{100}\left(1 - \tfrac{1}{21}\right)\text{ X.}$$
etc.

§. 16. Quo uſus harum formularum magis elucescat, quaeramus valorem ipſius π usque ad 16 figuras, et calculus erit:

Pro

Pro parte priore.

I. - - - VIII.	=	2,57400442723143523
term. IX.	=	- - - - 718892088
—— X.	=	- - - - 68105566
—— XI.	=	- - - - - 6486244
—— XII.	=	- - - - - 620423
—— XIII.	=	- - - - - 59561
—— XIV.	=	- - - - - - 5735
—— XV.	=	- - - - - - 554
—— XVI.	=	- - - - - - - 54
Summa	=	2,57400443517313748.

Pro parte posteriore.

I. - - - VIII.	=	0,56758821841665131
term. IX.	=	- - - - - - - 429
—— X.	=	- - - - - - - 8
Pars II.	=	0,56758821841665567
Pars I.	=	2,57400443517313748
hinc π	=	3,14159265358979315

§. 17. Possunt vero etiam aliae huiusmodi formulae pro π inveniri, quae adhuc magis convergant ac pariter per potestates denarii procedant. Cum enim in genere sit

$$\text{A tang.}\, \frac{\alpha}{a} = \text{A tang.}\, \frac{\beta}{b} + \text{A tang.}\, \frac{\alpha b - \beta a}{\alpha\beta + ab},$$

si sumamus $t = \frac{\alpha}{a}$, vel $\frac{\beta}{b}$, erit $\frac{tt}{1+tt} = \frac{\alpha\alpha}{\alpha\alpha + aa}$ vel $\frac{\beta\beta}{\beta\beta + bb}$; sumto vero $t = \frac{\alpha b - \beta a}{\alpha\beta + ab}$ fiet $\frac{tt}{1+tt} = \frac{(\alpha b - \beta a)^2}{(\alpha\alpha + aa)(\beta\beta + bb)}$. Vnde patet, si priores denominatores $\alpha\alpha + aa$ et $\beta\beta + bb$ fuerint potestates denarii, vel eo saltem reduci queant, quod

eve-

evenit, quando alios factores non involvunt praeter 2 et 5, tum etiam tertium denominatorem certe ad potestatem denarii reduci posse.

§. 18. Quoniam igitur habuimus hanc formulam:

$$\pi = 8\,A\,\text{tang.}\,\tfrac{1}{3} + 4\,A\,\text{tang.}\,\tfrac{1}{7},$$

loco prioris arcus ope reductionis allatae duos alios introducamus, ponendo scilicet $\frac{\alpha}{a} = \frac{1}{3}$; et pro $\frac{\beta}{b}$ sumamus $\frac{1}{7}$, fietque tertius arcus $= A\,\text{tang.}\,\frac{2}{11}$, ita ut sit

$$A\,\text{tang.}\,\tfrac{1}{3} = A\,\text{tang.}\,\tfrac{1}{7} + A\,\text{tang.}\,\tfrac{2}{11},$$

quo valore substituto formula nostra erit

$$\pi = 12\,A\,\text{tang.}\,\tfrac{1}{7} + 8\,A\,\text{tang.}\,\tfrac{2}{11},$$

cuius arcum priorem iam ante evolvimus. At vero ob $\frac{tt}{1+tt} = \frac{4}{125} = \frac{32}{1000}$ pro altero habebimus:

$$A\,\text{tang.}\,\tfrac{2}{11} = \tfrac{22}{125}\,[1 + \tfrac{2}{3}(\tfrac{32}{1000}) + \tfrac{2\cdot4}{3\cdot5}(\tfrac{32}{1000})^2 + \tfrac{2\cdot4\cdot6}{3\cdot5\cdot7}(\tfrac{32}{1000})^3 + \text{etc.}]$$

Verum hic continua multiplicatio per numerum 32 non satis ad calculum est idonea, praecipue autem haec series minus convergit quam quae ex $\frac{1}{7}$ est deducta.

