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Leonhard Euler

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DISQVISITIO VLTERIOR SVPER SERIEBVS SECVNDVM MVLTIPLA CVIVSDAM ANGVLI PROGREDIENTIBVS.

Auctore
L. EVLERO.

Conventui exhib. die 26 Maii 1777.

§. 1.

Contemplabor hic denuo eiusmodi functiones cuiuspiam anguli Φ, quas in feries, quarum termini cofinus angulorum multiplorum ipfius Φ continent, evolvere liceat. Scilicet fi Φ denotet talem functionem anguli Φ, quae per evolutionem huiusmodi feriei oriatur:

$$\Phi = A + B \cos.\Phi + C \cos.2\Phi + D \cos.\Phi + E \cos.4\Phi + \text{etc.}$$

manifeftum eft talem refolutionem femper fuccedere, quando eadem functio Φ per folutionem communem in talem feriem converti poteft:

$$\Phi = \alpha + \beta \cos.\Phi + \gamma \cos.\Phi^2 + \delta \cos.\Phi^3 + \varepsilon \cos.\Phi^4 + \text{etc}$$

propterea quod omnes poteftates cofinuum in cofinus multiplorum eiusdem anguli refolvi poffunt, id quod in poteftatibus finuum non fuccedit, quoniam tantum poteftates pares in cofinus multiplorum refolvuntur, poteftates vero impares ad finus multiplorum perducuntur. Quia vero omne

finu

finus facillime ad cofinus revocantur, ea quae hic fum traditurus, pariter quoque ad finus pertinere funt cenfenda.

§. 2. Nifi autem functio propofita Φ fuerit rationalis et fatis fimplex, feriei quae ex eius evolutione nafcitur:

$$A + B \operatorname{cof.} \phi + C \operatorname{cof.} 2\,\phi + D \operatorname{cof.} 3\,\phi + E \operatorname{cof.} 4\,\phi + \text{etc.}$$

finguli termini ita deprehenduntur comparati, ut eorum valores non nifi per quantitates maxime transcendentes exhiberi queant. Veluti fi functio propofita fuerit

$$\Phi = (1 - n \operatorname{cof.} \phi)^{-\frac{3}{2}},$$

a cuius evolutione propemodum univerfa theoria Aftronomiae pendet, feriei inde oriundae primus terminus A per hanc feriem exprimi invenitur:

$$1 + \tfrac{3.5}{4.4} n n + \tfrac{3.5}{4.4} \cdot \tfrac{7.9}{8.8} n^4 + \tfrac{3.5}{4.4} \cdot \tfrac{7.9}{8.8} \cdot \tfrac{11.13}{12.12} \cdot n^6 + \text{etc.}$$

cuius fummatio omnia artificia analytica hactenus inventa eludit; hinc olim plurimum in hoc elaboravi, ut eius fummationem ad refolutionem aequationis differentialis reducerem, unde deinceps haec inveftigatio ad genera quantitatum transcendentium, five ad quadraturas curvarum magis cognitas, deduci poffet; verum etiam in hoc labore operam meam nequicquam confumfi. Nuper autem fe mihi obtulit idea, quae me ad formulas integrales fatis concinnas manuduxit, quibus non folum primus huius feriei terminus A, fed adeo omnes termini, fatis commode exprimi poffunt, quas in fequenti theoremate fum complexurus.

Theorema generale.

§. 3. Si functio Φ anguli ϕ ita fuerit comparata, ut in talem feriem refolvi fe patiatur:

$$\Phi = A + B \operatorname{cof.} \phi + C \operatorname{cof.} 2\phi + D \operatorname{cof.} 3\phi + E \operatorname{cof.} 4\phi + \text{etc.}$$

tum fingulae quantitates A, B, C, D, E, etc. per fequentes formulas integrales determinantur, fiquidem in fingulis integratio a termino $\phi = 0$, usque ad terminum $\phi = \pi$ extendatur, denotante π femiperipheriam circuli cuius radius $= 1$.

1. $A = \frac{1}{\pi} \int \Phi \partial \phi$.

2. $B = \frac{2}{\pi} \int \Phi \partial \phi \operatorname{cof.} \phi$.

3. $C = \frac{2}{\pi} \int \Phi \partial \phi \operatorname{cof.} 2\phi$.

4. $D = \frac{2}{\pi} \int \Phi \partial \phi \operatorname{cof.} 3\phi$.

5. $E = \frac{2}{\pi} \int \Phi \partial \phi \operatorname{cof.} 4\phi$.

etc. etc.

ubi notetur primum coëfficientem effe $\frac{1}{\pi}$ dum fequentes omnes funt $\frac{2}{\pi}$.

§. 4. Hic primum obfervaffe iuvabit omnes has formulas integrales facillime per quadraturas curvarum fatis fimplicium repraefentari poffe. Si enim fuper axe rectilineo A B abfciffae A P aequales capiantur arcubus, qui angulos ϕ metiuntur, ita ut fit $AP = \phi$, tum vero fuper hoc axe conftruatur curva E M F, cuius applicatae P M referant functionem propofitam Φ, tum formula $\int \Phi \partial \phi$ exprimet aream A E M P, cuius initium in A ftatuitur, ubi $\phi = 0$. Quodfi iam punctum P usque ad B promoveatur, ut fiat $AB = \pi$, tum area A E F B per $\frac{1}{\pi}$ multiplicata ftatim praebet primum terminum A feriei quam quaerimus. Simili modo fecundus terminus B, fimulque omnes fequentes,

Tab. I. Fig. 6.

con-

construi poterunt, si curva EMF ita describatur, ut pro secundo termino B capiatur applicata $PM = \Phi \cos. \Phi$; pro tertio vero $PM = \Phi \cos. 2\Phi$; pro quarto $PM = \Phi \cos. 3\Phi$, et ita porro; tum enim tota area AEMF in $\frac{2}{\pi}$ ducta has ipsas quantitates B, C, D, etc. exhibebit.

