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Formae generales differentialium, quae, etsi nulla substitutione rationales reddi possunt. tamen integrationem per logarithmos et arcus circulares admittunt

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FORMAE GENERALES DIFFERENTIALIVM, QVAE ETSI NVLLA SVBSTITVTIONE RATIONALES REDDI POSSVNT, TAMEN INTEGRATIONEM PER LOGARITHMOS ET ARCVS CIRCVLARES ADMITTVNT.

Auctore

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§. 1.

Quae non ita pridem de integratione huius formulae differentialis: $\frac{\partial z(3+zz)}{(1+zz)\sqrt[4]{(1+6zz+z^4)}}$ per logarithmos et arcus circulares in medium attuli, eo maiori attentione sunt digna, quod ista formula tam complicatam irrationalitatem involvit, ut nulla plane substitutione ad rationalitatem perduci queat. Est vero ista formula casus specialissimus formarum maxime generalium, in quibus tam obstrusa irrationalitas involvitur, ut nulla certe substitutio sufficiat iis ad rationalitatem reducendis, quarum tamen integralia in genere per logarithmos et arcus circulares exprimi possunt. Quoniam

niam igitur tales formae generales in Analysin maximi momenti incrementa afferre posse sunt censendae, eas hoc loco accuratius explicare constitui.

§. 2. Quo autem huius generis formulas clarius exponam, a formula irrationali, quae in iis inest, inchoari convenit quam hoc modo repraesento:

$$v = \sqrt[n]{[a(\alpha + \gamma z)^n + b(\beta + \delta z)^n]},$$

quae irrationalitas, statim atque exponens n binarium superat, tantopere est abstrusa, ut nullo plane modo ad rationalitatem revocari possit. Deinde denotent litterae maiusculae A, B, C, D sive quantitates constantes sive functiones quascunque rationales formulae $\frac{(\alpha + \gamma z)^n}{(\beta + \delta z)^n}$, atque binae formulae integrales sequentes:

$$\mathfrak{P} = \frac{\int \partial z(\alpha + \gamma z)^{n-1}(\beta + \delta z)^{m-1}[A(\alpha + \gamma z)^{n-m} + B(\beta + \delta z)^{n-m}]}{v^m[C(\alpha + \gamma z)^n + D(\beta + \delta z)^n]}$$

$$\mathfrak{Q} = \frac{\int \partial z[A(\alpha + \gamma z)^{n-m} + B(\beta + \delta z)^{n-m}]\, v^m}{(\alpha + \gamma z)(\beta + \delta z)[C(\alpha + \gamma z)^n + D(\beta + \delta z)^n]}$$

semper per logarithmos et arcus circulares expediri possunt. Harum scilicet formularum prior $\mathfrak{P}$ irrationalitatem v^m in denominatore, posterior vero $\mathfrak{Q}$ in numeratore complectitur; hae igitur duae formae theorema maxime memorabile Analyticum constituunt, cuius veritatem duplici demonstratione sum ostensurus.

Demonstratio prima formularum ante propositarum.

§. 3. Ponatur $\beta + \delta z = x(\alpha + \gamma z)$, eritque formula irrationalis

$v =$

$$v = (\alpha + \gamma z)\sqrt[n]{(a + b x^n)}$$

ideoque

$$v^m = (\alpha + \gamma z)^m \sqrt[n]{(a + b x^n)^m}.$$

Deinde vero hinc erit $z = \frac{\alpha x - \beta}{\delta - \gamma x}$, ideoque $\partial z = \frac{(\alpha\delta - \beta\gamma)\partial x}{(\delta - \gamma x)^2}$, vel etiam cum fit $x = \frac{\beta + \delta z}{\alpha + \gamma z}$, erit $\partial x = \frac{(\alpha\delta - \beta\gamma)\partial z}{(\alpha + \gamma z)^2}$, unde fit $\partial z = \frac{\partial x (\alpha + \gamma z)^2}{\alpha\delta - \beta\gamma}$.

§. 4. Quodfi iam hi valores in priori forma ♄ fubftituantur, ea fequenti modo fatis commode per folam variabilem x exprimi reperietur

$$\text{♄} = \frac{1}{\alpha\delta - \beta\gamma}\int \frac{x^{m-1}\partial x (A + B x^{n-m})}{(C + D x^n)\sqrt[n]{(a + b x^n)^m}}.$$

Simili vero etiam modo altera forma ♃ per folam variabilem x commode exprimetur

$$\text{♃} = \frac{1}{\alpha\delta - \beta\gamma}\int \frac{\partial x (A + B x^{n-m})\sqrt[n]{(a + b x^n)^m}}{x (C + D x^n)}$$

ubi litterae A, B, C, D, nifi fuerint conftantes, erunt functiones rationales huius formulae $\frac{1}{x^n}$, five ipfius x^n.

Evolutio formae prioris ♄.

§. 5. Pofito brevitatis gratia $\alpha\delta - \beta\gamma = \theta$, haec forma in duas partes refolvatur, quae erunt

♄ =

$$\mathfrak{z} = \frac{1}{\theta}\int \frac{A\, x^{m-1}\, \partial x}{(C + D\, x^n)\, \sqrt[n]{(a + b\, x^n)^m}}$$
$$+ \frac{1}{\theta}\int \frac{B\, x^{n-1}\, \partial x}{(C + D\, x^n)\, \sqrt[n]{(a + b\, x^n)^m}}$$

quarum prior rationalis reddetur, ponendo $\frac{x}{\sqrt[n]{(a + b\, x^n)}} = t$; erit enim $\frac{x^n}{a + b\, x^n} = t^n$, unde elicitur $x^n = \frac{a\, t^n}{1 - b\, t^n}$; unde patet litteras A, B, C, D, quae in hac parte occurrunt, fore functiones rationales ipsius t^n, porro vero ob

$$n\, l\, x = l\, a + n\, l\, t - l\,(1 - b\, t^n),$$

erit differentiando

$$\frac{\partial x}{x} = \frac{\partial t}{t} + \frac{b\, t^{n-1}\, \partial t}{1 - b\, t^n} = \frac{\partial t}{t\,(1 - b\, t^n)}.$$

Cum igitur sit

$$\frac{x^m}{\sqrt[n]{(a + b\, x^n)^m}} = t^m \text{ et}$$

$$C + D\, x^n = \frac{C + t^n\,(a\, D - b\, C)}{1 - b\, t^n}$$

his substitutis pars prior formulae $\mathfrak{z}$ erit

$$= \frac{1}{\theta}\int \frac{A\, t^{m-1}\, \partial t}{C + t^n\,(a\, D - b\, C)}$$

quae ergo est rationalis, eiusque propterea integrale per logarithmos atque arcus circulares exhiberi potest.

§. 6.

§. 6. Pro altera autem parte formulae ♄ primo notetur, eam per praecedentem substitutionem rationalem reddi non posse; verum hoc multo facilius praestabitur ponendo $\sqrt[n]{(a+bx^n)} = u$, unde cum fiat $a + bx^n = u^n$, erit

$$x^n = \frac{u^n - a}{b} \text{ atque } x^{n-1}\partial x = \frac{u^{n-1}\partial u}{b}$$

et iam litterae B, C, et D erunt functiones rationales ipsius u^n; quam ob rem cum fit

$$\sqrt[n]{(a+bx^n)^m} = u^m \text{ et } C + Dx^n = \frac{bC - aD + Du^n}{b},$$

his valoribus substitutis pars posterior formulae ♄ erit

$$= \frac{1}{\theta}\int \frac{Bu^{n-m-1}\partial u}{bC - aD + Du^n}$$

quae cum etiam sit rationalis, pariter per logarithmos atque arcus circulares exhiberi poterit.

Evolutio formae posterioris ♃.

§. 7. Haec forma pariter in duas partes resoluta ita repraesentetur:

$$♃ = \frac{1}{\theta}\int \frac{A\partial x \sqrt[n]{(a+bx^n)^m}}{x(C+Dx^n)}$$

$$+ \frac{1}{\theta}\int \frac{Bx^{n-m-1}\partial x \sqrt[n]{(a+bx^n)^m}}{C+Dx^n}.$$

Prior autem pars statim rationalis redditur ponendo

$$\sqrt[n]{(a+bx^n)} = u, \text{ unde fit } x^n = \frac{u^n - a}{b}$$

sumfumtisque logarithmis $n\,l\,x = l\,(u^n - a) - l\,b$, ideoque

$$\frac{\partial x}{x} = \frac{u^{n-1}\,\partial u}{u^n - a} \text{ et } C + D\,x^n = \frac{b\,C - a\,D + D\,u^n}{b}$$

quibus fubftitutis pars prior evadit

$$= \frac{1}{\theta}\int \frac{A\,b\,u^{n+m-1}\,\partial u}{(u^n - a)\,(b\,C - a\,D + D\,u^n)}$$

quae forma, ob A, C, D functiones rationales ipfius u^n, utique ipfa eft rationalis.

§. 8. Altera autem pars formae ♃, quae eft

$$\frac{1}{\theta}\int \frac{B\,x^{n-m-1}\,\partial x\,\sqrt[n]{(a + b\,x^n)^m}}{C + D\,x^n}$$

ita repraefentetur

$$\frac{1}{\theta}\int \frac{\partial x}{x} \cdot \frac{B\,x^n}{C + D\,x^n} \cdot \frac{\sqrt[n]{(a + b\,x^n)^m}}{x^m},$$

et nunc manifeftum eft fcopum propofitum obtentum iri ope prioris fubftitutionis ante ufurpatae $\frac{x}{\sqrt[n]{(a + b\,x^n)}} = t$; fic enim poftremus factor erit $= \frac{1}{t^m}$. Deinde fupra vidimus fore $\frac{\partial x}{x} = \frac{\partial t}{t\,(1 - b\,t^n)}$. Denique vero fiet

$$\frac{B\,x^n}{C + D\,x^n} = \frac{a\,B\,t^n}{C + t^n\,(a\,D - b\,C)};$$

his autem fubftitutis altera pars ipfius ♃ erit

=

$$= \frac{1}{\theta} \int \frac{a B t^{n-m-1} \partial t}{(1 - b t^n)[C + t^n (a D - b C)]}$$

quae etiam eſt rationalis, ob litteras B, C, D functiones rationales ipſius t^n.

§. 9. Ex hac evolutione liquet, ſi litterarum A et B altera evaneſcat, formulas propoſitas ope idoneae ſubſtitutionis utique ad rationalitatem perduci poſſe, ita ut his caſibus noſtrae formulae nihil, quod memoratu eſſet adeo dignum, continerent; at vero ſi harum litterarum neutra evaneſcat, quoniam utraque peculiarem poſtulat ſubſtitutionem, evidens eſt, totum negotium ope unicae ſubſtitutionis nullo modo confici poſſe, atque ob hanc ipſam cauſam noſtrae formulae generales eo maiori attentione dignae ſunt cenſendae.

Demonſtratio alia, methodo prorſus mirabili innixa.

