1798

De formulis differentialibus secundi gradus quae integrationem admittunt

Leonhard Euler

DE
FORMVLIS DIFFERENTIALIBVS
SECVNDI GRADVS,
QVAE INTEGRATIONEM ADMITTVNT.

Auctore
L. EVLERO.

Conventui exhib. die 24 April. 1777.

§. 1.

Inter tales formulas differentiales secundi gradus, quae integrationem admittunt, imprimis notatu digna est haec formula: $\frac{(x \partial x + y \partial y)(\partial y \partial \partial x - \partial x \partial \partial y)}{(\partial x^2 + \partial y^2)^{\frac{3}{2}}}$, quae, si x et y designent coordinatas orthogonales lineae curvae, oritur, si elementum $x \partial x + y \partial y$ dividatur per radium osculi huius curvae; quandoquidem constat istius formulae integrale esse $\frac{y \partial x - x \partial y}{\sqrt{(\partial x^2 + \partial y^2)}}$, quemadmodum calculum instituenti, dum huius formulae differentiale quaeritur, facile patebit. Cum igitur haec integratio neutiquam sit obvia et plures ambages postulet, hoc argumentum hic accuratius pertractare constitui.

ſtitui, unde intelligi poterit, quemadmodum plures aliae huiusmodi formulae inveniri queant, quae pariter integrationem admittant.

§. 2. Quod quo facilius fieri poſſit, differentialia ſecundi gradus ex calculo eliminemus, quod commodiſſime fiet, ponendo $\partial y = p \partial x$, ita ut loco differentialium ſecundorum in calculum introducatur iſta nova quantitas $p = \frac{\partial y}{\partial x}$, quippe quae rationem differentialium primorum continet. Tum igitur erit

$$x \partial x + y \partial y = \partial x (x + p y) \text{ atque}$$

$$\partial x^2 + \partial y^2 = \partial x^2 (1 + p p),$$

ideoque denominator formulae propoſitae fit

$$(\partial x^2 + \partial y^2)^{\frac{3}{2}} = \partial x^3 (1 + p p)^{\frac{3}{2}};$$

denique pro altero numeratoris factore habetur

$$\partial y \partial \partial x = p \partial x \partial \partial x, \text{ et ob}$$

$$\partial \partial y = p \partial \partial x + \partial p \partial x, \text{ erit}$$

$$\partial x \partial \partial y = p \partial x \partial \partial x + \partial p \partial x^2$$

ſicque alter ille factor erit

$$\partial x \partial \partial x - \partial x \partial \partial y = - \partial p \partial x^2,$$

quibus ſubſtitutis formula propoſita hanc induet formam: $- \frac{\partial p (x + p y)}{(1 + p p)^{\frac{3}{2}}}$, cuius ergo integrale erit

$$\frac{y \partial x - x \partial y}{\sqrt{(\partial x^2 + \partial y^2)}} = \frac{y - p x}{\sqrt{(1 + p p)}}$$

quippe cuius differentiale ſuperiorem praebet formulam.

§. 3.

§. 3. Cum igitur facta hac substitutione in formulam differentio-differentialem unicum ingrediatur differentiale ∂p, inquisitu-sic in genere contemplabor hanc formulam: $V \partial p$, inquisiturus cuiusmodi valores isti litterae V tribui debeant, ut formulae $V \partial p$ integrale exhiberi queat; ubi quidem evidens est, hanc quantitatem V certam esse oportere functionem trium variabilium x, y et p, quae ergo quomodo comparata esse debeat, ut integratio succedat, hic accuratius investigare constitui.

