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Methodus facilis inveniendi series per sinus sosinusve angulorum multiplorum procedentes, quarum usus in universa theoria astronomiae est amplissimus

Leonhard Euler

METHODVS FACILIS INVENIENDI SERIES PER SINVS COSINVSVE ANGVLORVM MVLTIPLORVM PROCEDENTES, QVARVM VSVS IN VNIVERSA THEORIA ASTRONOMIAE EST AMPLISSIMVS.

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§. 1.

Notum eſt in Aſtronomiae theoria omnes expreſſiones Analyticas in huiusmodi ſeries converti ſolere, quarum termini ſive per ſinus ſive coſinus multiplorum cuiuspiam anguli procedant. Veluti in motu planetarum regulari, ſi anomalia vera fuerit $= \Phi$, orbitae excentricitas $= e$ et ſemiparameter $= b$, diſtantia planetae a ſole eſt $= \frac{b}{1 + e \,\text{coſ.}\, \Phi}$, quae formula in talem ſeriem:

$$A + B \,\text{coſ.}\, \Phi + C \,\text{coſ.}\, 2\Phi + D \,\text{coſ.}\, 3\Phi + E \,\text{coſ.}\, 4\Phi + \text{etc.}$$

evolvi ſolet, propterea quod eius uſus in tabulis Aſtronomicis condendis maximum adiumentum affert. Deinde etiam expreſſio temporis tali formula integrali: $\int \frac{\partial \Phi}{(1 + e \,\text{coſ.}\, \Phi)^2}$ expreſſa in huiusmodi ſeriem converti convenit. Ac ſi omnia elementa per

per anomaliam mediam exprimere velimus, quae ſit $= \theta$, totum negotium, quod non parum laboris poſtulat, ſemper ad inventionem talis ſeriei:

$$A + B \operatorname{cof.} \theta + C \operatorname{cof.} 2\theta + D \operatorname{cof.} 3\theta + E \operatorname{cof.} 4\theta + \text{etc.}$$

reducitur; unde deinceps, ſive per differentiationem, ſive per integrationem, ſimiles ſeries ſinus eorundem angulorum multiplorum eliciuntur.

§. 2. Quin etiam pro inveſtigatione motuum irregularium, qui oriuntur ex mutua planetarum actione, cardo rei in evolutione idonea huiusmodi formulae: $(1 + a \operatorname{cof.} \Phi)^{-\frac{3}{2}}$ verſari ſolet, quam formulam pariter in eiusmodi ſeriem:

$$A + B \operatorname{cof.} \Phi + C \operatorname{cof.} 2\Phi + D \operatorname{cof.} 3\Phi + \text{etc.}$$

evolvi convenit, ubi Φ denotat angulum, quo bini planetae ſe mutuo attrahentes, ex ſole viſi, diſtare a ſe invicem videntur. Tum vero etiam ſi tales inveſtigationes ulterius proſequi velimus, occurrunt quoque altiores poteſtates, veluti $(1 + a \operatorname{cof.} \Phi)^{-\frac{5}{2}}$ et $(1 + a \operatorname{cof.} \Phi)^{-\frac{7}{2}}$ etc. quarum evolutio in tales ſeries profundiſſimas inveſtigationes et calculos plerumque valde taedioſos poſtulat. Evenire quoque poteſt in huiusmodi inveſtigationibus ut formulae irrationales adhuc magis complicatae in calculum invehantur, quas ſimili modo in tales ſeries evolvi oporteat.

§. 3. Cum igitur talium formularum evolutio non ſolum in Aſtronomia maximi ſit momenti, ſed etiam per methodos vulgo uſitatas ſummum laborem exigant, aliam methodum novam ac facilem proponam, cuius ope omnes huius-

huiusmodi formulae, quantumvis fuerint perplexae, facili labore in tales feries converti queant. Loquor autem hic potiffimum de eiusmodi formulis, quae more folito in huiusmodi feries refolvuntur;

$$\alpha + \beta \operatorname{cof.} \Phi + \gamma \operatorname{cof.} \Phi^2 + \delta \operatorname{cof.} \Phi^3 + \varepsilon \operatorname{cof.} \Phi^4 + \text{etc.}$$

cuius fcilicet termini per poteftatas ipfius cof. Φ procedant. Quoniam enim quaelibet poteftas ipfius cof. Φ in cofinus multiplorum eiusdem anguli evolvi poteft, evidens eft per evolutionem fingulorum terminorum feriem eiusdem formae oriri debere, qualem hic tractare conftitui.

§. 4. Quaecunque autem formula hoc modo evolvenda proponatur, quoniam eam femper ut certam functionem anguli Φ fpectare licet, eam more iam recepto hoc charactere $\Gamma:\Phi$ defignabo, quam quidem functionem ita comparatam effe affumo, ut more folito in talem feriem:

$$\alpha + \beta \operatorname{cof.} \Phi + \gamma \operatorname{cof.} \Phi^2 + \delta \operatorname{cof.} \Phi^3 + \varepsilon \operatorname{cof.} \Phi^4 + \text{etc.}$$

evolvi queat; quam tamen in praefenti negotio exhibere nihil attinet. Generaliter igitur ftatuamus fieri

$$\Gamma:\Phi = A + B \operatorname{cof.} \Phi + C \operatorname{cof.} 2\Phi + D \operatorname{cof.} 3\Phi + E \operatorname{cof.} 4\Phi + \text{etc.}$$

ita ut totum negotium iam huc redeat, quemadmodum valores litterarum A, B, C, D, etc. inveftigari oportet, id quod methodo, quam hic fum expofiturus, fine ullis calculi ambagibus praeftari poterit.

§. 5. Ante autem quam hoc negotium adgrediar, plurimum intererit obfervaffe, iftas litteras neceffario feriem valde convergentem conftituere debere, ita ut quo longius procedamus, earum valores continuo magis decrescant. Quoniam

niam enim, si angulus Φ exigua particula augeatur, etiam ipse valor functionis $\Gamma:\Phi$ tantum parvam mutationem patitur, facile ostendi potest, litteram verbi gratia millesimam in hoc ordine valde paruam esse debere. Namque si iste terminus fuerit $M \cos. 1000\,\Phi$, eius valor pro casu $\Phi = 0$ erit $= M$. Augeamus nunc hunc angulum millesima parte semiperipheriae π, eritque $1000\,\Phi = \pi$, ideoque iste terminus $= -M$, ita ut facta hac exigua mutatione eius valor a $+M$ usque ad $-M$ immutetur. Vnde patet, ne ista mutatio in valore functionis $\Gamma:\Phi$ enorme discrimen pariat, illam litteram M vehementer parvam esse debere, id quod multo magis evenire debet in litteris quae cosinus adhuc maiorum multiplorum afficiunt. Haec autem tantum a terminis ab initio maxime remotis sunt intelligenda, quandoquidem utique evenire potest, ut ab initio seriei termini A, B, C, D, E, etc. adeo crescendo procedant, et progressionem maxime regularem constituant.