§. 10. Quoniam vidimus ambas noſtras formas tantum diſtribui debere, loco variabilis z ſtatim duas novas variabiles p et q in calculum introducamus, ponendo

$$p = \frac{\alpha + \gamma z}{v} \text{ et } q = \frac{\beta + \delta z}{v}.$$

Hinc autem primo erit $\delta p - \gamma q = \frac{\alpha \delta - \beta \gamma}{v}$; unde ſi ut ante ponamus $\alpha \delta - \beta \gamma = \theta$, erit $v = \frac{\theta}{\delta p - \gamma q}$. Deinde vero erit

$$\alpha q - \beta p = \frac{z(\alpha \delta - \beta \gamma)}{v} = \frac{\theta z}{v},$$

unde colligimus

$$z = \frac{v(\alpha q - \beta p)}{\theta} = \frac{\alpha q - \beta p}{\delta p - \gamma q},$$

unde differentiando colligitur

$$\partial z = \frac{\theta (p \partial q - q \partial p)}{(\delta p - \gamma q)^2},$$

quae expreſſio, ob $\delta p - \gamma q = \frac{\theta}{v}$, concinne ita refertur:

$$\partial z = \frac{vv}{\theta}(p \partial q - q \partial p).$$

§. 11. Deinde vero ex poſitionibus factis colligitur

$$a p^n + b q^n = \frac{a(\alpha + \gamma z)^n + b(\beta + \delta z)^n}{v^n} = 1$$

ob $v^n = a(\alpha + \gamma z^n) + b(\beta + \delta z)^n$, unde facile ſive p per q ſive q per p definiri poteſt, cum ſit

$$\text{vel } p^n = \frac{1 - b q^n}{a} \text{ vel } q^n = \frac{1 - a p^n}{b}.$$

Porro vero quia eſt

$$a p^{n-1} \partial p + b q^{n-1} \partial q = c, \text{ erit}$$

$$\partial p = -\frac{b q^{n+1} \partial q}{a p^{n-1}} \text{ et } \partial q = -\frac{a p^{n-1} \partial p}{b q^{n-1}}.$$

Hinc iam formula $p \partial q - q \partial p$ pro lubitu ſive per ∂q ſive per ∂p exhiberi poterit: priori ſcilicet modo erit

$$p \partial q - q \partial p = \frac{\partial q (a p^n + b q^n)}{a p^{n-1}} = \frac{\partial q}{a p^{n-1}},$$

poſteriore vero modo erit

$$p \partial q - q \partial p = -\frac{\partial p (a p^n + b q^n)}{b q^{n-1}} = -\frac{\partial p}{b q^{n-1}}.$$

Quovis igitur caſu ſive priore ſive poſteriore valore uti licebit, prouti commodius fuerit viſum.

§. 12. Nunc igitur hos novos valores in calculum introducamus, eliminando litteram z, veruntamen ipſam litteram v in calculo retineamus, quippe quae tandem ſponte te ex calculo excedet. Primo igitur, ut iam vidimus, erit $\partial z = \frac{vv}{\theta}(p\,\partial q - q\,\partial p)$, atque ob $\alpha + \gamma z = pv$ et $\beta + \delta z = qv$, erit

$$(\alpha + \gamma z)(\beta + \delta z) = pqvv;$$

$$A(\alpha + \gamma z)^{n-m} + B(\beta + \delta z)^{n-m}$$
$$= v^{n-m}(Ap^{n-m} + Bq^{n-m})$$

ac denique

$$C(\alpha + \gamma z)^n + D(\beta + \delta z)^n = v^n(Cp^n + Dq^n),$$

quibus valoribus substitutis binae nostrae formae generales sequenti modo referentur:

$$\mathfrak{P} = \frac{1}{\theta}\int \frac{p^{m-1}q^{m-1}(p\,\partial q - q\,\partial p)(Ap^{n-m} + Bq^{n-m})}{Cp^n + Dq^n} \text{ et}$$

$$\mathfrak{Z} = \frac{1}{\theta}\int \frac{(p\,\partial q - q\,\partial p)(Ap^{n-m} + Bq^{n-m})}{pq(Cp^n + Dq^n)},$$

ubi notetur litteras A, B, C, D, nisi sint constantes, iam fore functiones rationales formulae $\frac{p^n}{q^n}$, ideoque ob $ap^n + bq^n = 1$, vel ipsius p^n vel ipsius q^n.

Evolutio formulae $\mathfrak{P}$.

§. 13. Hic iterum ista formula per suas partes repraesentetur:

$$\mathfrak{P} = \frac{1}{\theta}\int \frac{Ap^{n-1}q^{m-1}(p\,\partial q - q\,\partial p)}{Cp^n + Dq^n}$$
$$+ \frac{1}{\theta}\int \frac{Bq^{n-1}p^{m-1}(p\,\partial q - q\,\partial p)}{Cp^n + Dq^n}.$$

Et quoniam pro $p\,\partial q - q\,\partial p$ supra geminum valorem exhihibuimus, alterum per ∂q alterum vero per ∂p expressum, priori valore utamur pro parte priori, quae evadet

$$= \frac{1}{a\theta}\int \frac{A q^{m-1}\partial q}{C p^n + D q^n},$$

quae porro, ob $p^n = \frac{1 - b q^n}{a}$, transit in hanc formam:

$$\frac{1}{\theta}\int \frac{A q^{m-1}\partial q}{C + q^n(a D - b C)};$$

ubi cum A, C, D per solam q rationaliter exprimi queant, sola variabilis q inest, idque rationaliter, unde integrale per logarithmos et arcus circulares exprimi poterit.

§. 14. Pro parte autem secunda formulae $\mathfrak{p}$ utamur valore posteriore pro $p\partial q - q\partial p$, qui est $-\frac{\partial p}{b q^{n-1}}$. Hinc enim ista pars prodibit

$$= -\frac{1}{\theta b}\int \frac{B p^{m-1}\partial p}{C p^n + D q^n},$$

quae ob $q^n = \frac{1 - a p^n}{b}$ abit in hanc

$$-\frac{1}{\theta}\int \frac{B p^{m-1}\partial p}{D - p^n(a D - b C)}$$

quae expressio solam variabilem p rationaliter comprehendit, quandoquidem litterae B, C, D, nisi sint constantes, sunt functiones ipsius p^n. His igitur partibus iunctis erit:

$$\mathfrak{p} = \frac{1}{\theta}\int \frac{A q^{m-1}\partial q}{C + q^n(a D - b C)} - \frac{1}{\theta}\int \frac{B p^{m-1}\partial p}{D - p^n(a D - b C)}.$$

Evo-

Evolutio formulae $\mathfrak{Z}$.

§. 15. Haec formula fimili modo per fuas partes ita repraefentabitur:

$$\mathfrak{Z} = \frac{1}{\theta}\int \frac{A\, p^{n-m-1}(p\,\partial q - q\,\partial p)}{q\,(C\,p^n + D\,q^n)}$$
$$+ \frac{1}{\theta}\int \frac{B\, q^{n-m-1}(p\,\partial q - q\,\partial p)}{p\,(C\,p^n + D\,q^n)}.$$

Pro priore parte utamur valore

$$p\,\partial q - q\,\partial p = -\frac{\partial p}{b\,q^{n-1}};$$

unde ifta pars fiet

$$= -\frac{1}{\theta\,b}\int \frac{A\,p^{n-m-1}\,\partial p}{q^n\,(C\,p^n + D\,q^n)},$$

quae porro ob $q^n = \frac{1 - a\,p^n}{b}$ induet hanc formam:

$$-\frac{b}{\theta}\int \frac{A\,p^{n-m-1}\,\partial p}{(1 - a\,p^n)\,[D - p^n\,(a\,D - b\,C)]}.$$

§. 16. Pro parte autem pofteriore utamur altero valore $p\,\partial q - q\,\partial p = \frac{\partial q}{a\,p^{n-1}}$, ex quo ifta pars evadet

$$\frac{1}{a\,\theta}\int \frac{B\,q^{n-m-1}\,\partial q}{p^n\,(C\,p^n + D\,q^n)},$$

quae porro ob $p^n = \frac{1 - b\,q^n}{a}$ reducitur ad hanc formam:

$$\frac{a}{\theta}\int \frac{B\,q^{n-m-1}\,\partial q}{(1 - b\,q^n)\,C + q^n\,(a\,D - b\,C)}.$$

Hoc

Hoc igitur modo altera formula generalis $\mathfrak{z}$ ita repraesentetur:

$$\mathfrak{z} = -\frac{b}{\theta}\int\frac{A\,p^{n-m-1}\,\partial p}{(1-a\,p^n)[D-p^n(a\,D-b\,C)]}$$
$$+\frac{a}{\theta}\int\frac{B\,q^{n-m-1}\,\partial q}{(1-b\,q^n)[C+q^n(a\,D-b\,C)]}.$$

Quanquam haec posterior methodus a praecedente prorsus differt, tamen egregia harmonia elucet.

§. 17. Quoniam autem haec nimis sunt generalia, quam ut clare percipi queant, paulatim ad magis particularia descendamus, ac primo quidem sumamus litteris A, B, C, D, perpetuo quantitates constantes designari, hicque statim se offert casus memorabilis, quo $C = a$ et $D = b$, siquidem hinc oritur $C(\alpha+\gamma z)^n + D(\beta+\delta z)^n = v^n$, sicque binae nostrae formae erunt:

$$\mathfrak{y} = \frac{\int\partial z(\alpha+\gamma z)^{m-1}(\beta+\delta z)^{m-1}[A(\alpha+\gamma z)^{n-m}+B(\beta+\delta z)^{n-m}]}{v^{m+n}}\text{ et}$$

$$\mathfrak{z} = \frac{\int\partial z[A(\alpha+\gamma z)^{n-m}+B(\beta+\delta z)^{n-m}]}{v^{n-m}(\alpha+\gamma z)(\beta+\delta z)}.$$

§. 18. Hoc igitur casu si ponatur $p = \frac{\alpha+\gamma z}{v}$ et $q = \frac{\beta+\delta z}{v}$, integralia harum formarum hoc modo exprimentur:

$$\mathfrak{y} = \frac{1}{\theta}\int\frac{A\,q^{m-1}\,\partial q}{a} - \frac{1}{\theta}\int\frac{B\,p^{m-1}\,\partial p}{b},$$

sicque iste valor adeo algebraice exhiberi poterit: erit enim

$$\mathfrak{y} = \frac{A}{m\theta a}\,q^m - \frac{B}{m\theta b}\,p^m,$$

sive

ſive erit

$$\mathfrak{p} = \frac{A(\beta + \delta z)^m}{m\theta a v^m} - \frac{B(\alpha + \gamma z)^m}{m\theta b v^m}.$$

Pro altera autem forma habebimus

$$\mathfrak{q} = -\frac{1}{\theta}\int \frac{A p^{n-m-1}\partial p}{1 - a p^n} + \frac{1}{\theta}\int \frac{B q^{n-m-1}\partial q}{1 - b q^n},$$

quae quidem forma aliter integrari nequit, niſi per logarithmos et arcus circulares, ſed ob concinnitatem imprimis eſt notatu digna.

§. 19. Imprimis autem formulae notabiles prodibunt, ſi ſtatuamus $\alpha = 1$; $\beta = 1$; $\gamma = 1$ at $\delta = -1$; unde fit $\theta = -2$ et iam binae noſtrae formae generales ſequentem faciem induent:

$$\mathfrak{p} = \frac{\int \partial z (1 - zz)^{m-1}[A(1+z)^{n-m} + B(1-z)^{n-m}]}{v^n[C(1+z)^n + D(1-z)^n]} \text{ et}$$

$$\mathfrak{q} = \frac{\int \partial z [A(1+z)^{n-m} + B(1-z)^{n-m}]v^m}{(1-zz)C[C(1+z)^n + D(1-z)^n]},$$

ubi iam eſt $v = \sqrt[n]{[a(1+z)^n + b(1-z)^n]}$. Tum vero, poſito $p = \frac{1+z}{v}$ et $q = \frac{1-z}{v}$, valores harum formarum ſequenti modo exprimentur:

$$\mathfrak{p} = -\frac{A}{2}\int \frac{q^{n-1}\partial q}{C + q^n(aD - bC)} + \frac{B}{2}\int \frac{p^{m-1}\partial p}{D - p^n(aD - bC)} \text{ et}$$

$$\mathfrak{q} = \frac{Ab}{2}\int \frac{p^{n-m-1}\partial p}{(1 - a p^n)[D - p^n(aD - bC)]}$$

$$- \frac{Ba}{2}\int \frac{q^{n-m-1}\partial q}{(1 - b q^n)[C + q^n(aD - bC)]}.$$

§. 20.