§. 4. Ac primo quidem ex iis, quae olim circa integrabilitatem formularum differentialium altiorum ordinum tradidi, criteria haud difficulter exhiberi poterunt, unde dignosci queat, utrum talis formula $V \partial p$ integrationem admittat nec ne? Tum temporis autem contemplatus sum talem formam $\int Z \partial x$, ubi positis $\partial y = p \partial x$; $\partial p = q \partial x$; $\partial q = r \partial x$; $\partial r = s \partial x$; etc. littera Z denotabat functionem ex litteris x, y, p, q, r, s, etc. utcunque compositam, atque ostendi, quoties haec formula $\int Z \partial x$ fuerit integrabilis, tum semper fore

$$0 = \left(\frac{\partial Z}{\partial y}\right) - \frac{1}{\partial x} \partial . \left(\frac{\partial Z}{\partial p}\right) + \frac{1}{\partial x^2} . \partial \partial . \left(\frac{\partial Z}{\partial q}\right) - \frac{1}{\partial x^3} \partial^3 . \left(\frac{\partial Z}{\partial r}\right) + \frac{1}{\partial x^4} \partial^4 . \left(\frac{\partial Z}{\partial s}\right) \text{ etc.}$$

Sin autem ista quantitas non sponte nihilo evadat aequalis, tum ista aequatio eam relationem inter x et y exprimit, pro qua formula integralis $\int Z \partial x$ maximum minimumve valorem nanciscatur.

§. 5. Vt igitur formulam $\int V \partial p$, quam hic consideramus, ad istam formam: $\int Z \partial x$ reducamus, statuamus $\partial p = q \partial x$, ut formula nostra evadat $V q \partial x$, ideoque $Z = V q$; ubi notetur, quantitatem V tantum ternas litteras x, y et p com-

complecti, quo observato erit $(\frac{\partial Z}{\partial y}) = (\frac{q\partial V}{\partial y})$, deinde $(\frac{\partial Z}{\partial p}) =$ $(\frac{q\partial V}{\partial p})$ et $(\frac{\partial Z}{\partial q}) = V$, sicque criterium integrabilitatem ind cans erit

$$0 = (\frac{q\partial V}{\partial y}) - \frac{1}{\partial x}\partial.(\frac{q\partial V}{\partial p}) + \frac{1}{\partial x^2}\partial\partial.V$$

quam aequationem etiam ita referre licet:

$$0 = q(\frac{\partial V}{\partial y}) - \frac{1}{\partial x}\partial.[q(\frac{\partial V}{\partial p}) - \frac{1}{\partial x}\partial V],$$

tum vero etiam, ob $q\partial x = \partial p$, hac ratione ea repraese tari potest:

$$0 = \partial p(\frac{\partial V}{\partial y}) - \frac{1}{\partial x}\partial.[\partial p(\frac{\partial V}{\partial p}) - \partial V].$$

§. 6. Cum igitur in genere per huiusmodi charact res iam satis usu receptos sit

$$\partial V = \partial x(\frac{\partial V}{\partial x}) + \partial y(\frac{\partial V}{\partial y}) + \partial p(\frac{\partial V}{\partial p})$$

hoc valore substituto criterium desideratum hac exprimetu ratione:

$$0 = \partial p(\frac{\partial V}{\partial y}) + \frac{1}{\partial x}\partial.[\partial x(\frac{\partial V}{\partial x}) + \partial y(\frac{\partial V}{\partial y})]$$

quae ergo aequatio continet criterium desideratum; ita ut quoties ista formula revera nihilo evadit aequalis, tum sem per certi esse queamus, istam formulam propositam $V\partial$ esse integrabilem.

§. 7. Quoniam V per hypothesin est functio invo vens has tres variabiles x, y et p, fit differentiationem more solito instituendo

$$\partial V = M\partial x + N\partial y + P\partial p$$

atque criterium continebitur in hac aequatione:

$$0 = N\partial p + \partial.(M + Np)$$

quae

quae porro evolvitur in hanc:

$$0 = 2 N \partial p + \partial M + p \partial N.$$

Cuius vis quo clarius perſpiciatur, applicemus iſtud criterium ad formulam initio propoſitam $+ \frac{\partial p (x + p y)}{(1 + p p)^{\frac{3}{2}}}$, ubi cum ſit $V = \frac{x + p y}{(1 + p p)^{\frac{3}{2}}}$, ſumta ſola x variabili, reperitur $M = \frac{1}{(1 + p p)^{\frac{3}{2}}}$, ſumta autem ſola y variabili, fiet $N = \frac{p}{(1 + p p)^{\frac{3}{2}}}$; hinc ergo erit