§. 6. Quoniam igitur functio proposita $\Gamma:\Phi$ pro data accipitur, eius valores pro quolibet angulo determinato, qui loco Φ assumitur, actu exhiberi poterunt. Hinc si primo sumamus $\Phi = 0$, erit

$$\Gamma:0 = A + B + C + D + E + \text{etc.}$$

Sin autem sumamus $\Phi = \pi$, erit

$$\Gamma:\pi = A - B + C - D + E - F + \text{etc.}$$

unde sequitur fore

$$\tfrac{1}{2}\Gamma:0 + \tfrac{1}{2}\Gamma:\pi = A + C + E + G + I + \text{etc.}$$

ubi termini alterni iam sunt exclusi. Statuamus nunc Φ $\Phi = \frac{\pi}{2}$, eritque

$$\Gamma : \tfrac{\pi}{2} = A - C + E - G + I - L + N - \text{etc.}$$

Quare ſi iſta ſeries ad praecedentem addatur, ſemiſumma ambarum erit

$$\tfrac{1}{4}\Gamma : 0 + \tfrac{1}{4}\Gamma : \pi + \tfrac{1}{2}\Gamma\,\tfrac{\pi}{2} = A + E + I + N + Q + \text{etc.}$$

ſicque valor ipſius A iam prope innoteſceret, ſi litteras ſequentes E, I, N etc. negligere vellemus.

§. 7. Quoniam hoc modo ſaltus ab A usque ad E eſt factus, ſimili modo ſaltus adhuc ampliores effici poterunt. Sumamus enim $\Phi = \frac{\pi}{4}$, et ponamus brevitatis gratia $\cos.\frac{\pi}{4} = \alpha$, unde fit $\cos.\frac{3\pi}{4} = -\alpha$; $\cos.\frac{5}{4}\pi = -\alpha$; $\cos.\frac{7}{4}\pi = \alpha$; etc. eritque

$$\Gamma : \tfrac{1}{4}\pi = A + \alpha B - \alpha D - E - \alpha G + \alpha I + K + \alpha L \text{ etc.}$$

Faciamus nunc $\Phi = \frac{3}{4}\pi$, ac reperiemus

$$\Gamma : \tfrac{3}{4}\pi = A - \alpha B + \alpha D - E + \alpha G - \alpha I + K - \alpha L \text{ etc.}$$

Harum igitur duarum ſerierum ſemiſumma erit

$$\tfrac{1}{2}\Gamma : \tfrac{1}{4}\pi + \tfrac{1}{2}\Gamma : \tfrac{3}{4}\pi = A - E + K - N + R \text{ etc.}$$

Quodſi ergo huic ſeriei addamus eam, quam paragrapho praecedente invenimus, erit

$$\tfrac{1}{8}\Gamma : 0 + \tfrac{1}{8}\Gamma : \pi + \tfrac{1}{4}\Gamma : \tfrac{\pi}{2} + \tfrac{1}{4}\Gamma : \tfrac{1}{4}\pi + \tfrac{1}{4}\Gamma : \tfrac{3}{4}\pi = A + I + R + \text{etc.}$$

quae expreſſio iam multo propius dabit verum valorem ipſius A, ob parvitatem terminorum ſequentium.

§. 8. Simili modo ex valoribus $\Phi = \frac{1}{8}\pi$; $\Phi = \frac{3}{8}\pi$; $\Phi = \frac{5}{8}\pi$ et $\Phi = \frac{7}{8}\pi$, ſeries colligi poterit $A - I + R$, etc. quae cum illa coniuncta dabit $A + R$ etc. unde iam ſatis tuto verus valor ipſius A reſultabit. Verum quia hoc modo procedendo mox totum alphabetum exhauriremus, magis ido-

idonea ratione litteras A, B, C, D, etc. defignemus, unde ftatim liqueat, ad quodnam multiplum anguli Φ fingulae referantur. Ponamus fcilicet $A = (0)$; $B = (1)$; $C = (2)$; $D = (3)$; $E = (4)$; etc. hoc modo feries poftremo loco inventa erit

$$(0) + (8) + (16) + (24) + (32) + (40) + \text{etc.}$$

§. 9. Statuamus iam fucceffive pro Φ angulos $\frac{1}{8}\pi$; $\frac{3}{8}\pi$; $\frac{5}{8}\pi$; $\frac{7}{8}\pi$; ac ponendo brevitatis gratia $\text{cof.}\,\frac{1}{8}\pi = \beta$; $\text{cof.}\,\frac{3}{8}\pi = b$, ut fit $\beta\beta + bb = 1$, erit $\text{cof.}\,\frac{5}{8}\pi = -b$; $\text{cof.}\,\frac{7}{8}\pi = -\beta$; $\text{cof}\,\frac{9}{8}\pi = -\beta$; $\text{cof.}\,\frac{11}{8}\pi = -b$; $\text{cof.}\,\frac{13}{8}\pi = +b$; $\text{cof.}\,\frac{15}{8}\pi = +\beta$; $\text{cof.}\,\frac{17}{8}\pi = \beta$; $\text{cof.}\,\frac{19}{8}\pi = b$; etc. tum vero feries hinc refultantes erunt:

$$\begin{aligned}
\Gamma : \tfrac{1}{8}\pi = {} & (0) + \beta(1) + \alpha(2) + b(3) + 0(4) - b(5) - \alpha(6) \\
& - \beta(7) - 1(8) - \beta(9) - \alpha(10) - b(11) \\
& - 0(12) + b(13) + \alpha(14) + \beta(15) + 1(16).
\end{aligned}$$

$$\begin{aligned}
\Gamma : \tfrac{3}{8}\pi = {} & (0) + b(1) - \alpha(2) - \beta(3) + 0(4) + \beta(5) + \alpha(6) \\
& - b(7) - 1(8) - b(9) + \alpha(10) + \beta(11) + 0(12) \\
& - \beta(13) - \alpha(14) + b(15) + 1(16).
\end{aligned}$$

$$\begin{aligned}
\Gamma : \tfrac{5}{8}\pi = {} & (0) - b(1) - \alpha(2) + \beta(3) + 0(4) - \beta(5) + \alpha(6) \\
& + b(7) - 1(8) + b(9) + \alpha(10) - \beta(11) + 0(12) \\
& + \beta(13) - \alpha(14) - b(15) + 1(16).
\end{aligned}$$

$$\begin{aligned}
\Gamma : \tfrac{7}{8}\pi = {} & (0) - \beta(1) + \alpha(2) - b(3) + 0(4) + b(5) - \alpha(6) \\
& + \beta(7) - 1(8) + \beta(9) - \alpha(10) + b(11) - 0(12) \\
& - b(13) + \alpha(14) - \beta(15) + 1(16).
\end{aligned}$$