§. 20. Combinemus nunc hanc posteriorem hypothesin cum praecedente, qua erat $C = a$ et $D = b$ ac formae nostrae erunt:

$$\mathfrak{P} = \frac{\int \partial z (1 - zz)^{m-1} [A(1+z)^{n-m} + B(1-z)^{n-m}]}{v^{m+n}} \text{ et}$$

$$\mathfrak{Q} = \frac{\int \partial z [A(1+z)^{n-m} + B(1-z)^{n-m}]}{v^{n-m}(1-zz)},$$

tum autem per nostram reductionem erit

$$\mathfrak{P} = -\frac{A(1-z)^m}{2 m a v^m} + \frac{B(1+z)^m}{2 m b v^m}, \text{ et}$$

$$\mathfrak{Q} = \frac{1}{2}\int \frac{A p^{n-m-1} \partial p}{1 - a p^n} + \frac{1}{2}\int \frac{B q^{n-m-1} \partial q}{1 - b q^n}.$$

§. 21. Quoniam autem hic forma $\mathfrak{P}$, utpote algebraice integrabilis, nulla laborat difficultate, eius loco aliam contemplabimur affinem, ponendo $C = a$ at $D = -b$, ita ut iam sit $aD - bC = -2ab$, eritque

$$\mathfrak{P} = \int \frac{\partial z (1 - zz)^{m-1} [A(1+z)^{n-m} + B(1-z)^{n-m}]}{v^m [a(1+z)^n - b(1-z)^n]},$$

cuius valor per p et q ita exprimitur, ut sit

$$\mathfrak{P} = -\frac{A}{2}\int \frac{q^{m-1}\partial q}{a - 2abq^n} + \frac{B}{2}\int \frac{p^{m-1}\partial p}{2abp^n - b}, \text{ sive}$$

$$\mathfrak{P} = -\frac{A}{2}\int \frac{q^{m-1}\partial q}{a - 2abq^n} - \frac{B}{2}\int \frac{p^{m-1}\partial p}{b - 2abp^n}:$$

In sequentibus istam formam $\mathfrak{P}$ cum praecedente forma $\mathfrak{Q}$ coniunctim considerabimus, atque bini casus seorsim tractandi se offerunt.

Evo-

Evolutio casus, quo $a = \frac{1}{2}$ et $b = -\frac{1}{2}$.

§. 22. Hic igitur erit $v = \sqrt[n]{[\frac{1}{2}(1+z)^n - \frac{1}{2}(1-z)^n]}$; huius ergo valores pro simplicioribus exponentibus n erunt uti sequuntur:

Si $n = 2$, erit $v = \sqrt{2z}$.

Si $n = 3$, erit $v = \sqrt[3]{(3z + z^3)}$.

Si $n = 4$, erit $v = \sqrt[4]{(4z + 4z^3)}$.

Si $n = 5$, erit $v = \sqrt[5]{(5z + 10z^3 + z^5)}$.

Si $n = 6$, erit $v = \sqrt[6]{(6z + 20z^3 + 6z^5)}$.

Expediamus nunc primo postremam formam pro $\mathfrak{P}$ datam, et quoniam in eius denominatore occurrit forma $a(1+z)^n - b(1-z)^n$, eius loco scribamus brevitatis gratia s, ita ut ob $a = \frac{1}{2}$ et $b = -\frac{1}{2}$ fit

$$s = \frac{1}{2}(1+z)^n + \frac{1}{2}(1-z)^n, \text{ ideoque}$$

$$\mathfrak{P} = \int \frac{\partial z(1-zz)^{m-1}[A(1+z)^{n-m} + B(1-z)^{n-m}]}{v^m s},$$

atque per litteras p et q erit

$$\mathfrak{P} = -A\int \frac{q^{m-1}\partial q}{1+q^n} + B\int \frac{p^{m-1}\partial p}{1-p^n};$$

ubi notentur pro simplicioribus exponentibus n valores:

Si $n = 2$, erit $s = 1 + zz$.

Si $n = 3$, erit $s = 1 + 3zz$.

Si $n = 4$, erit $s = 1 + 6zz + z^4$.

Si $n = 5$, erit $s = 1 + 10zz + 5z^4$.

Si $n = 6$, erit $s = 1 + 15zz + 15z^4 + z^6$.

§. 23. Poſtrema autem forma ♃ hoc caſu evadit

$$♃ = \int \frac{\partial z\,[A\,(1+z)^{n-m} + B\,(1-z)^{n-m}]}{v^{n-m}\,(1-zz)},$$

cuius valor per p et q expreſſus erit

$$♃ = A \int \frac{p^{n-m-1}\,\partial p}{2-p^n} - B \int \frac{q^{n-m-1}\,\partial q}{2+q^n}.$$

Evolutio caſus, quo $a = \frac{1}{2}$ et $b = \frac{1}{2}$.

§. 24. Hic igitur erit

$$v = \sqrt[n]{[\tfrac{1}{2}(1+z)^n + \tfrac{1}{2}(1-z)^n]},$$

huius ergo valores pro ſimplicioribus exponentibus n erunt, ut ſequitur:

Si $n = 2$, erit $v = \sqrt[2]{(1+zz)}$.

Si $n = 3$, erit $v = \sqrt[3]{(1+3zz)}$.

Si $n = 4$, erit $v = \sqrt[4]{(1+6zz+z^4)}$.

Si $n = 5$, erit $v = \sqrt[5]{(1+10zz+5z^4)}$.

Si $n = 6$, erit $v = \sqrt[6]{(1+15zz+15z^4+z^6)}$.

§. 25. Expediamus nunc poſtremam formam pro ♃ datam, in qua loco $a(1+z)^n - b(1-z)^n$ ſcribamus brevitatis gratia T, ita ut ſit $T = \frac{1}{2}(1+z)^n - \frac{1}{2}(1-z)^n$, ſicque ipſa forma erit

$$♃ = \int \frac{\partial z\,(1-zz)^{m-1}\,[A\,(1+z)^{n+m} + B\,(1-z)^{n-m}]}{v^m\,T},$$

quae

quae per litteras p et q ita exprimitur:

$$\mathfrak{z} = -A\int\frac{q^{m-1}\partial q}{1-q^n} - B\int\frac{p^{m-1}\partial p}{1-p^n},$$

ubi pro exponentibus fimplicioribus erit ut fequitur:

Si $n = 2$, erit $T = 2z$.
Si $n = 3$, erit $T = 3z + z^3$.
Si $n = 4$, erit $T = 4z + 4z^3$.
Si $n = 5$, erit $T = 5z + 10z^3 + z^5$.
Si $n = 6$, erit $T = 6z + 20z^3 + 6z^5$.

Hoc autem cafu evadet

$$\mathfrak{z} = \int\frac{\partial z[A(1+z)^{n-m} + B(1-z)^{n-m}]}{v^{n-m}(1-zz)},$$

cuius valor per p et q expreffus erit

$$\mathfrak{z} = A\int\frac{p^{n-m-1}\partial p}{2-p^n} - B\int\frac{q^{n-m-1}\partial q}{2-q^n}.$$

§. 26. In his autem formulis perpetuo accipiamus

$$A = \tfrac{1}{2}f + \tfrac{1}{2}g \text{ et } B = \tfrac{1}{2}f - \tfrac{1}{2}g,$$

tum igitur formula, ubi hae litterae occurrunt, hanc induet fpeciem: $fF + gG$, eritque

$$F = \tfrac{1}{2}(1+z)^{n-m} + \tfrac{1}{2}(1-z)^{n-m} \text{ et}$$
$$G = \tfrac{1}{2}(1+z)^{n-m} - \tfrac{1}{2}(1-z)^{n-m},$$

unde ergo fequentes valores pro cafibus fimplicioribus emergunt:

 Si

Si $n - m = 1$, erit $F = 1$ et $G = z$.

Si $n - m = 2$, erit $F = 1 + zz$ et $G = 2z$.

Si $n - m = 3$, erit $F = 1 + 3zz$ et $G = 3z + z^3$.

Si $n - m = 4$, erit $F = 1 + 6zz + z^4$ et $G = 4z + 4z^3$.

Si $n - m = 5$, erit $F = 1 + 10zz + 5z^4$ et
$G = 5z + 10z^3 + z^5$.

Si $n - m = 6$, erit $F = 1 + 15zz + 15z^4 + z^6$ et
$G = 6z + 20z^3 + 6z^5$.

§. 27. Secundum iſtas quatuor formas iam ſatis particulares totidem ordines formularum ſpecialium conſtituamus, dum ſcilicet exponentibus indefinitis m et n valores determinati ſimpliciores aſſignabuntur, ubi quidem pro m numeri minores quam n capientur.

Ordo primus formularum ſpecialium ex forma

$$\mathfrak{z} = \int \frac{\partial z\,(1 - zz)^{m-1}\,(fF + gG)}{v^m\, s}.$$

§. 28. Cuiusmodi valores litteris F, G, v et s ſint tribuendi, ſupra iam eſt oſtenſum, ubi etiam vidimus, ſi ſtatuatur $p = \frac{1+z}{v}$ et $q = \frac{1-z}{v}$, fore

$$\mathfrak{z} = -\frac{(f+g)}{2}\int \frac{q^{m-1}\partial q}{1 + q^n} + \frac{(f-g)}{2}\int \frac{p^{m-1}\partial p}{1 - p^n}.$$

Hinc iam ſequentes formulas ſpeciales derivemus.

1°. Sit $n = 2$ et $m = 1$.

§. 29. Hic igitur erit $v = \sqrt{2z}$; $s = 1 + zz$, $F = 1$ et $G = z$, ideoque formula ſpecialis

$\mathfrak{z} =$

$\mathfrak{v} = \int \frac{\partial z (f + g z)}{(1 + z z) \sqrt{2 z}}$, hocque cafu erit

$\mathfrak{v} = - \frac{(f + g)}{2} \int \frac{\partial q}{1 + q q} + \frac{(f - g)}{2} \int \frac{\partial p}{1 - p p}$,

exiftente $p = \frac{1 + z}{\sqrt{2 z}}$ et $q = \frac{1 - z}{\sqrt{2 z}}$.

2°. Sit $n = 3$ et $m = 1$, ideoque $n - m = 2$.

§. 30. Hic igitur erit $v = \sqrt[3]{(3 z + z^3)}$; $s = 1 + 3 z z$, $F = 1 + z z$ et $G = 2 z$, ideoque formula fpecialis

$$\mathfrak{v} = \int \frac{\partial z [f (1 + z z) + 2 g z]}{(1 + 3 z z) \sqrt[3]{(3 z + z^3)}},$$

hocque cafu erit

$$\mathfrak{v} = - \frac{(f + g)}{2} \int \frac{\partial q}{1 + q^3} + \frac{(f - g)}{2} \int \frac{\partial p}{1 - p^3},$$

exiftente $p = \frac{1 + z}{\sqrt[3]{(3 z + z^3)}}$ et $q = \frac{1 - z}{\sqrt[3]{(3 z + z^3)}}$.

3°. Sit $n = 3$ et $m = 2$, ideoque $n - m = 1$.

§. 31. Hic igitur erit $v = \sqrt[3]{(2 z + z^3)}$; $s = 1 + 3 z z$, $F = 1$ et $G = z$, ideoque formula fpecialis:

$$\mathfrak{v} = \int \frac{\partial z (1 - z z) (f + g z)}{(1 + 3 z z) \sqrt[3]{(3 z + z^3)^2}},$$

hocque cafu erit

$$\mathfrak{v} = - \frac{(f + g)}{2} \int \frac{q \partial q}{1 + q^3} + \frac{(f - g)}{2} \int \frac{p \partial p}{1 - p^3},$$

exiftente $p = \frac{1 + z}{\sqrt[3]{(3 z + z^3)}}$ et $q = \frac{1 - z}{\sqrt[3]{(3 z + z^3)}}$.

4°. Sit

4°. Sit $n = 4$ et $m = 1$, ideoque $n - m = 3$.

§. 32. Hic igitur erit $v = \sqrt[4]{(4z + 4z^3)}$; $s = 1 + 6zz + z^4$; $F = 1 + zz$ et $G = 3z + z^3$, ideoque formula ſpecialis

$$\mathfrak{b} = \int \frac{\partial z\,[f(1 + 3zz) + g(3z + z^3)]}{(1 + 6zz + z^4)\sqrt[4]{(4z + 4z^3)}}.$$

Hoc caſu erit

$$\mathfrak{b} = -\tfrac{(f+g)}{2}\int \frac{\partial q}{1 + q^4} + \tfrac{(f-g)}{2}\int \frac{\partial p}{1 - p^4},$$

exiſtente $p = \frac{1 + z}{\sqrt[4]{(4z + 4z^3)}}$ et $q = \frac{1 - z}{\sqrt[4]{(4z + 4z^3)}}$.