$$\partial M = - \frac{3 p \partial p}{(1 + p p)^{\frac{5}{2}}} \text{ et } \partial N = \frac{(1 - 2 p p) \partial p}{(1 + p p)^{\frac{5}{2}}}$$

quibus valoribus ſubſtitutis, quia eſt

$$1.\quad 2 N \partial p = \frac{2 p \partial p}{(1 + p p)^{\frac{3}{2}}} = \frac{2 p \partial p (1 + p p)}{(1 + p p)^{\frac{5}{2}}},$$

$$2.\quad \partial M = - \frac{3 p \partial p}{(1 + p p)^{\frac{5}{2}}},$$

$$3.\quad p \partial N = \frac{p \partial p (1 - 2 p p)}{(1 + p p)^{\frac{5}{2}}}$$

harum formularum ſumma manifeſto ad nihilum redigitur. Ex quo intelligitur hanc formulam revera eſſe integrabilem, etiamſi integrale non conſtaret.

§. 8.

§. 8. Quoniam autem hic nobis potius est propositum in valores idoneos pro littera V sumendos inquirere, quibus formula differentialis $V \partial p$ integrationem admittit, criterium inventum nullum usum praestare potest; quam ob rem investigationem nostram a casibus simplicissimis exordiamur, quibus formula nobis proposita integrationem admittit, inter quos sine dubio omnium simplicissimus est, quando V denotat quantitatem constantem. Sit igitur $V = 1$, eritque $\int \partial p = p$. Hinc autem porro sequitur, si differentiale ∂p in functionem quamcunque ipsius integralis p, quae sit $\Delta : p$, ducatur, tum semper hanc formulam $\partial p \Delta : p$ fore integrabilem, quod quidem per se est perspicuum. Hic enim sub voce integrabilitatis non tantum intelligimus quicquid algebraice exhiberi poterit, sed in genere, quicquid per quantitates utcunque transcendentes assignari potest.

§. 9. Secundus casus simplicissimus, quo formula $V \partial p$ integrabilis evadit, est quando $V = x$, ita ut formula differentialis sit $= x \partial p$. Quoniam enim per reductionem notissimam fit $\int x \partial p = p x - \int p \partial x$, ob $p \partial x = \partial y$ erit hoc integrale $\int x \partial p = p x - y$. Hinc igitur si $\Delta : (p x - y)$ denotet functionem quamcunque formulae $p x - y$ semper quoque integrationem admittet haec formula differentialis multo latius patens: $x \partial p \Delta : (p x - y)$, quippe quae, posito $p x - y = V$, ob $\partial V = x \partial p$ induit hanc formam: $V \Delta : V$.

§. 10. Praeterea vero datur etiam tertius casus simplicissimus, quo formula nostra $V \partial p$ fit integrabilis, qui oritur ponendo $V = \frac{y}{p p}$. Per eandem enim reductionem, qua est $\int t \partial u = t u - \int u \partial t$, sumendo $t = y$ et $\partial u = \frac{\partial p}{p p}$, unde

fit

fit $\partial t = \partial y = p\,\partial x$ et $u = \frac{-1}{p}$, fiet

$$\int \frac{y\,\partial p}{pp} = \frac{-y}{p} + \int \partial x = x - \frac{y}{p}.$$

Si igitur porro $\Delta : (x - \frac{y}{p})$ denotet functionem quamcunque formulae $x - \frac{y}{p}$, etiam semper integrabilis erit haec formula differentialis multo generalior: $\frac{y\,\partial p}{pp}\,\Delta : (x - \frac{y}{p})$. Quodsi enim ponatur $x - \frac{y}{p} = V$, ob $\partial V = \frac{y\,\partial p}{pp}$, haec forma evadit $= \partial V\,\Delta : V$, quae manifesto semper est integrabilis.