§. 5. Quoniam hoc modo abscissae AP arcubus circularibus aequales sunt capiendae, istae curvae descriptae pro algebraicis haberi nequeunt; interim tamen harum curvarum loco algebraicae substitui poterunt, ita ut omnes nostrae quantitates adeo per quadraturas curvarum algebraicarum exhiberi queant; tantum enim ponatur $\cos. \Phi = x$, et cum functio Φ spectari possit tanquam functio ipsius Φ, erit nunc Φ functio algebraica ipsius x. Cum autem hinc fiat $\partial \Phi = \frac{-\partial x}{\sqrt{(1-xx)}}$, pro prima quantitate habebimus:

$$A = -\frac{1}{\pi} \int \frac{\Phi \partial x}{\sqrt{(1-xx)}},$$

unde constructio ita erit instituenda, ut singulis abscissis $AP = x$ respondeant applicatae $PM = \frac{\Phi}{\sqrt{(1-xx)}}$, ita ut iam futura sit area $AEMP = \int \frac{\Phi \partial x}{\sqrt{(1-xx)}}$, quam autem nunc a termino $x = 1$ usque ad terminum $x = -1$ extendi oportet. Hic ergo abscissas a puncto fixo medio C capi conveniet, statuique $CP = x$ et $PM = \frac{\Phi}{\sqrt{(1-xx)}}$; tum enim, si fuerit $CA = 1$ et $CB = -1$, area AEFB, toti basi AB imminens, proposito satisfaciet, ita ut omes istae determinationes per quadraturas linearum curvarum expediri queant.

§. 6. His praenotatis adgrediamur demonstrationem nostri theorematis, ac primo manifestum est, si i denotet numerum

rum integrum quemcunque, integrale

$$\int \partial \Phi \operatorname{cos.} i \Phi = \tfrac{1}{i} \operatorname{fin.} i \Phi,$$

annihilari tam posito $\Phi = 0$, quam posito $\Phi = \pi$, quod ergo pro omnibus numeris integris i valebit, solo casu $i = 0$ excepto, quippe quo prodit $\int \partial \Phi \operatorname{cos.} i \Phi = \pi$. Hoc observato, quoniam per hypothesin est

$$\Phi = A + B \operatorname{cos.} \Phi + C \operatorname{cos.} 2 \Phi + D \operatorname{cos.} 3 \Phi + \text{etc.}$$

erit $\int \Phi \partial \Phi = A \pi$, integralibus scilicet a $\Phi = 0$ usque ad $\Phi = \pi$ extensis. Hinc igitur iam evicta est pars prima nostrae theorematis, qua est $A = \tfrac{1}{\pi} \int \Phi \partial \Phi$.

§. 7. Pro reliquis partibus consideremus formulam differentialem $\partial \Phi \operatorname{cos.} i \Phi \operatorname{cos.} \lambda \Phi$, quae in simplices cosinus resoluta dat

$$\tfrac{1}{2} \partial \Phi [\operatorname{cos.} (i - \lambda) \Phi + \operatorname{cos.} (i + \lambda) \Phi],$$

unde eius integrale erit

$$\int \partial \Phi \operatorname{cos.} i \Phi \operatorname{cos.} \lambda \Phi = \frac{\operatorname{fin.} (i - \lambda) \Phi}{2 (i - \lambda)} + \frac{\operatorname{fin.} (i + \lambda) \Phi}{2 (i + \lambda)},$$

quod integrale utique evanescit, tam sumto $\Phi = 0$ quam sumto $\Phi = \pi$, ob i et λ numeros integros; si modo unicum casum excipiamus, quo $\lambda = i$, quippe quo casu reperitur

$$\int \partial \Phi \operatorname{cos.} i \Phi^2 = \tfrac{1}{2} \Phi + \tfrac{1}{4 i} \operatorname{fin.} 2 i \Phi.$$

qui valor sumto $\Phi = \pi$ abit in $\tfrac{1}{2} \pi$.

§. 7. Cum igitur, integrationem a $\Phi = 0$ usque ad $\Phi = \pi$ extendendo, semper fit $\int \partial \Phi \operatorname{cos.} i \Phi \operatorname{cos.} \lambda \Phi = 0$, solo casu excepto $\lambda = i$, quippe quo casu integrale erit $= \tfrac{\pi}{2}$, ex aequatione

$\Phi =$

$$\Phi = A + B \cos. \phi + C \cos. 2\phi + D \cos. 3\phi + E \cos. 4\phi + \text{etc.}$$

pro parte ſecunda theorematis noſtri reperiemus

$$\int \Phi \partial \phi \cos. \phi = \tfrac{1}{2} \pi B,$$

propterea quod ex omnibus reliquis formulis nihil oritur; hinc viciſſim concluditur fore

$$B = \tfrac{2}{\pi} \int \Phi \partial \phi \cos. \phi,$$

ſi quidem integrale a $\phi = 0$ ad $\phi = \pi$ extendatur.

§. 8. Simili modo pro parte tertia reperiemus

$$\int \Phi \partial \phi \cos. 2\phi = \tfrac{1}{2} \pi C,$$

ideoque viciſſim habebimus:

$$C = \tfrac{2}{\pi} \int \Phi \partial \phi \cos. 2\phi.$$

Pari modo pro partibus ſequentibus prodibit

$$D = \tfrac{2}{\pi} \int \Phi \partial \phi \cos. 3\phi.$$

$$E = \tfrac{2}{\pi} \int \Phi \partial \phi \cos. 4\phi,$$

ſicque porro in infinitum: hocque ergo modo veritas noſtri theorematis perfecte eſt demonſtrata.

§. 9. Poſtquam veritatem noſtri theorematis extra omne dubium collocavimus, haud difficile erit, pro quovis caſu, quo functio propoſita Φ per ſeriem datur, cuius ſinguli termini ſecundum poteſtates ipſius $\cos. \phi$ progrediuntur, alteram ſeriem, quam intendimus

$$A + B \cos. \phi + C \cos. 2\phi + D \cos. 3\phi + \text{etc.}$$

formare atque dilucide oſtendere, quemadmodum ſinguli eius termini A, B, C, D, etc. exprimantur. Quoniam vero ſingulae

gulae litterae latinae maiores ab omnibus litteris graecis, sequentibus in infinitum pendent, ne ex ordine istarum litterarum confusio oriatur, loco litterarum graecarum sequentes characteres introducamus:

$$\Phi = (0) + (1)\cos.\Phi + (2)\cos.\Phi^2 + (3)\cos.\Phi^3 + (4)\cos.\Phi^4 + \text{etc.}$$

et iam quaestio huc redit, quomodo singulae litterae latinae A, B, C, D, etc. ex istis characteribus (0); (1); (2); (3); etc. definiri debeant.

§. 10. Incipiamus a primi littera A, cuius evolutio postulat sequens Lemma:

Lemma.