5°. Sit $n = 4$ et $m = 1$, ideoque $n - m = 2$.

§. 33. Hic igitur erit $v = \sqrt[4]{(4z + 4z^3)}$;
$s = 1 + 6zz + z^4$; $F = 1 + zz$ et $G = 2z$, ideoque formula ſpecialis

$$\mathfrak{b} = \int \frac{\partial z\,(1 - zz)\,[f(1 + zz) + 2gz]}{(1 + 6zz + z^4)\sqrt[4]{(4z + 4z^3)}}.$$

Hoc igitur caſu erit

$$\mathfrak{b} = -\tfrac{(f+g)}{2}\int \frac{q\,\partial q}{1 + q^4} + \tfrac{(f-g)}{2}\int \frac{p\,\partial p}{1 - p^4},$$

exiſtente

$$p = \frac{1 + z}{\sqrt[4]{(4z + 4z^3)}} \text{ et } q = \frac{1 - z}{\sqrt[4]{(4z + 4z^3)}}.$$

6°. Sit $n = 4$ et $m = 3$, ideoque $n - m = 1$.

§. 33.

§. 33. Hic manent ut ante $v = \sqrt[4]{(4z + 4z^3)}$; $s = 1 + 6zz + z^4$; at erit $F = 1$ et $G = z$, ideoque formula specialis

$$\text{þ} = \int \frac{\partial z (1 - zz)^2 (f + gz)}{(1 + 6zz + z^4) \sqrt[4]{(4z + 4z^3)^3}},$$

hocque casu erit

$$\text{þ} = -\frac{(f+g)}{2} \int \frac{qq\partial q}{1+q^4} + \frac{(f-g)}{2} \int \frac{pp\partial p}{1-p^4},$$

existente

$$p = \frac{1+z}{\sqrt[4]{(4z + 4z^3)}} \text{ et } q = \frac{1-z}{\sqrt[4]{(4z + 4z^3)}}.$$

7. Sit $n = 5$ et $m = 1$, ideoque $n - m = 4$.

§. 34. Hic igitur erit $v = \sqrt[5]{(5z + 10z^3 + z^5)}$; $s = 1 + 10zz + 5z^4$; $F = 1 + 6zz + z^4$ et $G = 4z + 4z^3$; ex quibus oritur formula specialis

$$\text{þ} = \int \frac{\partial z [f(1 + 6zz + z^4) + 4g(z + z^3)]}{(1 + 10zz + 5z^4) \sqrt[5]{(5z + 10z^3 + z^5)}},$$

cuius valor hoc casu erit

$$\text{þ} = -\frac{(f+g)}{2} \int \frac{\partial q}{1+q^5} + \frac{(f-g)}{2} \int \frac{\partial p}{1-p^5},$$

existente

$$p = \frac{1+z}{\sqrt[5]{(5z + 10z^3 + z^5)}} \text{ et } q = \frac{1-z}{\sqrt[5]{(5z + 10z^3 + z^5)}}.$$

8°. Sit $n = 5$ et $m = 2$, ideoque $n - m = 3$.

§. 35. Hic erit $v = \sqrt[5]{(5z + 10z^3 + z^5)}$; $s = 1 + 10zz + 5z^4$; $F = 1 + zz$ et $G = 3z + z^3$, hinc formula specialis

$$\mathfrak{z} = \int \frac{\partial z (1 - zz)[f(1 + 3zz) + g(3z + z^3)]}{(1 + 10zz + 5z^4)\sqrt[5]{(5z + 10z^3 + z^5)^2}},$$

hocque casu erit

$$\mathfrak{z} = -\frac{(f+g)}{2}\int \frac{q\,\partial q}{1 + q^5} + \frac{(f-g)}{2}\int \frac{p\,\partial p}{1 - p^5},$$

existente

$$p = \frac{1 + z}{\sqrt[5]{(5z + 10z^3 + z^5)}} \text{ et } q = \frac{1 - z}{\sqrt[5]{(5z + 10z^3 + z^5)}};$$

9°. Sit $n = 5$ et $m = 3$, ideoque $n - m = 2$.

§. 36. Hic igitur erit $v = \sqrt[5]{(5z + 10z^3 + z^5)}$; $s = 1 + 10zz + 5z^4$; $F = 1 + zz$ et $G = 2z$, ideoque formula specialis

$$\mathfrak{z} = \int \frac{\partial z (1 - zz)^2 [f(1 + zz) + 2gz]}{(1 + zz)\sqrt[5]{(5z + 10z^3 + z^5)^3}},$$

hocque casu erit

$$\mathfrak{z} = -\frac{(f+g)}{2}\int \frac{qq\,\partial q}{1 + q^5} + \frac{(f-g)}{2}\int \frac{pp\,\partial p}{1 - p^5},$$

existente ut ante

$$p = \frac{1 + z}{\sqrt[5]{(5z + 10z^3 + z^5)}} \text{ et } q = \frac{1 - z}{\sqrt[5]{(5z + 10z^3 + z^5)}}.$$

10°.

10. Sit $n = 5$ et $m = 4$, ideoque $n - m = 1$.

§. 37. Hic igitur erit $v = \sqrt[5]{(5z + 10z^3 + z^5)}$; $s = 1 + 10zz + 5z^4$; $F = 1$ et $G = z$, ideoque formula specialis

$$\mathfrak{z} = \int \frac{\partial z (1 - zz)^3 (f + gz)}{(1 + 10zz + 5z^4) \sqrt[5]{(5z + 10z^3 + z^5)}},$$

hocque casu erit

$$\mathfrak{z} = -\frac{(f+g)}{2} \int \frac{q^3 \partial q}{1 + q^5} + \frac{(f+g)}{2} \int \frac{p^3 \partial p}{1 - p^5},$$

existente

$$p = \frac{1 + z}{\sqrt[5]{(5z + 10z^3 + z^5)}} \text{ et } q = \frac{1 - z}{\sqrt[5]{(5z + 10z^3 + z^5)}}.$$

11. Sit $n = 6$ et $m = 1$, ideoque $n - m = 5$.

§. 38. Hic igitur erit

$v = \sqrt[6]{(6z + 20z^3 + 6z^5)}$; $s = 1 + 15zz + 15z^4 + z^6$;
$F = 1 + 10zz + 5z^4$ et $G = 5z + 10z^3 + z^5$,

ideoque formula specialis

$$\mathfrak{z} = \int \frac{\partial z [f(1 + 10zz + 5z^4) + g](5z + 10z^3 + z^5)}{(1 + 15zz + 15z^4 + z^6) \sqrt[6]{(6z + 20z^3 + 6z^5)}},$$

hocque casu erit

$$\mathfrak{z} = -\tfrac{1}{2}(f + g) \int \frac{\partial q}{1 + q^6} + \tfrac{1}{2}(f - g) \int \frac{\partial p}{1 - p^6},$$

existente

$$p = \frac{1 + z}{\sqrt[6]{(6z + 20z^3 + 6z^5)}} \text{ et } q = \frac{1 - z}{\sqrt[6]{(6z + 20z^3 + 6z^5)}}.$$

12. Sit $n = 6$ et $m = 2$, ideoque $n - m = 4$.

§. 39. Hic igitur erit

$$v = \sqrt[6]{6z + 20z^3 + 6z^5};\quad s = 1 + 15zz + 15z^4 + z^6;$$
$$F = 1 + 6zz + z^4 \text{ et } G = 4z + 4z^3,$$

ideoque formula ſpecialis

$$\mathfrak{h} = \int \frac{\partial z(1 - zz) f(1 + 6zz + z^4) + 4g(z + z^3)}{(1 + 15zz + 15z^4 + z^6)\sqrt[3]{(6z + 20z^3 + 6z^5)}},$$

cuius valor eſt

$$\mathfrak{h} = \tfrac{1}{2}(f + g)\int \tfrac{q\,\partial q}{1 + q^6} + \tfrac{1}{2}(f - g)\int \tfrac{p\,\partial p}{1 - p^6},$$

exiſtente

$$p = \frac{1 + z}{\sqrt[6]{(6z + 20z^3 + 6z^5)}} \text{ et } q = \frac{1 - z}{\sqrt[6]{(6z + 20z^3 + 6z^5)}}.$$

13. Sit $n = 6$ et $m = 3$, ideoque $n - m = 3$.

§. 40. Hic igitur erit

$$v = \sqrt[6]{6z + 20z^3 + 6z^5};\quad s = 1 + 15zz + 15z^4 + z^6;$$
$$F = 1 + zz \text{ et } G = 3z + z^3,$$

ideoque formula ſpecialis

$$\mathfrak{h} = \int \frac{\partial z(1 - zz)^2 [f(1 + zz) + g(3z + z^3)]}{(1 + 15zz + 15z^4 + z^6)\sqrt{(6z + 20z^3 + 6z^5)}},$$

cuius valor eſt

$$\mathfrak{h} = \tfrac{1}{2}(f + g)\int \tfrac{qq\,\partial q}{1 + q^6} + \tfrac{1}{2}(f - g)\int \tfrac{pp\,\partial p}{1 - p^6},$$

exiſtente

$$p = \frac{1 + z}{\sqrt[6]{(6z + 20z^3 + 6z^5)}} \text{ et } q = \frac{1 - z}{\sqrt[6]{(6z + 20z^3 + 6z^5)}}!$$

14.

14. Sit $n = 6$ et $m = 4$, ideoque $n - m = 2$.

§. 41. Hic igitur erit

$$v = \sqrt[6]{(6z + 20z^3 + 6z^5)};\ s = 1 + 15zz + 15z^4 + z^6;$$
$$F = 1 + zz \text{ et } G = 2z,$$

hincque formula ſpecialis

$$\mathfrak{p} = \int \frac{\partial z (1 - zz)^3 [f(1 + zz) + 2gz]}{(1 + 15zz + 15z^4 + z^6)\sqrt[6]{(6z + 20z^3 + 6z^5)^2}},$$

cuius valor eſt

$$\mathfrak{p} = \tfrac{1}{2}(f + g)\int \tfrac{q^3 \partial q}{1 + q^6} + \tfrac{1}{2}(f - g)\int \tfrac{p^3 \partial p}{1 - p^6},$$

exiſtente

$$p = \frac{1 + z}{\sqrt[6]{(6z + 20z^3 + 6z^5)}} \text{ et } q = \frac{1 - z}{\sqrt[6]{(6z + 20z^3 + 6z^5)}}.$$

15. Sit $n = 6$ et $m = 5$, ideoque $n - m = 1$.

§. 42. Hic igitur erit

$$v = \sqrt[6]{(6z + 20z^3 + 6z^5)};\ s = 1 + 15zz + 15z^4 + z^6;$$

$F = 1$ et $G = z$, ideoque formula ſpecialis

$$\mathfrak{p} = \int \frac{\partial z (1 - zz)^4 (f + gz)}{(1 + 15zz + 15z^4 + z^6)\sqrt[6]{(6z + 20z^3 + 6z^5)^5}},$$

cuius ergo valor eſt

$$\mathfrak{p} = \tfrac{1}{2}(f + g)\int \tfrac{q^4 \partial q}{1 + q^6} + \tfrac{1}{2}(f - g)\int \tfrac{p^4 \partial p}{1 - p^6},$$

exiſtente

$$p = \frac{1 + z}{\sqrt[6]{(6z + 20z^3 + 6z^5)}} \text{ et } q = \frac{1 - z}{\sqrt[6]{(6z + 20z^3 + 6z^5)}}.$$

 Ob-

Obſervatio in has formulas.