§. 11. His casibus principalibus constitutis inquiramus quoque in casus magis compositos, quibus formula generalis $V\,\partial p$ itidem fiet integrabilis, quem in finem sequentia problemata pertractemus.

Problema 1.

Quaerantur duae functiones ipsius p, quae sint P et Q, ita comparatae, ut ista formula differentialis: $\partial p\,(Px + Qy)$ evadat integrabilis.

Solutio.

§. 12. Quoniam haec formula duas involvit partes, eas per allatam reductionem seorsim evolvamus, ac primo quidem erit $\int Px\,\partial p = x\int P\,\partial p - \int \partial x \int P\,\partial p$; ubi quidem integrale $\int P\,\partial p$ ut quantitas cognita spectari potest, propterea quod P denotat functionem ipsius p. Simili modo pro altera parte erit $\int Qy\,\partial p = y\int Q\,\partial p - \int \partial y \int Q\,\partial p$, ubi postrema membra utrinque continent formulas per se non integrabiles, unde necesse est, ut binis formulis in unam summam collectis haec duo membra postrema se mutuo tollant. Fiat igitur $\int \partial x \int P\,\partial p + \int \partial y \int Q\,\partial p = 0$, ideoque differentiando, ob $\partial y = p\,dx$, erit $\int P \partial p + p \int Q \partial p = 0$. Nunc denuo differentiemus atque obtinebimus

$$P + \int Q \partial p + Q p = 0,$$

quae iterum differentiata praebet

$$\partial P + p \partial Q + 2 Q \partial p = 0,$$

in qua aequatione relatio quaesita inter binas functiones P et Q continetur.

§. 13. Quodsi haec ultima aequatio ducatur in p, prodibit $p \partial P + \partial . Q p p = 0$; unde patet, si altera harum duarum functionum P et Q fuerit cognita, hinc alteram determinari posse. Si enim verbi gratia data fuerit functio P, ob $\int p \partial P + Q p p = C$, erit $Q = \frac{C - \int p \partial P}{p p}$. Sin autem altera functio Q fuerit data, ex priore formula erit $\partial P = - p \partial Q - 2 Q \partial p$, ideoque integrando

$$P = C - \int (p \partial Q + 2 Q \partial p)$$

sive etiam

$$P = C - Q p - \int Q \partial p.$$

§. 14. Quando vero istae duae functiones P et Q hoc modo rite fuerint determinatae, tum integrale formulae differentialis propositae $\partial p (P x + Q y)$ ita exprimetur, ut sit $= x \int P \partial p + y \int Q \partial p$. Atque iam notavimus, alterutram functionum P et Q pro lubitu assumi posse. Quin etiam certa quaedam relatio inter P et Q statui potest. Veluti si velimus ut sit $P = n Q p$, hoc valore in aequatione differentiali substituto fiet

$$(n + 2) Q \partial p + (n + 1) p \partial Q = 0,$$

unde

unde porro deducitur

$$\frac{(n+2)\partial p}{p} + \frac{(n+1)\partial Q}{Q} = 0,$$

cuius integrale est

$$(n+2)\, l\, p + (n+1)\, l\, Q = l\, C,$$

hincque porro $p^{n+2} Q^{n+1} = C$, ex quo deducitur

$$Q = \frac{C}{p^{\frac{n+2}{n+1}}}, \text{ consequenter } P = \frac{n\, C}{p^{\frac{1}{n+1}}}.$$

§. 15. Quoniam integrale inventum est $x \int P \partial p + y \int Q \partial p$, hae duae formulae integrales duas constantes accipere sunt censendae, ita ut integrale verum ita prodeat expressum: $x \int P \partial p + y \int Q \partial p + \alpha x + \beta y$, ubi constantes α et β quovis casu ita determinari oportet, ut sumtis differentialibus elementum ∂x ex calculo excedat, id quod fit si fuerit

$$\partial x \int P \partial p + p\, \partial x \int Q \partial p + \alpha\, \partial x + \beta\, p\, \partial x = 0,$$

unde prodit, uti iam invenimus,

$$P \partial p + \partial p \int Q \partial p + Q p \partial p + \beta\, \partial p = 0,$$

quae per ∂p divisa et denuo differentiata praebet

$$\partial P + 2\, Q \partial p + p\, \partial Q = 0,$$

quae aequatio exprimit relationem requisitam inter P et Q.