Si integralia a $\Phi = 0$ usque ad $\Phi = \pi$, extendantur, semper erit

$$\int \partial\Phi \cos.\Phi^\lambda = \frac{\lambda-1}{\lambda}\int\partial\Phi\cos.\Phi^{\lambda-2}.$$

Ad hoc demonstrandum ponatur

$$\int\partial\Phi\cos.\Phi^\lambda = f\sin.\Phi\cos.\Phi^{\lambda-1} + g\int\partial\Phi\cos.\Phi^{\lambda-2},$$

et sumtis differentialibus erit

$$\cos.\Phi^\lambda = f\cos.\Phi^\lambda - f(\lambda-1)\sin.\Phi^2\cos.\Phi^{\lambda-2} + g\cos.\Phi^{\lambda-2},$$

quae aequatio, ob $\sin.\Phi^2 = 1 - \cos.\Phi^2$, induet hanc formam:

$$\cos.\Phi^\lambda = \lambda f\cos.\Phi^\lambda - f(\lambda-1)\cos.\Phi^{\lambda-2} + g\cos.\Phi^{\lambda-2},$$

unde primo fit $g = f(\lambda-1)$ et $f = \frac{1}{\lambda}$, ideoque $g = \frac{\lambda-1}{\lambda}$. sicque in genere habemus hanc reductionem:

$$\int\partial\Phi\cos.\Phi^\lambda = \frac{1}{\lambda}\sin.\Phi\cos.\Phi^{\lambda-1} + \frac{\lambda-1}{\lambda}\int\partial\Phi\cos.\Phi^{\lambda-2},$$

quod integrale ita capi debet, ut posito $\Phi = 0$ evanescat.

Quam-

Quamobrem si statuamus $\Phi = \pi$, unde fit sin. $\Phi = 0$; casu Lemmatis habebimus:

$$\int \partial \Phi \cos. \Phi^{\lambda} = \frac{\lambda - 1}{\lambda} \int \partial \Phi \cos. \Phi^{\lambda - 2}.$$

§. 11. Quoniam igitur a casibus simplicissimis incipiendo habemus:

I. $\int \partial \Phi \cos. \Phi^{0} = \pi.$

II. $\int \partial \Phi \cos. \Phi^{1} = 0,$

hinc omnes sequentes formulas assignare possumus:

III. $\int \partial \Phi \cos. \Phi^{2} = \frac{1}{2} \pi.$ IV. $\int \partial \Phi \cos. \Phi^{3} = 0.$

V. $\int \partial \Phi \cos. \Phi^{4} = \frac{1}{2} \cdot \frac{3}{4} \pi.$ VI. $\int \partial \Phi \cos. \Phi^{5} = 0.$

VII. $\int \partial \Phi \cos. \Phi^{6} = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \pi.$ VIII. $\int \partial \Phi \cos. \Phi^{7} = 0.$

IX. $\int \partial \Phi \cos. \Phi^{8} = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \pi.$ X. $\int \partial \Phi \cos. \Phi^{9} = 0.$

etc. etc.

§. 12. Quia igitur supra invenimus esse

$\pi A = \int \Phi \partial \Phi$, ob

$$\Phi = (0) + (1) \cos. \Phi + (2) \cos. \Phi^{2} + (3) \cos. \Phi^{3} + \text{etc.}$$

integrationes modo assignatae praebent

$$\pi A = (0) \pi + (2) \frac{1}{2} \pi + (4) \frac{1}{2} \cdot \frac{3}{4} \pi + (6) \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \pi + (8) \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \pi + \text{etc.}$$

divisione ergo per π facta nanciscimur hanc determinationem:

$$A = (0) + \frac{1}{2}(2) + \frac{1}{2} \cdot \frac{3}{4}(4) + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6}(6) + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8}(8) + \text{etc.}$$

sive elegantius

$$A = (0) + \frac{2}{4}(2) \frac{4\cdot 3}{4\cdot 8}(4) + \frac{6\cdot 5\cdot 4}{4\cdot 8\cdot 12}(6) + \frac{8\cdot 7\cdot 6\cdot 5}{4\cdot 8\cdot 12\cdot 16}(8) + \text{etc.}$$

quae est eadem series, quam olim per satis longas ambages sum adeptus.

§. 13. Ope eiusdem Lemmatis etiam secunda littera B definiri poterit. Quia enim invenimus $\frac{1}{2}\pi B = \int \Phi \partial \Phi \,\text{cos.}\, \phi$, si loco Φ seriem cognitam substituamus, integrationes Lemmatis nobis dabunt:

$$\frac{1}{2}\pi B = + \frac{1}{2}\pi (1) + \frac{1}{2}\cdot\frac{3}{4}\pi (3) + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\pi (5) + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\pi (7) + \text{etc.}$$

unde per π dividendo erit:

$$\frac{1}{2}B = \frac{1}{2}(1) + \frac{1}{2}\cdot\frac{3}{4}(3) + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}(5) + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}(7) + \text{etc.}$$

sive

$$B = (1) + \frac{3}{4}(3) + \frac{5\cdot 4}{4\cdot 8}(5) + \frac{7\cdot 6\cdot 5}{4\cdot 8\cdot 12}(7) + \frac{9\cdot 8\cdot 7\cdot 6}{4\cdot 8\cdot 12\cdot 16}(9) + \text{etc.}$$

§. 14. Pro tertia littera C peculiari Lemmate opus erit, quo est:

$$\int \partial \Phi \,\text{cos.}\, 2\Phi \,\text{cos.}\, \Phi^\lambda = \frac{\lambda(\lambda-1)}{\lambda\lambda-4} \int \partial \Phi \,\text{cos.}\, 2\Phi \,\text{cos.}\, \Phi^{\lambda-2},$$

siquidem integralia a $\Phi = 0$ usque $\Phi = \pi$ extendantur. Ad hoc demonstrandum ponamus in genere esse:

$$\int \partial \Phi \,\text{cos.}\, 2\Phi \,\text{cos.}\, \Phi^\lambda = f \,\text{fin.}\, 2\Phi \,\text{cos.}\, \Phi^\lambda + g \,\text{cos.}\, 2\Phi \,\text{fin.}\, \Phi \,\text{cos.}\, \Phi^{\lambda-1} + h \int \partial \Phi \,\text{cos.}\, 2\Phi \,\text{cos.}\, \Phi^{\lambda-2},$$

unde differentiatio praebet hanc aequationem:

$$\text{cos.}\, 2\Phi \,\text{cos.}\, \Phi^\lambda = 2f \,\text{cos.}\, 2\Phi \,\text{cos.}\, \Phi^\lambda + g \,\text{cos.}\, 2\Phi \,\text{cos.}\, \Phi^\lambda - g(\lambda-1) \,\text{cos.}\, 2\Phi \,\text{fin.}\, \Phi^2 \,\text{cos.}\, \Phi^{\lambda-2} - \lambda f \,\text{fin.}\, 2\Phi \,\text{fin.}\, \Phi \,\text{cos.}\, \Phi^{\lambda-1} - 2g \,\text{fin.}\, 2\Phi \,\text{fin.}\, \Phi \,\text{cos.}\, \Phi^{\lambda-1} + h \,\text{cos.}\, 2\Phi \,\text{cos.}\, \Phi^{\lambda-2}.$$