§. 43. Hic ii caſus imprimis notatu ſunt digni, quibus $n = 2m$, propterea quod tum in formulam integralem tantum ſignum $\sqrt{}$ quadraticum ingreditur; hos ergo caſus evolviſſe operae erit pretium. Poſito igitur $n = 2m$ habebitur

$$v = \sqrt[2m]{\left[\tfrac{1}{2}(1+z)^{2m} - \tfrac{1}{2}(1-z)^{2m}\right]}.$$

Ac ſi loco $\frac{1}{2}(f+g)$ et $\frac{1}{2}(f-g)$, litteras A et B reſtituamus, erit formula noſtra

$$\mathfrak{P} = \int \frac{\partial z(1-zz)^{m-1}[A(1+z)^m + B(1-z)^m]}{[\frac{1}{2}(1+z)^{2m}+\frac{1}{2}(1-z)^{2m}]\sqrt{[\frac{1}{2}(1+z)^{2m}-\frac{1}{2}(1-z)^{2m}]}},$$

cuius integrale, ſumtis $p = \frac{1+z}{v}$ et $q = \frac{1-z}{v}$, erit

$$\mathfrak{P} = -A\int \frac{q^{m-1}\partial q}{1+q^{2m}} + B\int \frac{p^{m-1}\partial p}{1-p^{2m}}.$$

§. 44. Has autem formulas in genere integrare licet. Pro priore enim ponamus $q^m = t$, eritque $q^{m-1}\partial q = \frac{\partial t}{m}$, ſicque pars prior erit

$$-\frac{A}{m}\int \frac{\partial t}{1+tt} = -\frac{A}{m}\,\text{Ar. tang. } t = -\frac{A}{m}\,\text{Ar. tang. } q^m.$$

Pro altera forma ſi ponamus $p^m = u$, erit altera pars

$$= \frac{B}{m}\int \frac{\partial u}{1-uu} = \frac{B}{2m}\, l\, \frac{1+p^m}{1-p^m},$$

ſicque ipſum integrale erit

$$= \frac{B}{2m}\, l\, \frac{1+p^m}{1-p^m} - \frac{A}{m}\,\text{Ar. tang. } q^m, \text{ ſive}$$

$$\mathfrak{P} = \frac{B}{2m}\, l\, \frac{v^m + (1+z)^m}{v^m - (1+z)^m} - \frac{A}{m}\,\text{Ar. tang. } \frac{(1-z)^m}{v^m}.$$

Ordo

Ordo fecundus formularum fpecialium ex forma

$$\text{þ} = \int \frac{\partial z (1 - zz)^{m-1} (f F + g G)}{v^m T}.$$

§. 45. Pro hac formula valores litterarum v et T fupra in §. 24. et 25, litterarum vero F et G in §. 26. funt affignati, ubi etiam vidimus, fi ponatur $p = \frac{1+z}{v}$ et $q = \frac{1-z}{v}$, tum valorem integralem fore

$$\text{þ} = -\tfrac{1}{2}(f+g)\int \frac{q^{m-1}\partial q}{1-q^n} - \tfrac{1}{2}(f-g)\int \frac{p^{m-1}\partial p}{1-q^n}.$$

Hinc iam fequentes formulas fpeciales derivemus.

1. Sit $n = 2$ et $m = 1$, ideoque $n - m = 1$.

§. 46. Hic igitur erit $v = \sqrt{(1+zz)}$; $T = 2z$; $F = 1$ et $G = z$, hinc iam formula fpecialis erit

$$\text{þ} = \int \frac{\partial z (f + gz)}{2z\sqrt{(1+zz)}},$$

cuius ergo integrale eft

$$\text{þ} = -\tfrac{1}{2}(f+g)\frac{\partial q}{1-qq} - \tfrac{1}{2}(f-g)\int \frac{\partial p}{1-pp},$$

exiftente

$$p = \frac{1+z}{\sqrt{(1+zz)}} \text{ et } q = \frac{1-z}{\sqrt{(1+zz)}}.$$

2. Sit $n = 3$ et $m = 1$, ideoque $n - m = 2$.

§. 47. Hic igitur erit $v = \sqrt[3]{(1+3zz)}$; $T = 3z + z^3$; $F = 1 + zz$ et $G = 2z$, hinc formula fpecialis

$$\text{þ} = \int \frac{\partial z [f(1+zz) + 2gz]}{(3z+z^3)\sqrt[3]{(1+3zz)}},$$

cuius

cuius valor est

$$\mathfrak{z} = -\tfrac{1}{2}(f+g)\int \tfrac{\partial q}{1-q^3} - \tfrac{1}{2}(f-g)\int \tfrac{\partial p}{1-p^3},$$

existente

$$p = \frac{1+z}{\sqrt[3]{(1+3zz)}} \text{ et } q = \frac{1-z}{\sqrt[3]{(1+3zz)}}.$$

3. Sit $n = 3$ et $m = 2$, ideoque $n - m = 1$.

§. 48. Hic igitur erit $v = \sqrt[3]{(1+3zz)}$; $T = 3z + z^3$; $F = 1$ et $G = z$, hincque formula specialis

$$\mathfrak{z} = \int \frac{\partial z(1-zz)(f+gz)}{(3z+z^3)\sqrt[3]{(1+3zz)^2}},$$

cuius integrale est

$$\mathfrak{z} = -\tfrac{1}{2}(f+g)\int \tfrac{q\,\partial q}{1-q^3} - \tfrac{1}{2}(f-g)\int \tfrac{p\,\partial p}{1-p^3},$$

existente

$$p = \frac{1+z}{\sqrt[3]{(1+3zz)}} \text{ et } q = \frac{1-z}{\sqrt[3]{(1+3zz)}}.$$

4. Sit $n = 4$ et $m = 1$, ideoque $n - m = 3$.

§. 49. Hic igitur erit $v = \sqrt[4]{(1+6zz+z^4)}$; $T = 4z + 4z^3$; $F = 1 + 3zz$ et $G = 3z + z^3$; hincque formula specialis

$$\mathfrak{z} = \int \frac{\partial z[f(1+3zz) + g(3z+z^3)]}{(4z+4z^3)\sqrt[4]{(1+6zz+z^4)}},$$

cuius ergo valor erit

$$\mathfrak{z} = -\tfrac{1}{2}(f+g)\int \tfrac{\partial q}{1-q^4} - \tfrac{1}{2}(f-g)\int \tfrac{\partial p}{1-p^4}$$

exi-

existente

$$p = \frac{1+z}{\sqrt[4]{(1+6zz+z^4)}} \text{ et } q = \frac{1-z}{\sqrt[4]{(1+6zz+z^4)}}.$$

5. Sit $n = 4$ et $m = 2$, ideoque $n - m = 2$.

§. 50. Hic erit $v = \sqrt[4]{(1+6zz+z^4)}$; $T = 4z + 4z^3$; $F = 1 + zz$ et $G = 2z$, hincque formula specialis

$$\mathfrak{p} = \int \frac{\partial z(1-zz)[f(1+zz)+2gz]}{(4z+4z^3)\sqrt[4]{(1+6zz+z^4)}},$$

cuius valor est

$$\mathfrak{p} = -\tfrac{1}{2}(f+g)\int \tfrac{q\,\partial q}{1-q^4} - \tfrac{1}{2}(f-g)\int \tfrac{p\,\partial q}{1-p^4};$$

existente

$$p = \frac{1+z}{\sqrt[4]{(1+6zz+z^4)}} \text{ et } q = \frac{1-z}{\sqrt[4]{(1+6zz+z^4)}}.$$

6. Sit $n = 4$ et $m = 3$, ideoque $n - m = 1$.

§. 51. Hic igitur erit $v = \sqrt[4]{(1+6zz+z^4)}$; $T = 4z + 4z^3$; $F = 1$ et $G = z$, hincque formula specialis

$$\mathfrak{p} = \int \frac{\partial z(1-zz)^2(f+gz)}{(4z+4z^3)\sqrt[4]{(1+6zz+z^4)^3}},$$

cuius ergo valor erit

$$\mathfrak{p} = -\tfrac{1}{2}(f+g)\int \tfrac{qq\,\partial q}{1-q^4} - \tfrac{1}{2}(f-g)\int \tfrac{pp\,\partial p}{1-p^4},$$

existente

$$p = \frac{1+z}{\sqrt[4]{(1+6zz+z^4)}} \text{ et } q = \frac{1-z}{\sqrt[4]{(1+6zz+z^4)}}.$$

7. Sit $n = 5$ et $m = 1$, ideoque $n - m = 4$.

§. 52. Hic igitur eft $v = \sqrt[5]{(1 + 10zz + 5z^4)}$; $T = 5z + 10z^3 + z^5$; $F = 1 + 6zz + z^4$; $G = 4z + 4z^3$; hincque formula fpecialis

$$\mathfrak{p} = \int \frac{\partial z\,[f(1 + 6zz + z^4) + 4g(z + z^3)]}{(5z + 10z^3 + z^5)\sqrt[5]{(1 + 10zz + 5z^4)}},$$

cuius valor eft

$$\mathfrak{p} = -\tfrac{1}{2}(f + g)\int \tfrac{\partial q}{1 - q^5} - \tfrac{1}{2}(f - g)\int \tfrac{\partial p}{1 - p^5};$$

exiftente

$$p = \frac{1 + z}{\sqrt[5]{(1 + 10zz + 5z^4)}} \text{ et } q = \frac{1 - z}{\sqrt[5]{(1 + 10zz + 5z^4)}}.$$

8. Sit $n = 5$ et $m = 2$, ideoque $n - m = 3$.

§. 53. Hic igitur erit $v = \sqrt[5]{(1 + 10zz + 5z^4)}$; $T = 5z + 10z^3 + z^5$; $F = 1 + 3zz$ et $G = 3z + z^3$, hincque formula fpecialis

$$\mathfrak{p} = \int \frac{\partial z\,(1 - zz)\,[f(1 + 3zz) + g(3z + z^3)]}{(5z + 10z^3 + z^5)\sqrt[5]{(1 + 10zz + 5z^4)^2}},$$

cuius valor eft

$$\mathfrak{p} = -\tfrac{1}{2}(f + g)\int \tfrac{q\,\partial q}{1 - q^5} - \tfrac{1}{2}(f - g)\int \tfrac{p\,\partial p}{1 - p^5};$$

exiftente

$$p = \frac{1 + z}{\sqrt[5]{(1 + 10zz + 5z^4)}} \text{ et } q = \frac{1 - z}{\sqrt[5]{(1 + 10zz + 5z^4)}}.$$

9.