§. 19. Hanc ob caussam penitus reiiciamus istum arcum, eiusque loco ope reductionis supra datae substituamus duos novos arcus, quorum alter sit $\frac{1}{7}$, statuendo $\frac{\alpha}{a} = \frac{2}{11}$ et $\frac{\beta}{b} = \frac{1}{7}$, hincque fiet $\frac{\alpha b - \beta a}{\alpha\beta + ab} = \frac{3}{79}$, ita ut sit

$$A\,\text{tang.}\,\tfrac{2}{11} = A\,\text{tang.}\,\tfrac{1}{7} + A\,\text{tang.}\,\tfrac{3}{79},$$

hincque

$$\pi = 20\,A\,\text{tang.}\,\tfrac{1}{7} + 8\,A\,\text{tang.}\,\tfrac{3}{79}.$$

Vbi

Vbi notetur, pofito $t = \frac{3}{79}$ fore

$$\frac{tt}{1+tt} = \frac{9}{6250} = \frac{144}{100000},$$

quae fractio propemodum eft $\frac{1}{700}$; unde patet, hanc feriem:

$$A\ \text{tang.}\ \frac{3}{79} = \frac{237}{6250}\left[1 + \frac{2}{3}\left(\frac{144}{100000}\right) + \frac{2\cdot 4}{3\cdot 5}\left(\frac{144}{100000}\right)^2 + \text{etc}\right]$$

maxime convergere eiusque terminos propemodum feptingenties fieri minores.

§. 20. Ifta igitur feries maxime eft notatu digna, propter infignem convergentiam, atque adeo plurimum operae pretium erit multiplicatione per 144 non deterreri, quippe quae, bis per 12 multiplicando, facile abfolvi poteft. Per 12 autem multiplicare vix difficilius eft quam per 2. Evolvamus igitur ambos iftos arcus per noftram novam feriem, atque impetrabimus fequentem formam:

$$\pi = \begin{cases} + \frac{28}{10}\left[1 + \frac{2}{3}\left(\frac{2}{100}\right) + \frac{2\cdot 4}{3\cdot 5}\left(\frac{2}{100}\right)^2 + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\left(\frac{2}{100}\right)^3 + \text{etc.}\right] \\ + \frac{30336}{100000}\left[1 + \frac{2}{3}\left(\frac{144}{100000}\right) + \frac{2\cdot 4}{3\cdot 5}\left(\frac{144}{100000}\right) + \text{etc}\right] \end{cases}$$

Hic igitur coëfficiens prioris feriei quinquies maior eft quam fupra, unde etiam finguli termini ibi exhibiti toties maiores funt capiendi, unde fumma octo priorum terminorum erit:

2, 837941092083 2565 | 706293 | 706293 | 706 etc.

octavus autem terminus:

0, 0000000000011 4064 | 211344 | 211344 | 211 etc.

ex quo iam fequentes termini facile colliguntur.

§. 21. Quo autem pro altera ferie calculus commodius inftitui poffit, primo conveniet divifiones per 100000 prorfus praetermitti, ita ut ex quolibet termino fequens obtineatur, dum ille bis per 12 multiplicetur et a produ debita pars subtrahatur, nullo respectu habito ad loca phrarum decimalium; quandoquidem ex hoc capite aber ri nequit, dum satis constat quoties quilibet terminus i nor est praecedente. Talem calculum pro sex prioribus t minis hic exhibeamus:

term. I. =	0,30336
	364032
3.)	4368384
	1456128
term. II. =	2912256
	34947072
5.)	419364864
	838729728
term. III. =	3354918912
	40259026944
7.)	483108323328
	69015474761, 142857, 142857, 142 etc.
term. IV. =	414092848566, 857142, 857142, 857 etc.
	4969114182802, 285714, 285714, 285 etc.
9.)	596293701936 27, 428571, 428571, 428 e
	66254855770 69, 714285, 714285, 714 e
term. V. =	5300388461 6557, 714285, 714285, 714 e
	63604661539 8692, 571428, 571428, 571 et

11.) $= 763255938478431\,0,\ 857142,\ 857142,\ 857$ etc.
$= 69386903498039\,1,\ 896103,\ 896103,\ 8961$ etc.

term. VI. $= 69386903498039\,1,\ 896103,\ 896103,\ 8961$ etc.

unde ipſos terminos deſumamus et in unam ſummam colligamus:

term. I. =	0, 30336
— II. =	2912256
— III. =	3354918912
— IV. =	41409284856 6, 857142, 857142
— V. =	53003884616557, 714285, 7
— VI. =	69386903498039 1 8, 96
Summa =	0, 303651561506514781282057700391 8, 961038, 961038, 961038 etc.

ubi imprimis notatu dignum occurrit, quod ſumma quinque priorum terminorum abſolute exhiberi poteſt, dum ſcilicet fractio decimalis in figura 26^{ma} abrumpitur, haecque poſtrema formula pro π data ad calculum maxime videtur accommodata.