Alia Solutio eiusdem problematis.

§. 16. Cum sit $x \partial p$ differentiale formulae $p x - y$, erit per reductionem

$$\int P x \partial p = P (p x - y) - \int (p x - y) \partial P;$$

 dein-

deinde cum sit $\frac{y\,\partial p}{pp}$ differentiale formulae $x - \frac{y}{p}$, erit per reductionem:

$$\int Q y\,\partial p = \int Q p p \,.\, \frac{y\,\partial p}{pp} = Q p p \left(x - \frac{y}{p}\right)$$
$$- \int \left(x - \frac{y}{p}\right) \partial \,.\, Q p p.$$

His igitur coniungendis integrale formulae propositae erit

$$P(px - y) + Qpp\left(x - \frac{y}{p}\right) - \int (px - y)\,\partial P - \int \left(x - \frac{y}{p}\right) \partial \,.\, Qpp,$$

unde evidens est partes postremas integrales nihilo aequales fieri debere. Hinc sumtis differentialibus statui debet

$$(px - y)\,\partial P + \left(x - \frac{y}{p}\right) \partial \,.\, Q p p = 0,$$

quae aequatio per $px - y$ divisa dat $\partial P + \frac{1}{p}\,\partial \,.\, Q p p = 0$, sive $\partial P + p\,\partial Q + 2\,Q\,\partial p = \quad$, quae est eadem aequatio inter P et Q, quam prior solutio suppeditavit.

§. 17. Quoniam supra vidimus hanc formulam $\frac{(x + p y)\,\partial p}{(1 + p p)^{\frac{3}{2}}}$ integrationem admittere, facta applicatione hic erit $P = \frac{1}{(1 + p p)^{\frac{3}{2}}}$ et $Q = \frac{p}{(1 + p p)^{\frac{3}{2}}}$. Spectemus nunc quantitatem P tanquam cognitam et videamus an pro Q eundem valorem reperiamus. Cum igitur $\partial P = \frac{-3\,p\,\partial p}{(1 + p p)^{\frac{5}{2}}}$, aequatio inventa evadet

$$\frac{-3\,p\,\partial p}{(1 + p p)^{\frac{5}{2}}} + p\,\partial Q + 2\,Q\,\partial p = 0,$$

quae ducta in p praebet

∂.

$$\partial \cdot Q p p = \frac{3 p p \partial p}{(1 + p p)^{\frac{5}{2}}}, \text{ ideoque } Q p p = \int \frac{3 p p \partial p}{(1 + p p)^{\frac{5}{2}}}.$$

Levi autem attentione adhibita patebit esse

$$\int \frac{3 p p \partial p}{(1 + p p)^{\frac{5}{2}}} = \frac{p^3}{(1 + p p)^{\frac{3}{2}}}, \text{ sicque erit}$$

$$Q p p = \frac{p^3}{(1 + p p)^{\frac{3}{2}}}, \text{ ideoque}$$

$$Q = \frac{p}{(1 + p p)^{\frac{3}{2}}} + \frac{C}{p p}.$$

§. 18. Hinc igitur videmus pro valore $P = \frac{1}{(1 + p p)^{\frac{3}{2}}}$ non solum esse $Q = \frac{p}{(1 + p p)^{\frac{3}{2}}}$, sed generalius sumi posse $Q = \frac{p}{(1 + p p)^{\frac{3}{2}}} + \frac{C}{p p}$, ita ut iam haec formula differentiationem admittat. Cum igitur in genere integrale inventum sit

$$P (p x - y) + Q p p (x - \tfrac{y}{p})$$

his valoribus substitutis integrale erit

$$\frac{p x - y}{(1 + p p)^{\frac{3}{2}}} + \frac{p p (p x - y)}{(1 + p p)^{\frac{3}{2}}} + \frac{C (p x - y)}{p},$$

quod reducitur ad hanc formam: $\frac{p x - y}{\sqrt{(1 + p p)}} + \frac{C (p x - y)}{p}$.