Hic

c iam primo termini, qui continent fin. 2 Φ, tolli debent, de fit $g = -\frac{\lambda f}{2}$; tum vero remanebit haec aequatio, post-am loco fin. Φ^2 fcriptum fuerit $1 -$ cof. Φ^2, divifione per 2 Φ facta,

$$\text{cof.}\,\Phi^\lambda = -\frac{f(\lambda\lambda-4)}{2}\,\text{cof.}\,\Phi^\lambda + \frac{\lambda f}{2}(\lambda-1)\,\text{cof.}\,\Phi^{\lambda-2} + h\,\text{cof.}\,\Phi^{\lambda-2},$$

de manifefto fit $f = \frac{-2}{\lambda\lambda-4}$, hincque $h = \frac{\lambda(\lambda-1)}{\lambda\lambda-4}$, ficque uctio generalis ita fe habebit:

$$\int\partial\Phi\,\text{cof.}\,2\Phi\,\text{cof.}\,\Phi^\lambda = \frac{-2}{\lambda\lambda-4}\,\text{fin.}\,2\Phi\,\text{cof.}\,\Phi^\lambda$$
$$+ \frac{\lambda}{\lambda\lambda-4}\,\text{cof.}\,2\Phi\,\text{fin.}\,\Phi\,\text{cof.}\,\Phi^{\lambda-1} + \frac{\lambda(\lambda-1)}{\lambda\lambda-4}\,\partial\Phi\,\text{cof.}\,2\Phi\,\text{cof.}\,\Phi^{\lambda-2}.$$

nc iam, pofito $\Phi = \pi$, erit fecundum Lemma

$$\int\partial\Phi\,\text{cof.}\,2\Phi\,\text{cof.}\,\Phi^\lambda = \frac{\lambda(\lambda-1)}{\lambda\lambda-4}\int\partial\Phi\,\text{cof.}\,2\Phi\,\text{cof.}\,\Phi^{\lambda-2}.$$

§. 15. Tribuamus nunc exponenti λ fucceffive ordi- valores 0, 1, 2, 3, 4, etc. ac pro $\lambda = 0$ erit

$$\int\partial\Phi\,\text{cof.}\,2\Phi = \tfrac{1}{2}\,\text{fin.}\,2\Phi = 0,$$

o cafu $\lambda = 1$ ipfum Lemma praebet $= 0$; at vero pro fu $\lambda = 2$ ufus Lemmatis ceffat: tractanda ergo erit ipfa mula $\int\partial\Phi\,\text{cof.}\,2\Phi\,\text{cof.}\,\Phi^2$, quae ob cof. $\Phi^2 = \frac{1}{2} + \frac{1}{2}$ cof. 2Φ it in hanc: $+\frac{1}{2}\int\partial\Phi\,\text{cof.}\,2\Phi^2$, quae ob cof. $2\Phi^2 = \frac{1}{2} + \frac{1}{2}$cof. 4Φ, it in $\frac{1}{4}\int\partial\Phi\,(1 + \text{cof.}\,4\Phi) = \frac{1}{4}\pi$, ficque pro cafu $\lambda = 2$ erit Φ cof. 2 Φ cof. $\Phi^2 = \frac{\pi}{4}$.

§. 16. His igitur cafibus fimplicioribus expeditis quentes ope Lemmatis facile conficiuntur; reperiemus enim:

1. $\int\partial\Phi\,\text{cof.}\,2\Phi\,\text{cof.}\,\Phi^3 = 0.$
2. $\int\partial\Phi\,\text{cof.}\,2\Phi\,\text{cof.}\,\Phi^4 = \frac{4\cdot3}{2\cdot6}\cdot\frac{\pi}{4}.$
3. $\int\partial\Phi\,\text{cof.}\,2\Phi\,\text{cof.}\,\Phi^5 = 0.$

4. $\int \partial \Phi \cos. 2\Phi \cos. \Phi^6 = \frac{4 \cdot 3}{2 \cdot 6} \cdot \frac{6 \cdot 5}{4 \cdot 8} \cdot \frac{\pi}{4}.$

5. $\int \partial \Phi \cos. 2\Phi \cos. \Phi^8 = \frac{4 \cdot 3}{2 \cdot 6} \cdot \frac{6 \cdot 5}{4 \cdot 8} \cdot \frac{8 \cdot 7}{6 \cdot 10} \cdot \frac{\pi}{4}.$

etc. etc.

Cum igitur invenerimus $\frac{1}{2}\pi C = \int \Phi \partial \Phi \cos. 2\Phi$, si integralia modo inventa introducantur, ac per π dividantur, reperietur:

$$\tfrac{1}{2}C = \tfrac{1}{4}(2) + \tfrac{1}{4} \cdot \tfrac{4 \cdot 3}{2 \cdot 6}(4) + \tfrac{1}{4} \cdot \tfrac{4 \cdot 3}{2 \cdot 6} \cdot \tfrac{6 \cdot 5}{4 \cdot 8}(6) + \tfrac{1}{4} \cdot \tfrac{4 \cdot 3}{2 \cdot 6} \cdot \tfrac{6 \cdot 5}{4 \cdot 8} \cdot \tfrac{8 \cdot 7}{6 \cdot 10}(8) + \text{etc.}$$

quae concinnius hoc modo exprimi potest:

$$\tfrac{1}{2}C = \tfrac{1}{2}(2) + \tfrac{3}{2 \cdot 6}(4) + \tfrac{3 \cdot 5}{2 \cdot 4 \cdot 8}(6) + \tfrac{3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 10}(8) + \tfrac{3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 12}(10) + \text{etc.}$$