9. Sit $n = 5$ et $m = 3$, ideoque $n - m = 2$.

§. 54. Hic igitur erit $v = \sqrt[5]{(1 + 10zz + 5z^4)}$; $T = 5z + 10z^3 + z^5$; $F = 1 + zz$ et $G = 2z$; hincque formula specialis

$$\mathfrak{p} = \int \frac{\partial z (1 - zz)^2 [f(1 + zz) + 2gz]}{(5z + 10z^3 + z^5) \sqrt[5]{(1 + 10zz + 5z^4)^3}},$$

cuius valor est

$$\mathfrak{p} = -\tfrac{1}{2}(f + g) \int \tfrac{qq\partial q}{1 - q^5} - \tfrac{1}{2}(f - g) \int \tfrac{pp\partial p}{1 - p^5},$$

existente

$$p = \frac{1 + z}{\sqrt[5]{(1 + 10zz + 5z^4)}} \text{ et } q = \frac{1 - z}{\sqrt[5]{(1 + 10zz + 5z^4)}}.$$

10. Sit $n = 5$ et $m = 4$, ideoque $n - m = 1$.

§. 55. Hic igitur est $v = \sqrt[5]{(1 + 10zz + 5z^4)}$; $T = 5z + 10z^3 + z^5$; $F = 1$ et $G = z$; hincque formula specialis

$$\mathfrak{p} = \int \frac{\partial z (1 - zz)^3 (f + gz)}{(5z + 10z^3 + z^5) \sqrt[5]{(1 + 10zz + 5z^4)^4}},$$

cuius ergo valor erit

$$\mathfrak{p} = -\tfrac{1}{2}(f + g) \int \tfrac{q^3\partial q}{1 - q^5} - \tfrac{1}{2}(f + g) \int \tfrac{p^3\partial p}{1 - p^5},$$

existente ut ante

$$p = \frac{1 + z}{\sqrt[5]{(1 + 10zz + 5z^4)}} \text{ et } q = \frac{1 - z}{\sqrt[5]{(1 + 10zz + 5z^4)}}.$$

11. Sit $n = 6$ et $m = 1$, ideoque $n - m = 5$.

§. 65. Hic igitur erit $v = \sqrt[6]{(1 + 15zz + 15z^4 + z^6)}$;

 $T =$

$T = 6z + 20z^3 + 6z^5$; $F = 1 + 10zz + 5z^4$ et $G = 5z + 10z^3 + z^5$; hincque formula ſpecialis

$$\mathfrak{p} = \int \frac{\partial z\,[f(1 + 10zz + 5z^4) + g(5z + 10z^3 + z^5)]}{(6z + 20z^3 + 6z^5)\sqrt[6]{(1 + 15zz + 15z^4 + z^6)}},$$

cuius ergo valor erit

$$\mathfrak{p} = -\tfrac{1}{2}(f + g)\int \frac{\partial q}{1 - q^6} - \tfrac{1}{2}(f - g)\int \frac{\partial p}{1 - p^6},$$

exiſtente

$$p = \frac{1 + z}{\sqrt[6]{(1 + 15zz + 15z^4 + z^6)}} \text{ et } q = \frac{1 - z}{\sqrt[6]{(1 + 15zz + 15z^4 + z^6)}}.$$

12. Sit $n = 6$ et $m = 2$, ideoque $n - m = 4$.

§. 57. Hic erit $v = \sqrt[6]{(1 + 15zz + 15z^4 + z^6)}$; $T = 6z + 20z^3 + 6z^5$; $F = 1 + 6zz + z^4$ et $G = 4z + 4z^3$; hincque formula ſpecialis

$$\mathfrak{p} = \int \frac{\partial z(1 - zz)[f(1 + 6zz + z^4) + 4g(z + z^3)]}{(6z + 20z^3 + 6z^5)\sqrt[3]{(1 + 15zz + 15z^4 + z^6)}},$$

cuius valor erit

$$\mathfrak{p} = -\tfrac{1}{2}(f + g)\int \frac{q\,\partial q}{1 - q^6} - \tfrac{1}{2}(f - g)\int \frac{p\,\partial p}{1 - p^6},$$

exiſtente ut ante

$$p = \frac{1 + z}{\sqrt[6]{(1 + 15zz + 15z^4 + z^6)}} \text{ et}$$

$$q = \frac{1 - z}{\sqrt[6]{(1 + 15zz + 15z^4 + z^6)}}.$$

13.

13. Sit $n = 6$ et $m = 3$, ideoque $n - m = 3$.

§. 58. Hic erit $v = \sqrt[6]{(1 + 15zz + 15z^4 + z^6)}$; $T = 6z + 20z^3 + 6z^5$; $F = 1 + 3zz$; $G = 3z + z^3$; hincque formula specialis

$$\Pi = \int \frac{\partial z(1 - zz)^2 [f(1 + 3zz) + g(3z + z^3)]}{(6z + 20z^3 + 6z^5)\sqrt{(1 + 15zz + 15z^4 + z^6)}},$$

cuius valor est

$$\Pi = -\tfrac{1}{2}(f + g)\int \tfrac{qq\partial q}{1 + q^6} - \tfrac{1}{2}(f - g)\int \tfrac{pp\partial p}{1 - p^6},$$

existente

$$p = \frac{1 + z}{\sqrt[6]{(1 + 15zz + 15z^4 + z^6)}} \text{ et}$$

$$q = \frac{1 - z}{\sqrt[6]{(1 + 15zz + 15z^4 + z^6)}}.$$

14. Sit $n = 6$ et $m = 4$, ideoque $n - m = 2$.

§. 59. Hic erit $v = \sqrt[4]{(1 + 15zz + 15z^4 + z^6)}$; $T = 6z^3 + 20z^3 + 6z^5$; $F = 1 + zz$ et $G = 2z$; hincque formula specialis

$$\Pi = \int \frac{\partial z(1 - zz)^3 [f(1 + zz) + 2gz]}{(6z + 20z^3 + 6z^5)\sqrt[4]{(1 + 15zz + 15z^4 + z^5)^2}},$$

cuius valor est

$$\Pi = -\tfrac{1}{2}(f + g)\int \tfrac{q^3\partial q}{1 + q^6} - \tfrac{1}{2}(f - g)\int \tfrac{p^3\partial p}{1 - p^6},$$

existente

$$p = \frac{1+z}{\sqrt[6]{(1+15zz+15z^4+z^6)}} \text{ et}$$

$$q = \frac{1-z}{\sqrt[6]{(1+15zz+15z^4+z^6)}}.$$

15. Sit $n=6$ et $m=5$, ideoque $n-m=1$.

§. 60. Hic erit $v=\sqrt[6]{(1+15zz+15z^4+z^6)}$; $T=6z+20z^3+6z^5$; $F=1$ et $G=z$; hincque formula specialis

$$\mathfrak{p} = \int \frac{\partial z(1-zz)^4(f+gz)}{(6z+20z^3+6z^5)\sqrt[6]{(1+15zz+15z^4+z^6)^5}},$$

cuius valor est

$$\mathfrak{p} = -\tfrac{1}{2}(f+g)\int\tfrac{q^4\partial q}{1-q^6} - \tfrac{1}{2}(f-g)\int\tfrac{p^4\partial p}{1-p^6},$$

existente

$$p = \frac{1+z}{\sqrt[6]{(1+15zz+15z^4+z^6)}} \text{ et}$$

$$q = \frac{1-z}{\sqrt[6]{(1+15zz+15z^4+z^6)}}.$$

Observatio in has formulas.

§. 61. Hic igitur etiam casus notatu dignus occurrit, si $n=2m$; quo fit

$$v = \sqrt[2m]{[\tfrac{1}{2}(1+z)^{2m}+\tfrac{1}{2}(1-z)^{2m}]}, \text{ ideoque}$$

$$v^m = \sqrt{[\tfrac{1}{2}(1+z)^{2m}+\tfrac{1}{2}(1-z)^{2m}]}.$$

At

At ſi loco $\frac{1}{2}(f+g)$ et $\frac{1}{2}(f-g)$ reſtituantur litterae A et B, erit formula noſtra:

$$\mathfrak{b} = \int \frac{\partial z\,(1 - zz)^{m-1}\,[A\,(1+z)^m + B\,(1-z)^m]}{[\frac{1}{2}(1+z)^{2m} - \frac{1}{2}(1-z)^{2m}]\,\sqrt{[\frac{1}{2}(1+z)^{2m} + \frac{1}{2}(1-z)^{2m}]}},$$

cuius integrale, ſumtis $p = \frac{1+z}{v}$ et $q = \frac{1-z}{v}$, erit

$$\mathfrak{b} = -A\int \frac{q^{m-1}\,\partial q}{1 - q^{2m}} - B\int \frac{p^{m-1}\,\partial p}{1 - p^{2m}}.$$

Quodſi ergo faciamus ut ante $q^m = t$ et $p^m = u$, integrale quaeſitum erit

$$\mathfrak{b} = -\frac{A}{m}\int \frac{\partial t}{1 - tt} - \frac{B}{m}\int \frac{\partial u}{1 - uu},\ \text{ſive}$$

$$\mathfrak{b} = -\frac{A}{2m}\, l\, \frac{1 + q^m}{1 - q^m} - \frac{B}{2m}\, l\, \frac{1 + p^m}{1 - p^m},\ \text{ſive}$$

$$\mathfrak{b} = -\frac{A}{2m}\, l\, \frac{v^m + (1-z)^m}{v^m - (1-z)^m} - \frac{B}{2m}\, l\, \frac{v^m + (1+z)^m}{v^m - (1+z)^m}.$$

Ordo tertius.

Formularum ſpecialium ex forma

$$\mathfrak{4} = \int \frac{\partial z\,(f\,F + g\,G)}{v^{n-m}\,(1 - zz)}.$$

§. 62. Hoc igitur caſu eſt

$$F = \tfrac{1}{2}(1+z)^{n-m} + \tfrac{1}{2}(1-z)^{n-m} \text{ et}$$
$$G = \tfrac{1}{2}(1+z)^{n-m} - \tfrac{1}{2}(1-z)^{n-m},$$

tum vero

$$v = \sqrt[n]{[\tfrac{1}{2}(1+z)^n - \tfrac{1}{2}(1-z)^n]}$$

unde

unde pofitis $p = \frac{1+z}{v}$ et $q = \frac{1-z}{v}$ integrale inventum eft

$$\mathfrak{Z} = \tfrac{1}{2}(f+g)\int \frac{p^{n-m-1}\partial p}{2-p^n} - \tfrac{1}{2}(f-g)\int \frac{q^{n-m-1}\partial q}{2+q^n}.$$

Hinc ergo formulae fpeciales prodibunt fequentes:

1. Sit $n = 2$ et $m = 1$, ideoque $n - m = 1$.

§. 63. Hic igitur erit $v = \sqrt{2z}$; $F = 1$ et $G = z$; hinc formula fpecialis $\mathfrak{Z} = \int \frac{\partial z(f+gz)}{(1-zz)\sqrt{2z}}$, cuius integrale eft

$$\mathfrak{Z} = \tfrac{1}{2}(f+g)\int \frac{\partial p}{2-pp} - \tfrac{1}{2}(f-g)\int \frac{\partial q}{2+qq},$$

exiftente

$$p = \frac{1+z}{\sqrt{2z}} \text{ et } q = \frac{1-z}{\sqrt{2z}}.$$

2. Sit $n = 3$ et $m = 1$, ideoque $n - m = 2$.

§. 64. Hic igitur erit $v = \sqrt[3]{(3z+z^3)}$; $F = 1 + zz$ et $G = 2z$; hinc formula fpecialis

$$\mathfrak{Z} = \int \frac{\partial z\,[f(1+zz) + 2gz]}{(1-zz)\sqrt[3]{(3z+z^3)^2}},$$

cuius integrale eft

$$\mathfrak{Z} = \tfrac{1}{2}(f+g)\int \frac{p\,\partial p}{2-p^3} - \tfrac{1}{2}(f-g)\int \frac{q\,\partial q}{2+q^3},$$

exiftente

$$p = \frac{1+z}{\sqrt[3]{(3z+z^3)}} \text{ et } q = \frac{1-z}{\sqrt[3]{(3z-z^3)}}.$$

3. Sit $n = 3$ et $m = 2$, ideoque $n - m = 1$.

§. 65. Hic igitur erit $v = \sqrt[3]{(3z+z^3)}$; $F = 1$ et

$G = z$

$G = z$; hinc formula ſpecialis

$$\mathfrak{Z} = \int \frac{\partial z\,(f + g z)}{(1 - zz)\sqrt[3]{(3z + z^3)}},$$

cuius integrale eſt

$$\mathfrak{Z} = \tfrac{1}{2}(f + g)\int \frac{\partial p}{2 - p^3} - \tfrac{1}{2}(f - g)\int \frac{\partial q}{2 + q^3},$$

exiſtente

$$p = \frac{1 + z}{\sqrt[3]{(3z + z^3)}} \text{ et } q = \frac{1 - z}{\sqrt[3]{(3z + z^3)}}.$$

4. Sit $n = 4$ et $m = 1$, ideoque $n - m = 3$.

§. 66. Hic igitur erit $v = \sqrt[4]{(4z + 4z^3)}$; $F = 1 + 3zz$ et $G = 3z + z^3$; hinc formula ſpecialis

$$\mathfrak{Z} = \int \frac{\partial z\,[f(1 + 3zz) + g(3z + z^3)]}{(1 - zz)\sqrt[4]{(4z + 4z^3)^3}},$$

cuius integrale eſt

$$\mathfrak{Z} = \tfrac{1}{2}(f + g)\int \frac{pp\,\partial p}{1 - p^4} - \tfrac{1}{2}(f - g)\int \frac{qq\,\partial q}{1 - q^4},$$

exiſtente

$$p = \frac{1 + z}{\sqrt[4]{(4z + 4z^3)}} \text{ et } q = \frac{1 - z}{\sqrt[4]{(4z + 4z^3)}}.$$