Iam harum quatuor ferierum fumma per quatuor divifa erit

$$(0)-(8)+(16)-(24)+(32)-(40)+(48)-(56) \text{ etc.}$$

cuius ergo ſumma erit

$$\tfrac{1}{4}\Gamma:\tfrac{1}{8}\pi+\tfrac{1}{4}\Gamma:\tfrac{3}{8}\pi+\tfrac{1}{4}\Gamma:\tfrac{5}{8}\pi+\tfrac{1}{4}\Gamma:\tfrac{7}{8}\pi.$$

§. 10. Quodſi ergo huic ſeriei addatur illa, qu
§. 8. fuerat inventa, diviſione facta per 2 habebimus

$$(0)+(16)+(32)+(48)+(64)+ \text{ etc.}$$

cuius ergo valor erit

$$\tfrac{1}{16}\Gamma:0+\tfrac{1}{16}\Gamma:\pi+\tfrac{1}{8}\Gamma:\tfrac{1}{2}\pi+\tfrac{1}{8}\Gamma:\tfrac{1}{4}\pi+\tfrac{1}{8}\Gamma:\tfrac{3}{4}\pi+\tfrac{1}{8}\Gamma:\tfrac{1}{8}$$
$$+\tfrac{1}{8}\Gamma:\tfrac{3}{8}\pi+\tfrac{1}{8}\Gamma:\tfrac{5}{8}\pi+\tfrac{1}{8}\Gamma:\tfrac{7}{8}\pi.$$

Quodſi porro angulos $\frac{1}{16}\pi$; $\frac{3}{16}\pi$; usque ad $\frac{15}{16}\pi$, ſubſidium vocare vellemus, ſerierum inde natarum ſum per 8 diviſa produceret ſeriem hanc:

$$(0)-(16)+(32)-(48)+(64) \text{ etc.}$$

quae cum praecedente coniuncta praebitura eſſet hanc riem: $(0)+(32)+(64)+(96)+$ etc. quae manifeſto i ſatis exacte valorem primi termini $(0)=A$ praebebit. terim tamen nihil impedit, quo minus ſimiles operatio ulterius proſequamur.

§. 11. Hoc autem modo tantum ad cognitione primi termini A pertingimus; at inventio ſecundi term B, pariter ac ſequentium C, D, E, etc. ex iisdem quid caſibus, ſed per longe alias combinationes, pluresque amb ges, concludi poſſet; quod cum nimis amplam deſcript nem poſtularet, plurimum intererit, certam et conſtante methodum attuliſſe, cuius beneficio per ſimiles formul non ſolum primus terminus A, ſed etiam omnes ſequent B, C, D, E, etc. tam exacte quam lubuerit indagari qu

an

ant. Omnes autem operationes instituendae innituntur sequenti Lemmati:

Lemma.

Si proposita fuerit ista series cosinuum:

$$1 + \operatorname{cos.} \Phi + \operatorname{cos.} 2\Phi + \operatorname{cos.} 3\Phi + \operatorname{cos.} 4\Phi \ldots \operatorname{cos.} n\Phi,$$

cuius ultimi termini angulus sit multiplum semiperipheriae π, invenire summam huius seriei.

Solutio.

§. 12. Ponamus summam quaesitam $= S$, ut sit

$$S = 1 + \operatorname{cos.} \Phi + \operatorname{cos.} 2\Phi + \operatorname{cos.} 3\Phi \ldots \operatorname{cos.} n\Phi,$$

et multiplicando per $\operatorname{sin.} \frac{1}{2}\Phi$, ob

$$2 \operatorname{sin.} \tfrac{1}{2}\Phi \operatorname{cos.} m\Phi = \operatorname{sin.} (m + \tfrac{1}{2})\Phi - \operatorname{sin.} (m - \tfrac{1}{2})\Phi,$$

nanciscemur:

$$2 S \operatorname{sin.} \tfrac{1}{2}\Phi = 2 \operatorname{sin.} \tfrac{1}{2}\Phi + \operatorname{sin.} \tfrac{3}{2}\Phi + \operatorname{sin.} \tfrac{5}{2}\Phi + \operatorname{sin.} \tfrac{7}{2}\Phi \ldots \operatorname{sin.} (n + \tfrac{1}{2})\Phi,$$
$$- \operatorname{sin.} \tfrac{1}{2}\Phi - \operatorname{sin.} \tfrac{3}{2}\Phi - \operatorname{sin.} \tfrac{5}{2}\Phi - \operatorname{sin.} (n - \tfrac{1}{2})\Phi,$$

ita ut sit

$$2 S \operatorname{sin.} \tfrac{1}{2}\Phi = \operatorname{sin.} \tfrac{1}{2}\Phi + \operatorname{sin.} (n + \tfrac{1}{2})\Phi.$$

Quodsi iam fuerit $n\Phi = \pi$, ultimus terminus erit

$$\operatorname{sin.} (\pi + \tfrac{1}{2})\Phi = - \operatorname{sin.} \tfrac{1}{2}\Phi,$$

ideoque $2 S \operatorname{sin.} \frac{1}{2}\Phi = 0$, quod idem valet si fuerit $n\Phi$ vel 3π, vel 5π, vel in genere $(2i - 1)\pi$; sin autem fuerit $n\Phi = 2\pi$, ultimus terminus erit $= \operatorname{sin.} \frac{1}{2}\Phi$, ideoque summa seriei erit $= 1$.

§. 13. Duo igitur casus hic occurrunt distinguendi, prouti fuerit vel $n\Phi = (2i - 1)\pi$, vel $n\Phi = 2i\pi$. Priori scili-

scilicet casu summa erit $= 0$; posteriore vero casu $= 1$. Interim tamen hae conclusiones quasdam exceptiones postulant, quando angulus $\frac{1}{2}\Phi$ iam ipse fuerit vel 0, vel π, vel 2π, vel 3π, vel 4π; horum enim casuum primo $\Phi = 0$ omnes termini seriei propositae erunt unitati aequales, quorum numerus cum sit $n + 1$, etiam summa erit $S = n + 1$. Hoc vero ipsum etiam ex solutione inventa concludi potest; cum enim per $2 \sin. \frac{1}{2}\Phi$ dividendo fit

$$S = \frac{1}{2} + \frac{\sin. (n + \frac{1}{2})\Phi}{2 \sin. \frac{1}{2}\Phi},$$

casu quo Φ infinite parvum, ideoque $\sin. (n + \frac{1}{2})\Phi = (n + \frac{1}{2})\Phi$ et $\sin. \frac{1}{2}\Phi = \frac{1}{2}\Phi$, his valoribus substitutis reperitur $S = n + 1$.