quae adhuc elegantius ita referri potest:

$$\tfrac{1}{2}C = \tfrac{1}{4} \cdot (2) + \tfrac{1}{6} \cdot \tfrac{3}{2}(4) + \tfrac{1}{8} \cdot \tfrac{3 \cdot 5}{2 \cdot 4}(6) + \tfrac{1}{10} \cdot \tfrac{3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6}(8) + \tfrac{1}{12} \cdot \tfrac{3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8}(10) + \text{etc.}$$

sive adhuc elegantius ita:

$$2C = (2) + \tfrac{4}{4}(4) + \tfrac{6 \cdot 5}{4 \cdot 8} \cdot (6) + \tfrac{7 \cdot 6 \cdot 5}{4 \cdot 8 \cdot 12}(8) + \tfrac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 8 \cdot 12 \cdot 16}(10) + \text{etc.}$$

§. 17. Pro sequentibus terminis stabiliamus istud Lemma generale:

$$\int \partial \Phi \cos. i\Phi \cos. \Phi^\lambda \left(\begin{matrix} a\ \Phi = 0 \\ ad\ \Phi = \pi \end{matrix}\right) = \frac{\lambda(\lambda - 1)}{\lambda\lambda - ii} \int \partial \Phi \cos. i\Phi \cos. \Phi^{\lambda - 2},$$

pro quo demonstrando statuamus in genere

$$\int \partial \Phi \cos. i\Phi \cos. \Phi^\lambda = f \sin. i\Phi \cos. \Phi^\lambda + g \cos. i\Phi \sin. \Phi \cos. \Phi^{\lambda - 1} + h \int \partial \Phi \cos. i\Phi \cos. \Phi^{\lambda - 2},$$

unde

unde differentiatio perducit ad hanc aequationem:

$$\cos. i\Phi \cos.\Phi^\lambda = (fi+g)\cos. i\Phi \cos.\Phi^\lambda - (\lambda f + gi)\sin. i\Phi \cos.\Phi^{\lambda-1} - g(\lambda-1)\cos. i\Phi \sin.\Phi^2 \cos.\Phi^{\lambda-2} + h\cos. i\Phi\cos.\Phi^{\lambda-2}$$

Hic iam primo termini, qui continent $\sin. i\Phi$, tolli debent, unde fit $g = -\frac{\lambda f}{i}$, quo valore substituto, per $\cos. i\Phi$ dividendo, postquam loco $\sin.\Phi^2$ scriptum fuerit $1 - \cos.\Phi^2$, prodit ista aequatio:

$$\cos.\Phi^\lambda = -\frac{f(\lambda\lambda - ii)}{i}\cos.\Phi^\lambda + \frac{\lambda f}{i}(\lambda-1)\cos.\Phi^{\lambda-2} + h\cos.\Phi^{\lambda-2},$$

unde manifesto fit $f = \frac{-i}{\lambda\lambda - ii}$, hincque $h = \frac{\lambda(\lambda-1)}{\lambda\lambda - ii}$, sicque reductio generalis ita se habebit:

$$\int\partial\Phi \cos. i\Phi \cos.\Phi^\lambda = \frac{-i}{\lambda\lambda - ii}\sin. i\Phi\cos.\Phi^\lambda + \frac{\lambda}{\lambda\lambda - ii}\cos.\Phi \sin. i\Phi \cos.\Phi^{\lambda-1} + \frac{\lambda(\lambda-1)}{\lambda\lambda - ii}\int\partial\Phi\cos. i\Phi\cos.\Phi^{\lambda-2},$$

quae saepenumero maximam utilitatem habere potest; posito autem $\Phi = \pi$ manifesto prodit effatum Lemmatis.

§. 18. Hoc Lemmate constituto pro littera D definienda sumi debet $i = 3$, eritque

$$\int\partial\Phi\cos. 3\Phi\cos.\Phi^\lambda = \frac{\lambda(\lambda-1)}{(\lambda-3)(\lambda+3)}\int\partial\Phi\cos. 3\Phi\cos.\Phi^{\lambda-2};$$

unde statim patet, casibus $\lambda = 0$ et $\lambda = 1$ formulam istam evanescere, posito scilicet $\Phi = \pi$, ita ut fit

$$\int\partial\Phi\cos. 3\Phi = 0 \text{ et } \int\partial\Phi\cos. 3\Phi\cos.\Phi = 0.$$

Hinc autem porro patet, casu quoque $\lambda = 2$ fore

$$\int\partial\Phi\cos. 3\Phi\cos.\Phi^2 = 0.$$

At vero casu $\lambda = 3$ Lemma dabit

$\int\partial\Phi$

$$\int \partial \Phi \operatorname{cof.} 3 \Phi \operatorname{cof.} \Phi^3 = \frac{0}{0},$$

cuius ergo valor peculiari modo inveftigari debet; neque vero artificia cognita hic ullum ufum praeftari poterunt.

§. 19. Ad ipfam ergo indolem formulae propofitae $\int \partial \Phi \operatorname{cof.} 3 \Phi \operatorname{cof.} \Phi^3$ refpicere debemus, refolvendo poteftatem $\operatorname{cof.} \Phi^3$ in hanc formam: $\frac{\operatorname{cof.} 3 \Phi + 3 \operatorname{cof.} \Phi}{4}$; tum igitur erit

$$\operatorname{cof.} 3 \Phi \operatorname{cof.} \Phi^3 = \tfrac{1}{8} + \tfrac{3}{8} \operatorname{cof.} 2 \Phi + \tfrac{3}{8} \operatorname{cof.} 4 \Phi + \tfrac{1}{8} \operatorname{cof.} 6 \Phi,$$

quae forma ducta in $\partial \Phi$ et integrata dat

$$\int \partial \Phi \operatorname{cof.} 3 \Phi \operatorname{cof.} \Phi^3 = \tfrac{1}{8} \Phi + \tfrac{3}{16} \operatorname{fin.} 2 \Phi + \tfrac{3}{32} \operatorname{fin.} 4 \Phi + \tfrac{1}{48} \operatorname{fin.} 6 \Phi,$$

unde fumto $\Phi = \pi$ valor exfurgit $= \frac{1}{8} \pi$, ita ut fit

$$\int \partial \Phi \operatorname{cof.} 3 \Phi \operatorname{cof.} \Phi^3 = \tfrac{1}{8} \pi.$$