5. Sit $n = 4$ et $m = 2$, ideoque $n - m = 2$.

§. 67. Hic igitur erit $v = \sqrt[4]{(4z + 4z^3)}$; $F = 1 + zz$ et $G = 2z$; hinc formula ſpecialis

$\mathfrak{Z} =$

$$\mathfrak{Y} = \int \frac{\partial z\,[f(1 + zz) + 2gz]}{(1 - zz)\sqrt{(4z + 4z^3)}},$$

cuius integrale eſt

$$\mathfrak{Y} = \tfrac{1}{2}(f + g)\int \tfrac{p\,\partial q}{1 - p^4} - \tfrac{1}{2}(f - g)\int \tfrac{q\,\partial q}{2 - q^4};$$

exiſtente

$$p = \frac{1 + z}{\sqrt[4]{(4z + 4z^3)}} \text{ et } q = \frac{1 - z}{\sqrt[4]{(4z + 4z^3)}}.$$

6. Sit $n = 4$ et $m = 3$, ideoque $n - m = 1$.

§. 68. Hic igitur erit $v = \sqrt[4]{(4z + 4z^3)}$; $F = 1$ et $G = z$, ideoque formula ſpecialis

$$\mathfrak{Y} = \frac{\partial z\,(f + gz)}{(1 - zz)\sqrt[4]{(4z + 4z^3)}},$$

cuius integrale

$$\mathfrak{Y} = \tfrac{1}{2}(f + g)\int \tfrac{\partial p}{2 - p^4} - \tfrac{1}{2}(f - g)\int \tfrac{\partial q}{2 + q^4},$$

exiſtente

$$p = \frac{1 + z}{\sqrt[4]{(4z + 4z^3)}} \text{ et } q = \frac{1 - z}{\sqrt[4]{(4z + 4z^3)}}.$$

7. Sit $n = 5$ et $m = 1$, ideoque $n - m = 4$.

§. 69. Hic igitur erit $v = \sqrt[5]{(5z + 10z^3 + z^5)}$; $F = 1 + 6zz + z^4$ et $G = 4z + 4z^3$; hinc formula ſpecialis

$$\mathfrak{Y} = \int \frac{\partial z\,[f(1 + 6zz + z^4)]}{(1 - zz)\sqrt[5]{(5z + 10z^3 + z^5)^4}},$$

cuius

cuius integrale eſt

$$\mathfrak{z} = \frac{1}{2}(f+g)\int\frac{p^3\,\partial p}{2-p^5} - \frac{1}{2}(f-g)\int\frac{q^3\,\partial q}{2+q^5},$$

exiſtente

$$p = \frac{1+z}{\sqrt[5]{(5z+10z^3+z^5)}} \text{ et } q = \frac{1-z}{\sqrt[5]{(5z+10z^3+z^5)}}.$$

8. Sit $n = 5$ et $m = 2$, ideoque $n - m = 3$.

§. 70. Hic igitur erit $v = \sqrt[5]{(5z+10z^3+z^5)}$; $F = 1 + 3zz$ et $G = 3z + z^3$; hinc formula ſpecialis

$$\mathfrak{z} = \int\frac{\partial z\,[f(1+3zz)+g(3z+z^3)]}{(1-zz)\sqrt[5]{(5z+10z^3+z^5)^3}},$$

cuius integrale eſt

$$\mathfrak{z} = \frac{1}{2}(f+g)\int\frac{pp\,\partial p}{2-p^5} - \frac{1}{2}(f-g)\int\frac{qq\,\partial q}{2+q^5},$$

exiſtente

$$p = \frac{1+z}{\sqrt[5]{(5z+10z^3+z^5)}} \text{ et } q = \frac{1-z}{\sqrt[5]{(5z+10z^3+z^5)}}.$$

9. Sit $n = 5$ et $m = 3$, ideoque $n - m = 2$.

§. 71. Hic igitur erit $v = \sqrt[5]{(5z+10z^3+z^5)}$; $F = 1 + zz$ et $G = 2z$; hinc formula ſpecialis

$$\mathfrak{z} = \frac{\partial z\,[f(1+zz)+2gz]}{(1+zz)\sqrt[5]{(5z+10z^3+z^5)^2}},$$

cuius integrale eſt

$$\mathfrak{z} = \frac{1}{2}(f+g)\int\frac{p\,\partial p}{2-p^5} - \frac{1}{2}(f-g)\int\frac{q\,\partial q}{2+q^5},$$

 exi-

existente

$$p = \frac{1+z}{\sqrt[5]{(5z+10z^3+z^5)}} \text{ et } q = \frac{1-z}{\sqrt[5]{(5z+10z^3+z^5)}}.$$

10. Sit $n=5$ et $m=4$, ideoque $n-m=1$.

§. 72. Hic igitur erit $v=\sqrt[5]{(5z+10z^3+z^5)}$; $F=1$ et $G=z$; hinc formula specialis

$$\mathfrak{Z} = \int \frac{\partial z(f+gz)}{(1-zz)\sqrt[5]{(5z+10z^3+z^5)}},$$

cuius integrale est

$$\mathfrak{Z} = \tfrac{1}{2}(f+g)\int\frac{\partial p}{2-p^5} - \tfrac{1}{2}(f-g)\int\frac{\partial q}{2+q^5};$$

existente

$$p = \frac{1+z}{\sqrt[5]{(5z+10z^3+z^5)}} \text{ et } q = \frac{1-z}{\sqrt[5]{(5z+10z^3+z^5)}}.$$

11. Sit $n=6$ et $m=1$, ideoque $n-m=5$.

§. 73. Hic igitur erit $v=\sqrt[6]{(6z+20z^3+6z^5)}$; $F=1+10zz+5z^4$ et $G=5z+10z^3+z^5$; hinc formula specialis

$$\mathfrak{Z} = \int \frac{\partial z[f(1+10zz+5z^4)+g(5z+10z^3+z^5)]}{(1-zz)\sqrt[6]{(6z+20z^3+6z^5)^5}},$$

cuius integrale

$$\mathfrak{Z} = -\tfrac{1}{2}(f+g)\int\frac{p^4\partial p}{2-p^6} - \tfrac{1}{2}(f-g)\int\frac{q^4\partial q}{2+q^6};$$

existente

$$p = \frac{1+z}{\sqrt[6]{(6z+20z^3+6z^5)}} \text{ et } q = \frac{1-z}{\sqrt[6]{(6z+20z^3+6z^5)}}.$$

12.

12. Sit $n = 6$ et $m = 2$, ideoque $n - m = 4$.

§. 74. Hic erit $v = \sqrt[6]{(6z + 20z^3 + 6z^5)}$; $F = 1 + 6zz + z^4$ et $G = 4z + 4z^3$; hinc formula

$$\mathfrak{Z} = \int \frac{\partial z[f(1 + 6zz + z^4) + 4gz(1 + zz)]}{(1 - zz)\sqrt[3]{(6z + 20z^3 + 6z^5)^2}},$$

cuius integrale

$$\mathfrak{Z} = \tfrac{1}{2}(f + g)\int \tfrac{p^3 \partial p}{2 - p^6} - \tfrac{1}{2}(f - g)\int \tfrac{q^3 \partial q}{2 + q^6},$$

existente

$$p = \tfrac{1 + z}{v} \text{ et } q = \tfrac{1 - z}{v}.$$

13. Sit $n = 6$ et $m = 3$, ideoque $n - m = 3$.

§. 75. Hic erit $v = \sqrt[6]{(\ z + 20z^3 + 6z^5)}$, $F = 1 + 3zz$ et $G = 3z + z^3$; hinc formula

$$\mathfrak{Z} = \int \frac{\partial z[f(1 + 3zz) + g(3z + z^3)]}{(1 - zz)\sqrt{(\ z + 20z^3 + 6z^5)}},$$

cuius integrale est

$$\mathfrak{Z} = \tfrac{1}{2}(f + g)\int \tfrac{pp\partial p}{2 - p^6} - \tfrac{1}{2}(f - g)\int \tfrac{qq\partial q}{2 + q^6},$$

existente

$$p = \tfrac{1 + z}{v} \text{ et } q = \tfrac{1 - z}{v}.$$

14. Sit $n = 6$ et $m = 4$, ideoque $n - m = 4$.

§. 76. Hic erit $v = \sqrt[6]{(6z + 20z^3 + 6z^5)}$; $F = 1 + zz$ et $G = 2z$; hinc formula

$$\mathfrak{Z} = \int \frac{\partial z[f(1 + zz) + 2gz]}{(1 - zz)\sqrt[6]{(6z + 10z^3 + 6z^5)}},$$

cuius integrale

$$\mathfrak{z} = \tfrac{1}{2}(f+g)\int \frac{p\,\partial p}{2-p^6} - \tfrac{1}{2}(f-g)\int \frac{q\,\partial q}{2+q^6}$$

existente $p = \frac{1+z}{v}$ et $q = \frac{1-z}{v}$.

15. Sit $n = 6$ et $m = 5$, ideoque $n - m = 1$.

§. 77. Hic erit $v = \sqrt[6]{(6z + 10z^3 + 6z^5)}$; $F = 1$; $G = z$, hinc formula specialis

$$\mathfrak{z} = \int \frac{\partial z\,(f + gz)}{(1 - zz)\sqrt[6]{(6z + 10z^3 + 6z^5)}},$$

cuius integrale est

$$\mathfrak{z} = \tfrac{1}{2}(f+g)\int \frac{\partial p}{2-p^6} - \tfrac{1}{2}(f-g)\int \frac{\partial p}{2+p^6},$$

existente

$$p = \frac{1+z}{v} \text{ et } q = \frac{1-z}{v}.$$

Observatio in has formulas.

§. 78. Consideremus hic iterum casum quo $n = 2m$, et quia

$$v = \sqrt[2m]{[\tfrac{1}{2}(1+z)^{2m} - \tfrac{1}{2}(1-z)^{2m}]}, \text{ erit}$$

$$v^{n-m} = v^m = \sqrt{[\tfrac{1}{2}(1+z)^{2m} - \tfrac{1}{2}(1-z)^{2m}]};$$

$$F = \tfrac{1}{2}(1+z)^m + \tfrac{1}{2}(1-z)^m \text{ et } G = \tfrac{1}{2}(1+z)^m - \tfrac{1}{2}(1-z)^m,$$

quo ergo casu erit

$$\mathfrak{z} = \int \frac{\partial z\,[A(1+z)^m + B(1-z)^m]}{(1-zz)\sqrt{[\tfrac{1}{2}(1+z)^{2m} - \tfrac{1}{2}(1-z)^{2m}]}},$$

tum vero posito $p = \frac{1+z}{v}$ et $q = \frac{1-z}{v}$, integrale erit

$\mathfrak{z} =$

$$\mathfrak{Z} = A\int \frac{p^{m-1}\partial p}{2-p^{2m}} - B\int \frac{q^{m-1}\partial q}{2+q^{2m}},$$

quae formula, posito $p^m = u$ et $q^m = t$, transit in hac formam:

$$\mathfrak{Z} = \frac{A}{m}\int \frac{\partial u}{2-uu} - \frac{B}{m}\int \frac{\partial t}{2+tt},$$

sive integrando erit

$$\mathfrak{Z} = \frac{A}{2m\sqrt{2}}\, l\, \frac{\sqrt{2}+p^m}{\sqrt{2}-p^m} - \frac{B}{m\sqrt{2}}\,\text{Ar. tang.}\,\frac{q^m}{\sqrt{2}},$$

Ordo quartus formularum specialium ex forma

$$\mathfrak{Z} = \frac{\partial z(fF+gG)}{v^{n-m}(1-zz)}.$$

Hic est ut ante

$$F = \tfrac{1}{2}(1+z)^{n-m} + \tfrac{1}{2}(1-z)^{n-m} \text{ et}$$
$$G = \tfrac{1}{2}(1+z)^{n-m} - \tfrac{1}{2}(1-z)^{n-m},$$

at vero

$$v = \sqrt[n]{[\tfrac{1}{2}(1+z)^n + \tfrac{1}{2}(1-z)^n]},$$

tum vero posito $p = \frac{1+z}{v}$ et $q = \frac{1-z}{v}$, integrale inventum est

$$\mathfrak{Z} = \tfrac{1}{2}(f+g)\int \frac{p^{n-m-1}\partial p}{2-p^n} - \tfrac{1}{2}(f-g)\int \frac{q^{n-m-1}\partial q}{2-q^n},$$

formulae ergo speciales sequuntur.