§. 14. Evolvamus nunc casum quo $\frac{1}{2}\Phi = \pi$, et quia in expressione nostra etiamnunc postremae fractionis numerator et denominator evanescunt, eorum loco substituantur sua differentialia, eritque

$$S = \frac{1}{2} + \frac{(n + \frac{1}{2}) \cos. (n + \frac{1}{2})\Phi}{\cos. \frac{1}{2}\Phi},$$

unde ob $\cos. \frac{1}{2}\Phi = -1$ et $\cos. (n + \frac{1}{2})\Phi = \cos. (2n\pi + \pi) = -1$, erit $S = n + 1$; id quod etiam evenit in genere, si fuerit $\frac{1}{2}\Phi = \lambda\pi$, tum enim erit

$$S = \frac{1}{2} + (n + \frac{1}{2}) \frac{\cos. 2n\lambda\pi + \lambda\pi}{\cos. \lambda\pi}.$$

Est vero $\cos. (2\lambda n\pi + \lambda\pi) = \cos. \lambda\pi$, unde erit $S = n + 1$. Hinc ergo sequuntur istae duae conclusiones:

I.

I. Pro caſibus $n\,\Phi = (2\,i - 1)\,\pi$.

His caſibus ſumma ſeriei propoſitae ſemper eſt nihilo aequalis, exceptis caſibus, quibus $\Phi = 2\,i\,\pi$, ſcilicet multiplum totius peripheriae, quippe quibus ſumma ſeriei ſemper eſt $= n + 1$.

II. Pro caſibus $n\,\Phi = 2\,i\,\pi$.

Pro his caſibus ſumma ſeriei propoſitae ſemper eſt $= 1$, *niſi fuerit* $\Phi = 2\,i\,\pi$, *ſcilicet multiplo totius peripheriae aequalis; quandoquidem his caſibus ſumma eſt* $= n + 1$.

§. 15. Ponamus brevitatis gratia $\frac{\pi}{n} = \omega$, atque huius progreſſionis:

$$1 + \operatorname{coſ.} \omega + \operatorname{coſ.} 2\,\omega \ldots\ldots + \operatorname{coſ.} \pi$$

cuius terminorum numerus eſt $n + 1$, ſumma ſemper erit $= 0$; at vero ſi ſinguli anguli duplicentur, huius expreſſionis:

$$1 + \operatorname{coſ.} 2\,\omega + \operatorname{coſ.} 4\,\omega \ldots\ldots + \operatorname{coſ.} 2\,\pi,$$

vi Lemmatis, ſumma ſemper eſt $= 1$. Sin autem angulos hos per numerum quemcunque λ multiplicemus, ut prodeat haec ſeries:

$$1 + \operatorname{coſ.} \lambda\,\omega + \operatorname{coſ.} 2\,\lambda\,\omega \ldots\ldots + \operatorname{coſ.} \lambda\,\pi$$

eius ſumma ſemper erit $= 0$, ſi λ fuerit numerus impar: ſin autem fuerit par, eius ſumma erit $= 1$, exceptis tamen caſibus, quibus eſt vel $\lambda = 2\,n$, vel $\lambda = 4\,n$, vel $\lambda = 6\,n$, etc. quippe quibus ſumma erit $= 1 + n$, quoniam ſecundus terminus erit $\operatorname{coſ.} 2\,n\,\omega = \operatorname{coſ.} 2\,\pi$. Hinc iam ſequentis problematis ſolutio dari poteſt.

Problema.

Poſito $\frac{\pi}{n} = \omega$, *ſi proponatur iſta progreſſio:*

$$\Gamma : 0\,\omega + \Gamma : \omega + \Gamma : 2\,\omega + \Gamma : 3\,\omega \ldots\ldots + \Gamma : \pi,$$

cuius

cuius terminorum numerus eſt $n + 1$, eius ſummam, quae ponatur $= S$, definire.

Solutio.

§. 16. Quoniam in genere ſeries ex formula $\Gamma : \phi$ poſita eſt

$$(0) + (1)\cos\phi + (2)\cos 2\phi + (3)\cos 3\phi + (4)\cos 4\phi + \text{etc.}$$

pro prima formula eſt $\phi = 0$; pro ſecunda $\phi = \omega$; pro tertia $\phi = 2\omega$; etc. Hinc ergo ex ſingulis formulis, ſeries, quae inde naſcuntur, verticaliter ſubſcribamus, ita ut prima ſeries horizontalis exhibeat terminos coëfficiente (0) affectos; ſecunda terminos coëfficiente (1) affectos; tertia coëfficienti (2) reſpondentes; etc. ſequenti modo:

	$\Gamma : 0\,\omega$	$\Gamma : \omega$	$\Gamma : 2\,\omega$	$\Gamma : 3\,\omega$	. . . .	$\Gamma : \pi$.
(0)	1	$+1$	$+1$	$+1$	. . .	$+1$
(1)	$1+$	$\cos\omega$	$+\cos 2\omega$	$+\cos 3\omega$	. . .	$+\cos\pi$
(2)	$1+$	$\cos 2\omega$	$+\cos 4\omega$	$+\cos 6\omega$	. . .	$+\cos 2\pi$
(3)	$1+$	$\cos 3\omega$	$+\cos 6\omega$	$+\cos 9\omega$	. . .	$+\cos 3\pi$
.	.	.	.	.		.
.	.	.	.	.		.
(λ)	$1+$	$\cos\lambda\omega$	$+\cos 2\lambda\omega$	$+\cos 3\lambda\omega$	. . .	$+\cos\lambda\pi$.

§. 17. Nunc igitur vi lemmatis ſumma primae ſeriei horizontalis erit $= (1 + n)(0)$; ſecundae vero ſumma erit $= 0$; tertiae $= (2)\,1$; tum vero omnium coëfficientium imparium ſummae evaneſcunt, parium vero ſummae unitati aequantur, exceptis coëfficientibus $(2n)$; $(4n)$; $(6n)$; $(8n)$ etc. quorum valores erunt $= n + 1$, ſicque ſumma quaeſita ita exhiberi poterit:

$$S = (1 + n)(0) + (n + 1)(2n) + (n + 1)(4n) \text{ etc.}$$
$$+ 1(2) + 1(4) + 1(6) + \text{etc.}$$

Sci-

Scilicet pro coëfficiente generali (λ) termini tantum occurrunt, quando λ eſt numerus par, quorum valor eſt $= 1$, exceptis caſibus, quibus eſt $\lambda = 2n$; vel $\lambda = 4n$; vel $\lambda = 6n$; etc. quippe pro quibus habemus $n = 1$.