Pro-

Problema 2.

Si M et N fuerint functiones quaecunque datae ipsius p, invenire eiusdem functionem Π, ut ista formula differentialis: $(Mx + Ny)\Pi\partial p$, integrationem admittat.

Solutio.

§. 19. Si hoc problema cum praecedente comparemus, facile patet functiones illas litteris P et Q designatas esse $M\Pi$ et $N\Pi$, ita ut sit $P = M\Pi$ et $Q = N\Pi$. Quare cum integrabilitas postulet hanc aequationem:

$$\partial P + 2Q\partial p + p\partial Q = 0,$$

facta hac substitutione nanciscemur sequentem aequationem:

$$M\partial\Pi + \Pi\partial M + 2N\Pi\partial p + Np\partial\Pi + \Pi p\partial N = 0,$$

ex qua, quia M et N sunt functiones cognitae ipsius p, elicimus $\frac{\partial\Pi}{\Pi} = \frac{-\partial M - 2N\partial p - p\partial N}{M + Np}$, unde colligimus integrando

$$l\Pi = -l(M + Np) - \frac{N\partial p}{M + Np}.$$

Ponamus igitur brevitatis gratia $\int\frac{N\partial p}{M + Np} = lK$, quandoquidem etiam haec formula K tanquam data spectari potest, sicque erit $l\Pi = -l(M + Np) - lK + lA$. Quocirca pro solutione nostri problematis habebimus:

$$\Pi = \frac{A}{K(M + Np)}, \text{ existente } lK = \int\frac{N\partial p}{M + Np}.$$

§. 20. Inuento autem hoc valore functionis quaesitae $\Pi = \frac{A}{K(M + Np)}$, quoniam supra integrale in genere prodiit

$$P(px - y) + Qpp(x - \tfrac{y}{p}) = (px - y)(P + Qp),$$

substitutis pro P et Q debitis valoribus integrale formulae diffe-

differentialis propositae $(Mx+Ny)\Pi\partial p$ erit

$$(px-y)(M\Pi+N\Pi p)=\frac{A(px-y)(M+Np)}{K(M+Np)},$$

quae commode ulterius reducitur ad hanc formam simplicissimam: $\frac{A(px-y)}{K}$, sicque erit $\int\frac{Mx+Ny)\partial p}{K(M+Np)}=\frac{px-y}{K}$, existente $lK=\int\frac{N\partial p}{M+Np}$, sive $K=e^{\int\frac{N\partial p}{M+Np}}$, id quod operae pretium erit exemplis illustrare.

Exemplum 1.

§. 21. Sit $M=1$ et $N=1$, ita ut proponatur haec formula differentialis: $(x+y)\Pi\partial p$. Hic igitur erit $lK=\int\frac{\partial p}{1+p}=l(1+p)$, ideoque $K=1+p$, ita ut iam functio quaesita fit $\Pi=\frac{A}{(1+p)^2}$, hincque formula differentialis integrationem admittens erit $\frac{(x+y)\partial p}{(1+p)^2}$, quippe cuius integrale est $\frac{px-y}{1+p}$. Quodsi enim haec formula differentietur, prodit $\frac{x\partial p}{1+p}-\frac{(px-y)\partial p}{(1+p)^2}$, quae reducitur ad hanc formam: $\frac{(x+y)\partial p}{(1+p)^2}$.

Exemplum 2.