1. Sit $n = 2$ et $m = 1$, ideoque $n - m = 1$.

§. 79. Hic igitur erit $v = \sqrt{(1+zz)}$; $F = 1$ et $G = z$; hinc formula specialis $\mathfrak{Z} = \int \frac{\partial z(f+gz)}{(1-zz)\sqrt{(1+zz)}}$, cuius in-

integrale eſt

$$\mathfrak{Z} = \tfrac{1}{2}(f+g)\int \frac{\partial p}{2-pp} - \tfrac{1}{2}(f-g)\int \frac{\partial q}{2-qq},$$

exiſtente

$$p = \frac{1+z}{\sqrt{(1+zz)}} \text{ et } q = \frac{1-z}{\sqrt{(1+zz)}}.$$

2. Sit $n = 3$ et $m = 1$, ideoque $n - m = 2$.

§. 80. Hic igitur erit $v = \sqrt[3]{(1+3zz)}$; $F = 1+z$ et $G = 2z$; hinc formula ſpecialis

$$\mathfrak{Z} = \int \frac{\partial z\,[f(1+zz)+2gz]}{(1-zz)\sqrt[3]{(1+3zz)^2}},$$

cuius integrale eſt

$$\mathfrak{Z} = \tfrac{1}{2}(f+g)\int \frac{p\,\partial p}{2-p^3} - \tfrac{1}{2}(f-g)\int \frac{q\,\partial q}{2-q^3},$$

exiſtente

$$p = \frac{1+z}{\sqrt[3]{(1+3zz)}} \text{ et } q = \frac{1-z}{\sqrt[3]{(1+3zz)}}.$$

3. Sit $n = 3$ et $m = 2$, ideoque $n - m = 1$.

§. 81. Hic igitur eſt $v = \sqrt[3]{(1+3zz)}$; $F = 1$ $G = z$; hinc formula ſpecialis

$$\mathfrak{Z} = \int \frac{\partial z\,(f+gz)}{(1-zz)\sqrt[3]{(1+3zz)}},$$

cuius integrale eſt

$$\mathfrak{Z} = \tfrac{1}{2}(f+g)\int \frac{\partial p}{2-p^3} - \tfrac{1}{2}(f-g)\int \frac{\partial q}{2-q^3},$$

exiſtente

$$p = \frac{1 + z}{\sqrt[3]{(1 + 3zz)}} \text{ et } q = \frac{1 - z}{\sqrt[3]{(1 + 3zz)}}.$$

4. Sit $n = 4$ et $m = 1$, ideoque $n - m = 3$.

§. 82. Hic igitur erit $v = \sqrt[4]{(1 + 6zz + z^4)}$; $F = 1 + 3zz$ et $G = 3z + z^3$; hinc formula

$$\mathfrak{U} = \int \frac{\partial z [f(1 + 3zz) + gz(3 + zz)]}{(1 - zz)\sqrt[4]{(1 + 6zz + z^4)^3}},$$

cuius integrale est

$$\mathfrak{U} = \tfrac{1}{2}(f+g)\int \tfrac{p}{2} \frac{p \partial p}{2 - p^4} - \tfrac{1}{2}(f - g)\int \frac{qq\partial q}{2 - q^4},$$

existente

$$p = \frac{1 + z}{\sqrt[4]{(1 + 6zz + z^4)}} \text{ et } q = \frac{1 - z}{\sqrt[4]{(1 + 6zz + z^4)}}.$$

5°. Sit $n = 4$ et $m = 2$, ideoque $n - m = 2$.

§. 83. Hic igitur erit $v = \sqrt[4]{(1 + 6zz + z^4)}$; $F = 1 + zz$ et $G = 2z$; hinc formula specialis:

$$\mathfrak{U} = \int \frac{\partial z [f(1 + zz) + 2gz]}{(1 - zz) + \sqrt{(1 + 6zz + z^4)}}.$$

ideoque eius integrale:

$$\mathfrak{U} = \tfrac{1}{2}(f + g)\int \frac{p\partial p}{2 - p^4} - \tfrac{1}{2}(f - g)\int \frac{q\partial q}{2 - q^4},$$

existente $p = \frac{1+z}{v}$ et $q = \frac{1-z}{v}$.

6°.

6°. Sit $n = 4$ et $m = 3$, ideoque $n - m = 1$.

§. 84. Hic igitur erit $v = \sqrt[4]{(1 + 6zz + 4z^4)}$; $F = 1$ et $G = z$; hinc formula:

$$y = \int \frac{\partial z (f + gz)}{(1 - zz)\sqrt[4]{(1 + 6zz + z^4)}},$$

cuius integrale

$$y = \tfrac{1}{2}(f + g)\int \tfrac{\partial p}{2 - p^4} - \tfrac{1}{2}(f - g)\int \tfrac{\partial q}{2 - q^4},$$

existente $p = \frac{1 + z}{v}$ et $q = \frac{1 - z}{v}$.

7°. Sit $n = 5$ et $m = 1$, ideoque $n - m = 4$.

§. 85. Hic igitur erit $v = \sqrt[5]{(1 + 10zz + 5z^4)}$;
$F = 1 + 6zz + z^4$ et $G = 4z(1 + zz)$;

hinc formula

$$y = \int \frac{\partial z [f(1 + 6zz + z^4) + 4gz(1 + zz)]}{(1 - zz)\sqrt[5]{(1 + 10zz + 5z^4)^4}},$$

ideoque eius integrale

$$y = \tfrac{1}{2}(f + g)\int \tfrac{p^3 \partial p}{2 - p^5} - \tfrac{1}{2}(f - g)\int \tfrac{q^3 \partial q}{2 - q^5},$$

existente $p = \frac{1 + z}{v}$ et $q = \frac{1 - z}{v}$.

8. Sit $n = 5$ et $m = 2$, ideoque $n - m = 3$.

§. 86. Hic igitur erit $v = \sqrt[5]{(1 + 10zz + 5z^4)}$;
$F = 1 + 3zz$ et $G = z(3 + zz)$;

hinc formula

$$\mathfrak{z} = \int \frac{\partial z\,[f(1 + 3zz) + gz(3 + zz)]}{(1 - zz)\sqrt[4]{(1 + 10zz + 5z^4)^3}},$$

cuius integrale

$$\mathfrak{z} = \tfrac{1}{2}(f + g)\int \tfrac{pp\partial p}{2 - p^5} - \tfrac{1}{2}(f - g)\int \tfrac{qq\partial q}{2 - q^5},$$

existente

$$p = \tfrac{1+z}{v} \text{ et } q = \tfrac{1-z}{v}.$$

9. Sit $n = 5$ et $m = 3$, ideoque $n - m = 2$.

§. 87. Hic erit $v = \sqrt[5]{(1 + 10zz + 5z^4)}$; $F = 1 + zz$ et $G = 2z$; hinc formula

$$\mathfrak{z} = \int \frac{\partial z\,[f(1 + zz) + 2gz]}{(1 - zz)\sqrt[5]{(1 + 10zz + 5z^4)^2}},$$

cuius integrale

$$\mathfrak{z} = \tfrac{1}{2}(f + g)\int \tfrac{p\partial p}{2 - p^5} - \tfrac{1}{2}(f - g)\int \tfrac{q\partial q}{2 - q^5},$$

existente

$$p = \tfrac{1+z}{v} \text{ et } q = \tfrac{1-z}{v}.$$

10. Sit $n = 5$ et $m = 4$, ideoque $n - m = 1$.

§. 88. Hic erit $v = \sqrt[5]{(1 + 10zz + 5z^4)}$; $F = 1$ et $G = z$; hinc formula

$$\mathfrak{z} = \int \frac{\partial z\,(f + gz)}{(1 - zz)\sqrt[5]{(1 + 10zz + 5z^4)}},$$

cuius integrale

$\mathfrak{z} =$

$$\mathfrak{Z} = \tfrac{1}{2}(f+g)\int \frac{\partial p}{2-p^5}\,\tfrac{1}{2} - (f-g)\int \frac{\partial q}{2-q^5},$$

exiſtente

$$p = \frac{1+z}{v} \text{ et } q = \frac{1-z}{v},$$

11. Sit $n = 6$ et $m = 1$, ideoque $n - m = 5$.

§. 89. Hic erit $v = \sqrt[6]{(1 + 15zz + 15z^4 + z^6)}$;

$F = 1 + 10zz + 5z^4$ et $G = 5z + 10z^3 + z^5$,

hinc formula

$$\mathfrak{Z} = \int \frac{\partial z\,[f(1+10zz+5z^4) + g(5z+10z^3+z^5)]}{(1-zz)\sqrt[6]{(1+15zz+15z^4+z^6)^5}},$$

cuius integrale

$$\mathfrak{Z} = \tfrac{1}{2}(f+g)\int \frac{p^4\,\partial p}{2-p^6} - \tfrac{1}{2}(f-g)\int \frac{q^4\,\partial q}{2-q^6},$$

exiſtente

$$p = \frac{1+z}{v} \text{ et } q = \frac{1-z}{v}.$$

12. Sit $n = 6$ et $m = 2$, ideoque $n - m = 4$.

§. 90. Hic erit $v = \sqrt[6]{(1 + 15zz + 15z^4 + z^6)}$,

$F = 1 + 6zz + z^4$ et $G = 4z(1+zz)$;

hinc formula

$$\mathfrak{Z} = \int \frac{\partial z\,[f(1+6zz+z^4) + 4gz(1+zz)]}{(1-zz)\sqrt[3]{(1+15zz+15z^4+z^6)^2}},$$

cuius integrale

$$\mathfrak{Z} = \tfrac{1}{2}(f+g)\int \frac{p^3\,\partial p}{2-p^6} - \tfrac{1}{2}(f-g)\int \frac{q^3\,\partial q}{2-q^6},$$

exiſtente

$$p = \frac{1+z}{v} \text{ et } q = \frac{1-z}{v}.$$

13. Sit $n = 6$ et $m = 3$, ideoque $n - m = 3$.

§. 91. Hic erit $v = \sqrt[6]{(1 + 15zz + 15z^4 + z^6)}$;
$F = 1 + 3zz$ et $G = 3z + z^3$;

hinc formula

$$\text{♃} = \int \frac{\partial z\,[f(1+3zz) + gz(3+zz)]}{(1-zz)\sqrt[6]{(1+15zz+15z^4+z^6)}},$$

cuius integrale

$$\text{♃} = \tfrac{1}{2}(f+g)\int \frac{pp\,\partial p}{2-p^6} - \tfrac{1}{2}(f-g)\int \frac{qq\,\partial q}{2-q^6},$$

exiſtente

$$p = \frac{1+z}{v} \text{ et } q = \frac{1-z}{v}.$$

14. Sit $n = 6$ et $m = 4$, ideoque $n - m = 2$.

§. 92. Hic erit $v = \sqrt[6]{(1 + 15zz + 15z^4 + z^6)}$;
$F = 1 + zz$ et $G = 2z$; hinc formula

$$\text{♃} = \int \frac{\partial z\,[f(1+zz) + 2gz]}{(1-zz)\sqrt[3]{(1+15zz+15z^4+z^6)}},$$

cuius integrale

$$\text{♃} = \tfrac{1}{2}(f+g)\int \frac{p\,\partial p}{2-p^6} - \tfrac{1}{2}(f-g)\int \frac{q\,\partial q}{2-q^6},$$

exiſtente

$$p = \frac{1+z}{v} \text{ et } q = \frac{1-z}{v}.$$