§. 18. Conſideremus nunc aggregata primi et ultimi termini cuiusque feriei horizontalis, ſive factorem formae $\Gamma : 0 + \Gamma : \pi$, ac pro ($0$) prodit 2; pro (1) fiet $= 0$; pro (2) iterum prodit 2; atque ita porro pro omnibus imparibus oritur 0, pro paribus vero 2; unde manifeſtum eſt fore

$$\tfrac{1}{2}\Gamma : 0 + \tfrac{1}{2}\Gamma : \pi = (0) + (2) + (4) + (6) + (8) + \text{etc.}$$

ideoque omnes coëfficientes pares hic occurrunt, ſine ulla exceptione. Quam ob rem ſi hanc feriem a valore ante pro S invento ſubtrahamus, omnes termini ſola unitate affecti tolluntur, atque habebimus ſequentem feriem memorabilem pro forma $S - \tfrac{1}{2}\Gamma : 0 - \tfrac{1}{2}\Gamma : \pi$, quae erit

$$n(0) + n(2n) + n(4n) + n(6n) + \text{etc.}$$

unde ſequens theorema:

Theorema.

Si in expreſſione praecedentis problematis primi et poſtremi termini tantum ſemiſſis capiatur, ita ut habeatur iſta forma:

$$\tfrac{1}{2}\Gamma : 0 + \Gamma : \omega + \Gamma : 2\omega \;.\;.\;.\;.\;.\;.\; \tfrac{1}{2}\Gamma : \pi = \Sigma,$$

exiſtente $\omega = \frac{\pi}{n}$, *eius valor per ſequentem feriem infinitam exprimetur:*

$$\Sigma = n(0) + n(2n) + n(4n) + n(6n) + n(8n) + \text{etc.}$$

 §. 19.

§. 19. Quicunque ergo numerus integer pro n accipiatur, fi ponatur $\frac{\pi}{n} = \omega$, huius feriei infinitae:

$$(0) + (2n) + (4n) + (6n) + (8n) + \text{etc.}$$

fumma erit $= \frac{1}{n}\Sigma$, exiftente

$$\Sigma = \tfrac{1}{2}\Gamma:0 + \Gamma:\omega + \Gamma:2\omega + \Gamma:3\omega \ldots\ldots + \tfrac{1}{2}\Gamma:\pi.$$

Ita fi fumamus $n = 12$, pofito $\frac{\pi}{12} = 15^\circ$, ita ut fit

$$\Sigma = \tfrac{1}{2}\Gamma:0 + \Gamma:\omega + \Gamma:2\omega + \Gamma:3\omega \ldots\ldots + \tfrac{1}{2}\Gamma:\pi,$$

eius pars duodecima, fcilicet $\frac{1}{12}\Sigma$, exprimet fummam huius feriei: $(0) + (24) + (48) + (72) +$ etc. Quare quia termini poft primum funt valde parvi, illa formula $\frac{1}{12}\Sigma$ fatis exacte exhibebit valorem formulae (0), qui initio per litteram A eft indicatus.

§. 20. Cum igitur hoc modo valor primi termini A, quem hic per (0) defignamus, tam exacte definiri queat quam lubuerit, (tantum enim opus eft pro n numeros maiores accipere), oftendamus etiam quomodo fimili ratione valor fecundi termini $B = (1)$ inveftigari poffit; quem in finem confiderari conveniet iftam expreffionem:

$$S = \Gamma:0 + \text{cof.}\,\omega\,\Gamma:\omega + \text{cof.}\,2\omega\,\Gamma:2\omega + \text{cof.}\,3\omega\,\Gamma:3\omega \ldots + \text{cof.}\,\pi\,\Gamma:\pi,$$

exiftente iterum $\omega = \frac{\pi}{n}$, ita ut terminorum numerus fit $= 1 + n$. Evolvamus igitur huius expreffionis membrum quodcunque $\text{cof.}\,\lambda\omega\,\Gamma:\lambda\omega$, et cum fit

$$\Gamma:\lambda\omega = (0) + (1)\,\text{cof.}\,\lambda\omega + (2)\,\text{cof.}\,2\lambda\omega + (3)\,\text{cof.}\,3\lambda\omega + \text{etc.}$$

erit

$$\begin{aligned}2\,\text{cof.}\,\lambda\omega\,\Gamma:\lambda\omega = {} & 2(0)\,\text{cof.}\,\lambda\omega + (1)(1 + \text{cof.}\,2\lambda\omega) \\ & + (2)(\text{cof.}\,\lambda\omega + \text{cof.}\,3\lambda\omega) + (3)(\text{cof.}\,2\lambda\omega + \text{cof.}\,4\lambda\omega) \\ & + (4)(\text{cof.}\,3\lambda\omega + \text{cof.}\,5\lambda\omega) + \text{etc.}\end{aligned}$$

§. 21.

§. 21. Quodsi iam hoc modo singulis terminis propositis bis sumtis series inde resultantes verticaliter subscribamus, erit ut sequitur:

$$2S = 2\Gamma:0 + 2\,\mathrm{cos.}\,\omega\,\Gamma:\omega + 2\,\mathrm{cos.}\,2\omega\,\Gamma:\mathrm{cos.}\,2\omega \;\ldots\; + 2\,\mathrm{cos.}\,\pi\,\Gamma:\pi.$$