§. 22. Sint ambae functiones M et N constantes, scilicet $M=m$ et $N=n$, ut proposita sit haec formula differentialis: $(mx+ny)\Pi\partial p$. Hic igitur erit primo

$$lK=\int\frac{n\partial p}{m+np}=l(m+np),$$

ita vt sit $K=m+np$. Hinc igitur functio quaesita Π erit $=\frac{A}{(m+np)^2}$, ita ut iam integrabilis sit haec formula: $\frac{(mx+ny)\partial p}{(m+np)^2}$, quippe cuius integrale erit $\frac{px-y}{m+np}$.

Exem-

Exemplum 3.

§. 23. Sumamus nunc $M = 1$ et $N = p$, ut formula integrabilis reddenda fit $(x + py)\Pi \partial p$. Hic igitur erit primo $l K = \int \frac{p \partial p}{1 + pp} = l \sqrt{(1 + pp)}$, ideoque $K = \sqrt{(1 + pp)}$; unde fit functio quaesita $\Pi = \frac{A}{(1 + pp)^{\frac{3}{2}}}$, hincque formula differentialis integrationem admittens erit $\frac{(x + py)\partial p}{(1 + pp)^{\frac{3}{2}}}$, quae est ea ipsa, quam initio sumus contemplati; cuius ergo integrale est $\frac{px + y}{\sqrt{(1 + pp)}}$.

Exemplum 4.

§. 24. Sit nunc $M = m$ et $N = np$, ut formula integrabilis reddenda fit $(mx + npy)\Pi \partial p$. Hic igitur erit

$$l K = \int \frac{np \partial p}{m + npp} = l \sqrt{(m + npp)},$$

ideoque $K = \sqrt{(m + npp)}$, unde functio quaesita erit $\Pi = \frac{A}{(m + npp)^{\frac{3}{2}}}$, ita ut iam integrabilis fit haec formula $\frac{(mx + npy)\partial p}{(m + npp)^{\frac{3}{2}}}$, cuius ergo integrale erit $= \frac{px - y}{\sqrt{(m + npp)}}$.

Exemplum 5.

§. 25. Sit nunc $M = m$ et $N = np^{\lambda - 1}$, ita ut formula integrabilis reddenda fit $(mx + np^{\lambda - 1}y)\Pi \partial p$. Hic igitur erit

$$l K = \int \frac{np^{\lambda - 1} \partial p}{m + np^{\lambda}} = \frac{1}{\lambda} l (m + np^{\lambda}),$$

ideo-

ideoque $K = (m + n p^{\lambda})^{\frac{1}{\lambda}}$, unde functio quaesita Π erit $= \frac{A}{(m + n p^{\lambda})^{\frac{\lambda+1}{1}}}$, ita ut iam integrabilis sit haec formula: $\frac{(m x + n p^{\lambda-1} y) \partial p}{(m + n p^{\lambda})^{\frac{\lambda+1}{\lambda}}}$, cuius ergo integrale erit $= \frac{p x - y}{(m + n p)^{\frac{1}{\lambda}}}$.

Exemplum 6.

§. 26. Sit nunc $M = m p$ et $N = n$, ita ut formula integrabilis reddenda sit $(m p x + n y) \Pi \partial p$. Hic igitur erit $l K = \int \frac{n \partial p}{m p + n p} = \frac{n}{m + n} l p$, ideoque $K = p^{\frac{n}{m+n}}$, ergo $\Pi = \frac{A}{(m + n) p^{\frac{m+2n}{m+n}}}$, sicque formula integrabilis nunc est $\frac{(m p x + n y) \partial p}{(m + n) p^{\frac{m+2n}{m+n}}}$, cuius ergo integrale erit $= \frac{p x - y}{p^{\frac{n}{m+n}}}$.