§. 20. Ab hoc autem valore $\lambda = 3$ pendent fequentes: $\lambda = 5$; $\lambda = 7$; $\lambda = 9$; etc., qui ergo ita fe habebunt:

$$\int \partial \Phi \operatorname{cof.} 3 \Phi \operatorname{cof.} \Phi^3 = \frac{\pi}{8},$$

$$\int \partial \Phi \operatorname{cof.} 3 \Phi \operatorname{cof.} \Phi^5 = \frac{5 \cdot 4}{2 \cdot 8} \cdot \frac{\pi}{8},$$

$$\int \partial \Phi \operatorname{cof.} 3 \Phi \operatorname{cof.} \Phi^7 = \frac{5 \cdot 4}{2 \cdot 8} \cdot \frac{7 \cdot 6}{4 \cdot 10} \cdot \frac{\pi}{8},$$

$$\int \partial \Phi \operatorname{cof.} 3 \Phi \operatorname{cof.} \Phi^9 = \frac{5 \cdot 4}{2 \cdot 8} \cdot \frac{7 \cdot 6}{4 \cdot 10} \cdot \frac{9 \cdot 8}{6 \cdot 12} \cdot \frac{\pi}{8},$$

$$\int \partial \Phi \operatorname{cof.} 3 \Phi \operatorname{cof.} \Phi^{11} = \frac{5 \cdot 4}{2 \cdot 8} \cdot \frac{7 \cdot 6}{4 \cdot 10} \cdot \frac{9 \cdot 8}{6 \cdot 12} \cdot \frac{11 \cdot 10}{8 \cdot 14} \cdot \frac{\pi}{8},$$

etc. etc.

reliqui vero cafus omnes evanefcunt.

§. 21.

§. 21. Cum nunc ſit $D = \frac{2}{\pi}\int \Phi\, \partial\phi \cos. 3\,\phi$, exiſtente

$$\Phi = (0) + (1)\cos.\phi + (2)\cos.\phi^2 + (3)\cos.\phi^3 + (4)\cos.\phi^4 + \text{etc.}$$

ſingulos valores integrales in unam ſummam colligendo reperietur:

$$D = \tfrac{1}{4}\left[1\,(3) + \tfrac{5\cdot4}{2\cdot8}(5) + \tfrac{5\cdot4}{2\cdot8}\cdot\tfrac{7\cdot6}{4\cdot10}(7) + \tfrac{5\cdot4}{2\cdot8}\cdot\tfrac{7\cdot6}{4\cdot10}\cdot\tfrac{9\cdot8}{6\cdot12}(9) + \text{etc.}\right]$$

quae quidem expreſſio in plures alias formas transfundi posſet, quarum elegantiſſima eſt haec:

$$4D = (3) + \tfrac{5}{4}(5) + \tfrac{7\cdot6}{4\cdot8}(7) + \tfrac{9\cdot3\cdot7}{4\cdot8\cdot12}(9) + \tfrac{11\cdot10\cdot9\cdot8}{4\cdot8\cdot12\cdot16}(11)$$
$$+ \tfrac{13\cdot12\cdot11\cdot10\cdot9}{4\cdot8\cdot12\cdot16\cdot20}(13) + \text{etc.}$$

§. 22. Pro littera porro E invenienda poni debet $i = 4$, et Lemma praemiſſum dabit:

$$\int \partial\phi \cos. 4\,\phi \cos.\phi^\lambda = \frac{\lambda(\lambda-1)}{(\lambda-4)(\lambda+4)} \int \partial\phi \cos. 4\,\phi \cos.\phi^{\lambda-2},$$

unde iterum patet caſibus $\lambda = 0$ et $\lambda = 1$ valorem evaneſcere, quod propterea etiam continget caſibus $\lambda = 2$ et $\lambda = 3$; at vero caſus $\lambda = 4$ peculiarem evolutionem poſtulat. Quoniam vero ante vidimus eſſe $\cos.\phi^3 = \frac{1}{4}\cos. 3\,\phi + \frac{3}{4}\cos.\phi$, ſi denuo per $\cos.\phi$ multiplicemus, prodibit $\cos.\phi^4 = \frac{3}{8} + \frac{1}{2}\cos. 2\,\phi + \frac{1}{8}\cos. 4\,\phi$, quae forma porro in $\cos. 4\phi$ ducta dabit:

$$\cos. 4\,\phi \cos.\phi^4 = \tfrac{1}{16} + \tfrac{4}{16}\cos. 2\,\phi + \tfrac{6}{16}\cos. 4\,\phi$$
$$+ \tfrac{4}{16}\cos. 6\,\phi + \tfrac{1}{16}\cos. 8\,\phi.$$

Haec iam formula ducatur in $\partial\phi$ et integretur, tum vero facto $\phi = \pi$ manifeſto reſultabit valor quaeſitus $= \frac{1}{16}\pi$, qui eatenus tantum prodiit, quatenus poteſtas $\cos.\phi^4$ per reſolutionem dederat $\cos. 4\,\phi$.

§. 23. Cum igitur caſu $\lambda = 4$ prodierat valor $\frac{\pi}{16}$, reliqui independentes vi Lemmatis ſequentes accipient valores:

$\int \partial\phi$

$$\int \partial \Phi \operatorname{cof.} 4 \Phi \operatorname{cof.} \Phi^4 = \frac{\pi}{16},$$
$$\int \partial \Phi \operatorname{cof.} 4 \Phi \operatorname{cof.} \Phi^6 = \frac{6.5}{2.10} \cdot \frac{\pi}{16},$$
$$\int \partial \Phi \operatorname{cof.} 4 \Phi \operatorname{cof.} \Phi^8 = \frac{6.5}{2.10} \cdot \frac{8.7}{4.12} \cdot \frac{\pi}{16},$$
$$\int \partial \Phi \operatorname{cof.} 4 \Phi \operatorname{cof.} \Phi^{10} = \frac{6.5}{2.10} \cdot \frac{8.7}{4.12} \cdot \frac{10.9}{6.14} \cdot \frac{\pi}{16}.$$
etc. etc.