$$\begin{array}{l|llcl}
(0) & 2+2\,\mathrm{cos.}\,\omega & +2\,\mathrm{cos.}\,2\omega & \ldots & +2\,\mathrm{cos.}\,\pi \\
(1) & \left\{\begin{array}{l} 1+\quad 1 \\ 1+\mathrm{cos.}\,2\omega \end{array}\right. & \begin{array}{l} +\quad 1 \\ +\mathrm{cos.}\,4\omega \end{array} & \begin{array}{c} \ldots \\ \ldots \end{array} & \begin{array}{l} +\quad 1 \\ +\mathrm{cos.}\,2\pi \end{array} \\
(2) & \left\{\begin{array}{l} 1+\mathrm{cos.}\,\omega \\ +1+\mathrm{cos.}\,3\omega \end{array}\right. & \begin{array}{l} +\mathrm{cos.}\,2\omega \\ +\mathrm{cos.}\,6\omega \end{array} & \begin{array}{c} \ldots \\ \ldots \end{array} & \begin{array}{l} +\mathrm{cos.}\,\pi \\ +\mathrm{cos.}\,3\pi \end{array} \\
(3) & \left\{\begin{array}{l} 1+\mathrm{cos.}\,2\omega \\ +1+\mathrm{cos.}\,4\omega \end{array}\right. & \begin{array}{l} +\mathrm{cos.}\,4\omega \\ +\mathrm{cos.}\,8\omega \end{array} & \begin{array}{c} \ldots \\ \ldots \end{array} & \begin{array}{l} +\mathrm{cos.}\,2\pi \\ +\mathrm{cos.}\,4\pi \end{array} \\
(4) & \left\{\begin{array}{l} 1+\mathrm{cos.}\,3\omega \\ +1+\mathrm{cos.}\,5\omega \end{array}\right. & \begin{array}{l} +\mathrm{cos.}\,6\omega \\ +\mathrm{cos.}\,10\omega \end{array} & \begin{array}{c} \ldots \\ \ldots \end{array} & \begin{array}{l} +\mathrm{cos.}\,3\pi \\ +\mathrm{cos.}\,5\pi \end{array} \\
\cdot & \quad\cdot & \quad\cdot & & \quad\cdot \\
\cdot & \quad\cdot & \quad\cdot & & \quad\cdot \\
(\lambda) & \left\{\begin{array}{l} 1+\mathrm{cos.}\,(\lambda-1)\omega \\ +1+\mathrm{cos.}\,(\lambda+1)\omega \end{array}\right. & \begin{array}{l} +\mathrm{cos.}\,2(\lambda-1)\omega \\ +\mathrm{cos.}\,2(\lambda+1)\omega \end{array} & \begin{array}{c} \ldots \\ \ldots \end{array} & \begin{array}{l} +\mathrm{cos.}\,(\lambda-1)\pi \\ +\mathrm{cos.}\,(\lambda+1)\pi. \end{array}
\end{array}$$

§. 22. Nunc ope Lemmatis supra expositi facile erit singulas has progressiones summare. Scilicet pro (0) erit summa $= 0$, pariter ac summa primi et ultimi. At pro (1) prioris seriei summa est $1+n$, posterioris vero summa est 1, ita ut pro (1) iunctim prodeat $2+n$. At vero primus et ultimus terminus simul dant 4, quorum semissis si inde auferatur, remanebit $(1)\,n$. Pro (2) superior series dat 0, et inferior pariter 0, ita ut hic terminus prorsus evanescat. Pro (3) superior series dat 1, inferior vero pariter 1, quare si semisumma primi et ultimi hinc auferatur, etiam iste terminus evanescit, id quod semper eveniet, nisi in secundis terminis occurrat vel $\mathrm{cos.}\,2\pi$; vel $\mathrm{cos.}\,4\pi$; vel $\mathrm{cos.}\,6\pi$; etc. quoniam tum seriei summa est $1+n$.

§. 23. Consideremus igitur indicem generalem (λ) pro quo superior series est:

$$1 + \cos.(\lambda - 1)\omega + \cos. 2(\lambda - 1)\omega + \ldots + \cos.(\lambda - 1)\pi$$

cuius summa, si fuerit $\lambda - 1 = 2n$, erit $1 + n$; inferioris autem seriei summa erit $1 + n$, quando fuerit $\lambda + 1 = 2n$. Hinc si semisumma primi et ultimi termini subtrahatur, utroque casu remanebit tantum n. Ex quo patet si perpetuo semisummam primi et ultimi termini subtrahamus, omnes terminos e medio tolli, praeter eos, quibus est vel $\lambda - 1 = 2in$ vel $\lambda + 1 = 2in$, denotante $2i$ numerum parem quemcunque, pro quibus valor erit $= n$, id quód ergo locum habet pro forma $2S - \Gamma : 0 - \cos.\pi\Gamma : \pi$. Quare si proponatur ista expressio:

$$\Sigma = \tfrac{1}{2}\Gamma : 0 + \cos.\omega\Gamma : \omega + \cos. 2\omega\Gamma : 2\omega + \cos. 3\omega\Gamma : 3\omega + \ldots$$
$$\ldots + \tfrac{1}{2}\cos.\pi\Gamma : \pi,$$

eius valor erit

$$\tfrac{1}{2}n(1) + \tfrac{1}{2}n(2n-1) + \tfrac{1}{2}n(2n+1) + \tfrac{1}{2}n(4n-1) + \ldots$$
$$\ldots + \tfrac{1}{2}n(4n+1) \text{ etc.}$$

unde concludimus huius seriei:

$$\left.\begin{array}{l}(1) + (2n-1) + (4n-1) + (6n-1) + (8n-1)\\ \quad + (2n+1) + (4n+1) + (6n+1) + (8n+1)\end{array}\right\} \text{etc.}$$

summam esse $\frac{2}{n}\Sigma$; unde si n fuerit numerus satis magnus, ista expressio $\frac{2}{n}\Sigma$ satis exacte praebebit valorem (1), quem supra littera B designavimus.

§. 24. Contemplemur hic quoque casum, quo $n = 12$ et $\omega = \frac{\pi}{12} = 15^\circ$, ideoque formula:

$$\Sigma = \tfrac{1}{2}\Gamma : 0 + \cos.\omega\Gamma : \omega + \cos. 2\omega\Gamma : 2\omega + \cos. 3\omega\Gamma : 3\omega + \ldots$$
$$\ldots + \tfrac{1}{2}\cos.\pi\Gamma : \pi,$$

ex

ex quo valore cognito innotefcet fumma fequentis feriei:

$$\left\{\begin{matrix}(1)+(23)+(47)+(71)\\ +(25)+(49)+(73)\end{matrix}\right\}\text{ etc.}=\tfrac{1}{6}\Sigma,$$

qui ergo valor iam fatis prope dabit valorem $(1)=B$. Quodfi fimili modo in valorem tertii termini $C=(2)$ inquirere velimus, talem expreffionem:

$$S=\Gamma:0+\text{cof.}\,2\omega\,\Gamma:\omega+\text{cof.}\,4\omega\,\Gamma:2\omega+\text{cof.}\,6\omega\,\Gamma:3\omega+\ldots$$
$$\ldots+\text{cof.}\,2\pi\,\Gamma:\pi$$

evolvere debemus, quae ut fractiones evitentur, duplicetur, ac finguli termini per feries verticales fequenti modo exhibeantur:

$$2S=2\Gamma:0+2\,\text{cof.}\,2\omega\Gamma:\omega+2\,\text{cof.}\,4\omega\Gamma:2\omega+2\,\text{cof.}\,6\omega\Gamma:3\omega\ldots+2\,\text{cof.}\,2\pi\Gamma:\pi$$