§. 22. Ex eodem principio, unde noſtram ſeriem deduximus, aliae ſimiles ſeries derivari poſſunt pariter maxime convergentes. Inchoando ſcilicet a ſerie vulgari:

$$\text{A tang. } t = t - \tfrac{1}{3}t^3 + \tfrac{1}{5}t^5 - \tfrac{1}{7}t^7 + \text{ etc.}$$

ponamus huius ſeriei iam n terminos actu eſſe collectos, quorum ſumma ſit

$$\Sigma = 1 - \tfrac{1}{3}t^3 + \tfrac{1}{5}t^5 - \ldots\ldots \pm \frac{t^{2n-1}}{2n-1}.$$

Sum-

Summam autem sequentium terminorum statuamus:

$$s = \frac{t^{2n+1}}{2n+1} - \frac{t^{2n+3}}{2n+3} + \frac{t^{2n+5}}{2n+5} - \text{etc.}$$

ita ut sit A tang. $t = \Sigma \pm s$, ubi ergo numerus Σ tanquam iam inventus spectatur, alter vero s investigari debeat.

§. 23. Ratiocinium igitur eodem modo instituamus, ut supra §. 4, quas operationes hic apponamus.

$$s = \frac{t^{2n+1}}{2n+1} - \frac{t^{2n+3}}{2n+3} + \frac{t^{2n+5}}{2n+5} - \text{etc.}$$

$$stt = \quad + \frac{t^{2n+3}}{2n+1} - \frac{t^{2n+5}}{2n+3} + \text{etc.}$$

$$s(1+tt) = \frac{t^{2n+1}}{2n+1} + \frac{2t^{2n+3}}{(2n+1)(2n+3)} - \frac{2t^{2n+5}}{(2n+3)(2n+5)} + \text{etc.}$$

$$= \frac{t^{2n+1}}{2n+1} + s'tt.$$

$$s(1+tt) = \frac{t^{2n+1}}{2n+1} + \frac{2t^{2n+3}}{(2n+1)(2n+3)} - \frac{2t^{2n+5}}{(2n+3)(2n+5)} + \text{etc.}$$

$$= \frac{t^{2n+1}}{2n+1} + s'tt, \text{ ergo}$$

$$s' = \frac{2t^{2n+1}}{(2n+1)(2n+3)} - \frac{2t^{2n+3}}{(2n+3)(2n+5)} + \text{etc.}$$

$$s'tt = \quad + \frac{2t^{2n+3}}{(2n+1)(2n+3)} - \text{etc.}$$

$$s'(1+tt) = \frac{2t^{2n+1}}{(2n+1)(2n+3)} + \frac{2 \cdot 4t^{2n+3}}{(2n+1)(2n+3)(2n+5)} - \text{etc.}$$

$$= \frac{2t^{2n+1}}{(2n+1)(2n+3)} + s''tt. \text{ etc.}$$

§. 24.

§. 24. Quod si iam valores introducti restituantur, facile patet tandem ad hanc seriem perventum iri:

$$s = \frac{t^{2n+1}}{(2n+1)(1+tt)} + \frac{2\,t^{2n+3}}{(2n+1)(2n+3)(1+tt)^2} + \frac{2\cdot 4\,t^{2n+5}}{(2n+1)(2n+3)(2n+5)(1+tt)^3} + \text{etc.}$$

quae expressio contrahitur in sequentem:

$$s = \frac{t^{2n+1}}{(2n+1)(1+tt)}\left(1 + \frac{2tt}{(2n+3)(1+tt)} + \frac{2\cdot 4\,t^4}{(2n+3)(2n+5)(1+tt)^2} + \text{etc.}\right)$$

haecque series utique aliquanto magis convergit quam praecedens, propterea quod denominatores multo maiores sunt quam numeratores; veruntamen formulae ante exhibitae his seriebus longissime anteferendae videntur, siquidem ad usum practicum respiciamus.

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