Cum igitur fit $E = \frac{2}{\pi} \int \Phi \partial \Phi \operatorname{cof.} 4 \Phi$, exiftente

$$\Phi = (0) + (1) \operatorname{cof.} \Phi + (2) \operatorname{cof.} \Phi^2 + (3) \operatorname{cof.} \Phi^3 + (4) \operatorname{cof.} \Phi^4 + \text{etc.}$$

praemiffae reductiones nobis fuppeditabunt fequentem valorem:

$$E = \frac{1}{8} [(4) + \frac{5.6}{2.10} (6) + \frac{5.6.\ 7.8}{2.10.4.12} (8) + \frac{5.6.\ 7.8.\ 9.10}{2.10.4.12.6.14} (10) + \text{etc.}]$$

quae forma haud difficulter in fequentem transfunditur:

$$8E = (4) + \frac{6}{4} (6) + \frac{8.7}{4.8} (8) + \frac{10.9.8}{4.\ 8.12} (10) + \frac{13.11.10.9}{4.\ 8.\ 12.16} (12) + \text{etc.}$$

His cafibus evolutis iam rem in genere exfequi poffumus pro quocunque numero i, ubi totum negotium ad cafum $\lambda = i$ reducitur, quem ergo in peculiari problemate refolvamus.

Problema.

Denotante i numerum integrum quemcunque inveftigare valorem huius formulae integralis: $\int \partial \Phi \operatorname{cof.} i \Phi \operatorname{cof.} \Phi^i$, fi quidem poft integrationem ftatuatur $\Phi = \pi$.

Solutio.

§. 24. Vt folutionem ex primis principiis repetamus, ponamus

$$\operatorname{cof.} \Phi + \sqrt{-1} \operatorname{fin.} \Phi = p \text{ et } \operatorname{cof.} \Phi - \sqrt{-1} \operatorname{fin.} \Phi = q,$$

erit-

eritque primo $pq = 1$, deinde vero erit $\cos.\Phi = \frac{p+q}{2}$, et quia porro eſt

$$p^n = \cos. n\Phi + \sqrt{-1} \sin. n\Phi \text{ et}$$
$$q^n = \cos. n\Phi - \sqrt{-1} \sin. n\Phi,$$

erit $\cos. i\Phi = \frac{p^i + q^i}{2}$, praeterea vero erit $\cos. \Phi^i = \frac{(p+q)^i}{2^i}$.

§. 25. Evolvatur iam poteſtas $(p+q)^i$ more ſolito; verum termini poſtremi cum primis iuncti repraeſententur hoc modo:

$$(p+q)^i = + p^i + \tfrac{i}{1} p^{i-1} q + \tfrac{i(i-1)}{1.2} p^{i-2} qq + \tfrac{i(i-1)(i-2)}{1.2.3} p^{i-3} q^3 + \text{etc.}$$
$$+ q^i + \tfrac{i}{1} p q^{i-1} + \tfrac{i(i-1)}{1.2} pp q^{i-2} + \tfrac{i(i-1)(i-2)}{1.2.3} p^3 q^{i-3} + \text{etc.}$$

quae ſeries, ob $pq = 1$, in hanc formam commodiorem redigitur:

$$(p+q)^i = p^i + q^i + \tfrac{i}{1}.(p^{i-2} + q^{i-2}) + \tfrac{i(i-1)}{1.2}(p^{i-4} + q^{i-4})$$
$$+ \tfrac{i(i-1)(i-2)}{1.2.3}(p^{i-6} + q^{i-6}) \text{ etc.}$$

Hic tantum notari oportet caſibus, quibus i eſt numerus par, terminum dari medium ſolitarium, qui continebit quantitatem conſtantem, quam ergo duplicare non decet.

§. 26. Cum igitur ad angulos regrediendo ſit in genere $p^n + q^n = 2 \cos. n\Phi$, erit nunc:

$$(p+q)^i = 2\cos. i\Phi + \tfrac{2i}{1}\cos.(i-2)\Phi + \tfrac{2i(i-1)}{1.2}\cos.(i-4)\Phi$$
$$+ \tfrac{2i(i-1)(i-2)}{1.2.3}\cos.(i-6)\Phi + \text{etc.}$$

quoniam vero eſt $p + q = 2\cos.\Phi$, erit

$$2^{i-1}\cos.\Phi^i = \cos. i\Phi + \tfrac{i}{1}\cos.(i-2)\Phi + \tfrac{i(i-1)}{1.2}\cos.(i-4)\Phi$$
$$+ \tfrac{i(i-1)(i-2)}{1.2.3}\cos.(i-6)\Phi + \text{etc.}$$

Multiplicetur nunc utrinque per $2\,\mathrm{cof.}\,i\Phi$ et per notiffimas reductiones reperietur:

$$2\,\mathrm{cof.}\,i\Phi\,\mathrm{cof.}\,\Phi^i = 1 + \tfrac{i}{1}\,\mathrm{cof.}\,2\Phi + \tfrac{i(i-1)}{1.\,2.}\,\mathrm{cof.}\,4\Phi + \tfrac{i(i-1)(i-2)}{1.\,2.\,3}\,\mathrm{cof.}\,6\Phi + \text{etc.}$$
$$+ \mathrm{cof.}\,2i\Phi + \tfrac{i}{1}\,\mathrm{cof.}(2i-2)\Phi + \tfrac{i(i-1)}{1.\,2}\,\mathrm{cof.}(2i-4)\Phi$$
$$+ \tfrac{i(i-1)(i-2)}{1.\,2.\,3.}\,\mathrm{cof.}(2i-6)\Phi + \text{etc.}$$

§. 27. Multiplicetur nunc utrinque per $\partial\Phi$ et integretur, prodibitque

$$2^i\int\partial\Phi\,\mathrm{cof.}\,i\Phi\,\mathrm{cof.}\,\Phi^i = \Phi + \tfrac{i}{2}.\,\mathrm{fin.}\,2\Phi + \tfrac{i(i-2)}{4.\,2.}\,\mathrm{fin.}\,4\Phi + \text{etc.}$$
$$+ \tfrac{1}{2i}\,\mathrm{fin.}\,2i\Phi + \tfrac{i}{2i-2}\,\mathrm{fin.}(2i-2)\Phi + \tfrac{i(i-2)}{4(2i-4)2}\,\mathrm{fin.}(2i-4)\Phi + \text{etc.}$$

quae formula iam evanefcit pofito $\Phi = 0$. Statuatur ergo $\Phi = \pi$, atque proveniet $2^i\int\partial\Phi\,\mathrm{cof.}\,i\Phi\,\mathrm{cof.}\,\Phi^i = \pi$, quocirca valor in problemate quaefitus erit

$$= \int\partial\Phi\,\mathrm{cof.}\,i\Phi\,\mathrm{cof.}\,\Phi^i = \frac{\pi}{2^i}.$$