$$\begin{array}{c|lllll}
(0) & 2+2\,\text{cof.}\,2\omega & +2\,\text{cof.}\,4\omega & +2\,\text{cof.}\,6\omega & \ldots & +2\,\text{cof.}\,2\pi.\\
(1) & \left\{\begin{array}{l}1+\text{cof.}\,\omega\\ +1+\text{cof.}\,3\omega\end{array}\right. & \begin{array}{l}+\text{cof.}\,2\omega\\ +\text{cof.}\,6\omega\end{array} & \begin{array}{l}+\text{cof.}\,3\omega\\ +\text{cof.}\,9\omega\end{array} & \begin{array}{l}\ldots\\ \ldots\end{array} & \begin{array}{l}\text{cof.}\,\pi.\\ +\text{cof.}\,3\pi.\end{array}\\
(2) & \left\{\begin{array}{l}1+\;\;1\\ +1+\text{cof.}\,4\omega\end{array}\right. & \begin{array}{l}+\;\;1\\ +\text{cof.}\,8\omega\end{array} & \begin{array}{l}+\;\;1\\ +\text{cof.}\,12\omega\end{array} & \begin{array}{l}\ldots\\ \ldots\end{array} & \begin{array}{l}+\;\;1.\\ +\text{cof.}\,4\pi.\end{array}\\
(3) & \left\{\begin{array}{l}1+\text{cof.}\,\omega\\ +1+\text{cof.}\,5\omega\end{array}\right. & \begin{array}{l}+\text{cof.}\,2\omega\\ +\text{cof.}\,10\omega\end{array} & \begin{array}{l}+\text{cof.}\,3\omega\\ +\text{cof.}\,15\omega\end{array} & \begin{array}{l}\ldots\\ \ldots\end{array} & \begin{array}{l}+\text{cof.}\,\pi.\\ +\text{cof.}\,5\pi.\end{array}\\
\cdot & \;\;\cdot & \;\;\cdot & \;\;\cdot & & \;\;\cdot\\
\cdot & \;\;\cdot & \;\;\cdot & \;\;\cdot & & \;\;\cdot\\
(\lambda) & \left\{\begin{array}{l}1+\text{cof.}(\lambda-2)\omega\\ +1+\text{cof.}(\lambda+2)\omega\end{array}\right. & \begin{array}{l}+\text{cof.}\,2(\lambda-2)\omega\\ +\text{cof.}\,2(\lambda+2)\omega\end{array} & \begin{array}{l}+\text{cof.}\,3(\lambda-2)\omega\\ +\text{cof.}\,3(\lambda+2)\omega\end{array} & \begin{array}{l}\ldots\\ \ldots\end{array} & \begin{array}{l}\text{cof.}(\lambda-2)\pi.\\ \text{cof.}(\lambda+2)\pi.\end{array}
\end{array}$$

§. 25. Hinc igitur formula (0) afficietur per 2; unde ergo fi femifumma primi et ultimi fubtrahatur, hic terminus prorfus e medio tolletur. Porro pro (1) fuperioris feriei fumma erit $=0$, inferioris vero pariter $=0$, quemadmodum etiam femifummae primi et ultimi termini evanefcunt, ſcunt, ita ut iſte terminus pariter auferatur. At pro (2) ſeries ſuperior dat $1+n$, inferior vero 1, hinc ſemiſumma primi et ultimi ablata relinquet n. Pro ſequente (3) ſuperior ſeries dat 0, pariter ac inferior, ita ut iſte terminus e medio tollatur. Eodem modo etiam in genere terminus (λ), e medio tolletur, ſiquidem ſemiſumma primi et ultimi ſubtrahatur, niſi fuerit vel $\lambda - 2 = 2in$, vel $\lambda + 2 = 2in$, quippe quibus caſibus ſumma ſit $= n$.

§. 26. Quodſi ergo ponamus

$$\Sigma = \tfrac{1}{2}\Gamma : 0 + \operatorname{cof.} 2\omega\Gamma : \omega + \operatorname{cof.} 4\omega\Gamma : 2\omega + \operatorname{cof.} 6\omega\Gamma : 3\omega \ldots \ldots + \tfrac{1}{2}\operatorname{cof.} 2\pi\Gamma : \pi,$$

exiſtente $\omega = \frac{\pi}{n}$, iſta quantitas per hanc ſeriem repraeſentabitur:

$$\Sigma = \left\{ \begin{array}{l} \tfrac{1}{2}n(2) + \tfrac{1}{2}n(2n-2) + \tfrac{1}{2}n(4n-2) + \tfrac{1}{2}n(6n-2) \\ \qquad + \tfrac{1}{2}n(2n+2) + \tfrac{1}{2}n(4n+2) + \tfrac{1}{2}n(6n+2) \end{array} \right\} \text{etc.}$$

unde viciſſem iſtius ſeriei

$$\left\{ \begin{array}{l} (2) + (2n-2) + (4n-2) + (6n-2) + (8n-2) \\ \qquad + (2n+2) + (4n+2) + (6n+2) + (8n+2) \end{array} \right\} \text{etc.}$$

ſumma erit $= \frac{2}{n}\Sigma$; ſicque hinc ſatis exacte valor termini $(2) = C$ definiri poterit, ſi modo pro n numerus ſatis magnus accipiatur. Ita ſi ſumamus $n = 12$, ut ſit $\omega = \frac{\pi}{12} = 15^0$, atque computemus valorem huius formae:

$$\Sigma = \tfrac{1}{2}\Gamma : 0 + \operatorname{cof.} 2\omega\Gamma : \omega + \operatorname{cof.} 4\omega\Gamma : 2\omega + \operatorname{cof.} 6\omega\Gamma : 3\omega \ldots \ldots + \tfrac{1}{2}\operatorname{cof.} 2\pi\Gamma : \pi,$$

quarum partium numerus eſt 13; huius ſeriei:

$$\left\{ \begin{array}{l} (2) + (22) + (46) + (70) + (94) + (118) \\ \qquad + (26) + (50) + (74) + (98) + (122) \end{array} \right\} \text{etc.}$$

ſumma erit $= \frac{1}{6}\Sigma$.

§. 27.

§. 27. Nunc iam ſine pluribus ambagibus ſimiles formulas pro ſequentibus terminis (3); (4); (5); etc. exhibere poterimus. Ita pro termino $(3) = C$, ſumto n numero ſatis magno, poſitoque $\frac{\pi}{n} = \omega$, inveſtigetur valor huius expreſſionis:

$$\Sigma = \tfrac{1}{2}\Gamma : 0 + \text{coſ.}\, 3\omega\Gamma : \omega + \text{coſ.}\, 6\omega\Gamma : 2\omega + \text{coſ.}\, 9\omega\Gamma : 3\omega \ldots$$
$$\ldots + \tfrac{1}{2}\text{coſ.}\, 3\pi\Gamma : \pi,$$

atque hinc conſequemur hanc ſummationem:

$$\left\{\begin{matrix}(3) + (2n-3) + (4n-3) + (6n-3) \\ + (2n+3) + (4n+3) + (6n+3)\end{matrix}\right\} \text{etc.} = \tfrac{2}{n}\Sigma.$$

Ita ſi fuerit $n = 12$, ideoque $\omega = 15°$, habebimus:

$$\left\{\begin{matrix}(3) + (21) + (45) + (69) \\ + (27) + (51) + (75)\end{matrix}\right\} \text{etc.} = \tfrac{1}{6}\Sigma.$$

§. 28. Eodem modo pro termino ſequente $(4) = D$ in ſubſidium vocetur iſta forma:

$$\Sigma = \tfrac{1}{2}\Gamma : 0 + \text{coſ.}\, 4\omega\Gamma : \omega + \text{coſ.}\, 8\omega\Gamma : 2\omega + \text{coſ.}\, 12\omega\Gamma : 3\omega \ldots$$
$$\ldots + \tfrac{1}{2}\text{coſ.}\, 4\pi\Gamma : \pi,$$

unde ſequens orietur ſummatio

$$\left\{\begin{matrix}(4) + (2n-4) + (4n-4) + (6n-4) + (8n-4) \\ + (2n+4) + (4n+4) + (6n+4) + (8n+4)\end{matrix}\right\} \text{etc.} = \tfrac{2}{n}\Sigma.$$

Hinc ſi ſumamus $n = 12$ et $\omega = 15°$, ſummatio inde orianda erit:

$$\left\{\begin{matrix}(4) + (20) + (44) + (68) + (92) \\ + (28) + (52) + (76) + (100)\end{matrix}\right\} \text{etc.} = \tfrac{1}{6}\Sigma.$$

Ex his iam ſatis intelligitur, ſi definire debeat in genere character (λ), computandam ante omnia eſſe formam:

$$\Sigma = \tfrac{1}{2}\Gamma : 0 + \text{coſ.}\, \lambda\omega\Gamma : \omega + \text{coſ.}\, 2\lambda\omega\Gamma : 2\omega \ldots + \tfrac{1}{2}\text{coſ.}\, \lambda\pi\Gamma : \pi,$$

tum

tum vero hinc iftam fummationem obtineri:

$$\left\{\begin{matrix}(\lambda)+(2n-\lambda)+(4n-\lambda)+(6n-\lambda)+(8n-\lambda)\\ +(2n+\lambda)+(4n+\lambda)+(6n+\lambda)+(8n+\lambda)\end{matrix}\right\}\text{ etc.}=\frac{2}{n}\Sigma.$$

Hinc ergo fi capiatur $\lambda = n$, feries hanc induet formam:

$$2(n)+2(3n)+2(5n)+2(7n)+2(9n)+2(11n)+\text{etc.}=\frac{2}{n}\Sigma,$$

ficque orietur ifta feries fatis fimplex:

$$(n)+(3n)+(5n)+(7n)+(9n)+(11n)+\text{ etc.}=\frac{1}{n}\Sigma,$$

in qua continetur infignis proprietas coëfficientium (0); (1); (2); (3); etc., quorum naturam hic inveftigamus.

§. 29. Sin autem hic ftatuere velimus $\lambda = 2n$, feries fummata tranfibit in hanc formam:

$$(0)+2(2n)+2(4n)+2(6n)+2(8n)\text{ etc.}=\frac{2}{n}\Sigma,$$

quae eft nova feries fatis memorabilis pro inveniendo primo termino (0), pro qua igitur erit:

$$\Sigma=\tfrac{1}{2}\Gamma:0+\text{cof.}\,2\pi\Gamma:\omega+\text{cof.}\,4\pi\Gamma:2\omega+\text{cof.}\,6\pi\Gamma:3\omega\;\ldots$$
$$\ldots+\tfrac{1}{2}\text{cof.}\,2n\pi\Gamma:\pi,$$

qui cofinus cum omnes fint unitati aequales, erit

$$\Sigma=\tfrac{1}{2}\Gamma:0+\Gamma:\omega+\Gamma:2\omega+\Gamma:3\omega+\Gamma:4\omega\;\ldots\;\tfrac{1}{2}\Gamma:\pi,$$

quae expreffio prorfus convenit cum ea, quam fupra §. 19. exhibuimus. Verum in ferie, quae inde eft deducta, ingens difcrimen fe exferit, cum hic primus terminus deberet effe duplo maior; huius autem difcriminis caufa in eo latet, quod in genere fuppofuimus fummam feriei pro (0) effe nihilo aequalem, cum cafu $\Gamma = 2n$ ifta feries

$$2+2\,\text{cof.}\,\gamma\omega+2\,\text{cof.}\,2\gamma\omega+2\,\text{cof.}\,3\gamma\omega+\text{etc.}$$

non evanefcat, fed evadat $= 2(1+n)$.

§. 30.

§. 30. His expofitis fatis perfpicuum eft, quemadmodum pro quovis cafu oblato, qui more vulgari in talem feriem:

$$\alpha + \beta \operatorname{cof.} \Phi + \gamma \operatorname{cof.} \Phi^2 + \delta \operatorname{cof.} \Phi^3 + \varepsilon \operatorname{cof.} \Phi^4 + \text{etc.}$$

converti poteft, altera feries ad ufum aftronomicum neceffaria et per cofinus angulorum multiplorum procedens, quam hac forma repraefentavimus:

$$A + B \operatorname{cof.} \Phi + C \operatorname{cof.} 2\Phi + D \operatorname{cof.} 3\Phi + E \operatorname{cof.} 4\Phi + \text{etc.}$$

formari debeat, idque ex fola indole functionis propofitae, fine ulla integratione, dum totum negotium eo tantum redit, ut pro certis cafibus, quibus ftatuitur vel $\Phi = 0$, vel $\Phi = \omega$, vel $\Phi = 2\omega$, etc. valores ipfius functionis refolvendae definiantur, id quod plerumque fine ullo labore fieri poteft. Ita fi formula propofita fuerit $(1 - e \operatorname{cof.} \Phi)^{-\frac{3}{2}} = \Gamma : \Phi$, per univerfam Aftronomiam frequentiffime occurrens, ftatim patet fore:

$$\Gamma : 0 = (1 - e)^{-\frac{3}{2}} = \frac{1}{1 - e\sqrt{(1-e)}} \text{ et } \Gamma : \pi = (1 + e)^{-\frac{3}{2}};$$

at vero $\Gamma : \frac{\pi}{2} = 1$, ita ut perpetuo fit $1 = A - C + E - G + I$ etc. five $(0) - (2) + (4) - (6) + (8) - (10)$ etc. in infinitum $= 1$. Praeterea vero modus, quo hoc argumentum tractavimus, plurimas alias infignes proprietates fuppeditavit, quibus determinatio fingulorum terminorum non mediocriter fublevari poteft.