§. 28. Quodfi iam ponamus in ferie quam quaerimus

$$A + B\,\mathrm{cof.}\,\Phi + C\,\mathrm{cof.}\,2\Phi + D\,\mathrm{cof.}\,3\Phi + E\,\mathrm{cof.}\,4\Phi + \text{etc.}$$

coëfficientem ipfius $\mathrm{cof.}\,i\Phi$ effe I, ita ut fit

$$I = \tfrac{2}{\pi}\int\Phi\,\partial\Phi\,\mathrm{cof.}\,i\Phi\,,\ \text{exiftente}$$
$$\Phi = (0) + (1)\,\mathrm{cof.}\,\Phi + (2)\,\mathrm{cof.}\,\Phi^2 + (3)\,\mathrm{cof.}\,\Phi^3 + (4)\,\mathrm{cof.}\,\Phi^4 + \text{etc.}$$

evidens eft ex fingulis terminis initialibus nihil prodire, donec perveniatur ad $\lambda = i$, quippe quo cafu modo vidimus effe $\int\partial\Phi\,\mathrm{cof.}\,i\Phi\,\mathrm{cof.}\,\Phi^i = \frac{\pi}{2^i}$, a quo valore pendent cafus fequentes per binarium afcendentes, $\lambda = i + 2$; $\lambda = i + 4$; $\lambda = i + 6$; etc. Scilicet vi Lemmatis erit:

$\int\partial\Phi$

$$\int \partial\Phi \cos. i\Phi \cos. \Phi^{i+2} = \frac{\pi}{2^i} \cdot \frac{\lambda(\lambda-1)}{(\lambda-i)(\lambda+i)} = \frac{(i+2)}{4} \cdot \frac{\pi}{2^i};$$

$$\int \partial\Phi \cos. i\Phi \cos. \Phi^{i+4} = \frac{(i+4)(i+3)}{4 \cdot 8} \cdot \frac{\pi}{2^i};$$

$$\int \partial\Phi \cos. i\Phi \cos. \Phi^{i+6} = \frac{(i+6)(i+5)(i+4)}{4 \cdot 8 \cdot 12} \cdot \frac{\pi}{2^i};$$

$$\int \partial\Phi \cos. i\Phi \cos. \Phi^{i+8} = \frac{(i+8)(i+7)(i+6)(i+5)}{4 \cdot 8 \cdot 12 \cdot 16} \cdot \frac{\pi}{2^i};$$

$$\int \partial\Phi \cos. i\Phi \cos. \Phi^{i+10} = \frac{(i+10)(i+9)(i+8)(i+7)(i+6)}{4 \cdot 8 \cdot 12 \cdot 16 \cdot 20} \cdot \frac{\pi}{2^i};$$

etc. etc.

§. 29. His iam terminis colligendis et multiplicando per 2^{i-1} orietur ſequens expreſſio:

$$2^{i-1}\text{I} = (i) + \frac{i+2}{4}(i+2) + \frac{(i+4)(i+3)}{4 \cdot 8}(i+4) + \frac{(i+6)(i+5)(i+4)}{4 \cdot 8 \cdot 12}(i+6) + \frac{(i+8)(i+7)(i+6)(i+5)}{4 \cdot 8 \cdot 12 \cdot 16}(i+8) + \text{etc.}$$

quae forma iam continet determinationem omnium terminorum feriei, in quam formulam:

$$\Phi = (0) + (1)\cos.\Phi + (2)\cos.\Phi^2 + (3)\cos.\Phi^3 + (4)\cos.\Phi^4 + \text{etc.}$$

evolvere erat propoſitum, quam hoc modo hactenus repraeſentavimus:

$$\Phi = \text{A} + \text{B}\cos.\Phi + \text{C}\cos.2\Phi + \text{D}\cos.3\Phi + \text{E}\cos.4\Phi + \text{etc.}$$

cuius ſinguli termini per ſequentes ſeries exprimentur:

$$\text{A} = (0) + \tfrac{2}{4}(2) + \tfrac{4.3}{4.8}(4) + \tfrac{6.5.4}{4.8.12}(6) + \tfrac{8.7.6.5}{4.8.12.16}(8) + \text{etc.}$$

$$1.\ \text{B} = (1) + \tfrac{3}{4}(3) + \tfrac{5.4}{4.8}(5) + \tfrac{7.6.5}{4.8.12}(7) + \tfrac{9.8.7.6}{4.8.12.16}(9) + \text{etc.}$$

$$2.\ \text{C} = (2) + \tfrac{4}{4}(4) + \tfrac{6.5}{4.8}(6) + \tfrac{8.7.6}{4.8.12}(8) + \tfrac{10.9.8.7}{4.8.12.16}(10) + \text{etc.}$$

$$4 \cdot D = (3) + \tfrac{5}{4}(5) + \tfrac{7.6}{4.8}(7) + \tfrac{9.3.7}{4.8.12}(9) + \tfrac{11.10.9.8}{4.8.12.16}(11) + \text{etc.}$$

$$8 \cdot E = (4) + \tfrac{6}{4}(6) + \tfrac{8.7}{4.8}(8) + \tfrac{10.9.8}{4.8.12}(10) + \tfrac{12.11.10.9}{4.8.12.16}(12) + \text{etc.}$$

$$16 \cdot F = (5) + \tfrac{7}{4}(7) + \tfrac{9.8}{4.8}(9) + \tfrac{11.10.9}{4.8.12}(11) + \tfrac{13.12.11.10}{4.8.12.16}(13) + \text{etc.}$$

etc. etc.

atque in genere, fi in ferie quaefita terminus indici i refpondens fuerit I cof. $i\,\Phi$ erit:

$$2^{i-1} I = (i) + \tfrac{i+2}{4}(i+2) + \tfrac{i+4}{4} \cdot \tfrac{i+3}{8}(i+4) + \tfrac{i+6}{4} \cdot \tfrac{i+5}{8} \cdot \tfrac{i+4}{12}(i+6)$$
$$+ \tfrac{i+8}{4} \cdot \tfrac{i+7}{8} \cdot \tfrac{i+6}{12} \cdot \tfrac{i+5}{16}(i+8) + \text{etc.}$$

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