§. 27. Hic casus imprimis notabilis occurrit, quo $m = -n$, sive $m + n = 0$; tum enim formula maxime incongrua resultat, ob exponentem ipsius p infinitum. Hic autem casus per se est obvius. Si enim quaeratur Π, ut ista formula $(p x - y) \Pi \partial p$ evadat integrabilis, quoniam est $\partial . (p x - y) = x \partial p$, evidens est nullam dari functionem ipsius p tantum, qua huic conditioni satisfieri queat. Statim autem ac non fuerit $m + n = 0$, solutio semper est possibilis.

Exemplum VII.

§. 28. Sumatur nunc $M = m p p$ et $N = n$, ut integrabilis reddi debeat haec formula: $(m p p x + n y) \Pi \partial p$. Hic ergo erit

$$l K = \int \frac{n \partial p}{n p + m p p} = l p - l (m p + n),$$

consequenter $K = \frac{p}{m p + n}$, hincque $\Pi = \frac{A}{p p}$, sicque formula integrabilis iam erit $\frac{(m p p x + n y) \partial p}{p p}$; eius enim integrale erit $\frac{(p x - y)(m p + n)}{p}$.

Exemplum VIII.

§. 29. Sit nunc $M = p^{\lambda + 1}$ et $N = 1$, ita ut formula integrabilis reddenda sit $(p^{\lambda + 1} x + y) \Pi \partial p$. Hic ergo erit

$$l K = \int \frac{\partial p}{p^{\lambda + 1} + p} = l p - \frac{1}{\lambda} l (p^{\lambda + 1} + 1),$$

ergo $K = \frac{p}{(p^{\lambda + 1} + 1)^{\frac{1}{\lambda}}}$, hincque $\Pi = \frac{A (p^{\lambda} + 1)^{\frac{1 - \lambda}{\lambda}}}{p p}$, unde formula integrabilis erit $\frac{(p^{\lambda} + 1)^{\frac{1 - \lambda}{\lambda}} (p^{\lambda + 1} x + y) \partial p}{p p}$, quippe cuius integrale est $= \frac{(p x - y)(p^{\lambda} + 1)^{\frac{1}{\lambda}}}{p}$.

Exemplum 9.

§. 30. Sit denique $M = m p^{\lambda + 1}$ et $N = n$, ut formula integrabilis reddenda sit $(m p^{\lambda + 1} x + n y) \Pi \partial p$. Hic ergo erit

$l K$

$$lK = \int \frac{n\,\partial p}{m\,p^{\lambda+1} + n\,p} = l\,p - \tfrac{1}{\lambda}\,l\,(m\,p^{\lambda} + n),$$

ideoque $K = \frac{p}{(m\,p^{\lambda} + n)^{\frac{1}{\lambda}}}$, hincque $\Pi = \frac{A\,(m\,p^{\lambda} + n)^{\frac{1-\lambda}{\lambda}}}{p\,p}$, unde formula integrabilis erit

$$= \frac{(m\,p^{\lambda+1}\,x + n\,y)\,(m\,p^{\lambda} + n)^{\frac{1-\lambda}{\lambda}}\,\partial p}{p\,p},$$

quippe cuius integrale erit $\frac{(p\,x - y)\,(m\,p^{\lambda} + n)^{\frac{1}{\lambda}}}{p}$.

Problema 3.

Invenire duas functiones ipsius p, quae sint P et Q, ut ista formula differentialis: $(p\,x - y)^{n-1}\,(P\,x + Q\,y)\,\partial p$ fiat integrabilis.

Solutio.

§. 31. Cum sit $x\,\partial p = \partial.(p\,x - y)$, erit

$$\int P\,x\,\partial p\,(p\,x - y)^{n-1} = \tfrac{1}{n}\,P\,(p\,x - y)^{n} - \tfrac{1}{n}\int (p\,x - y)^{n}\,\partial P.$$

Deinde cum sit $\frac{y\,\partial p}{p\,p} = \partial.(x - \frac{y}{p})$, loco $Q\,y\,\partial p$ scribamus $Q\,p\,p.\frac{y\,\partial p}{p\,p}$, tum vero loco $p\,x - y$ scribamus $p\,(x - \frac{y}{p})$, ideoque loco $(p\,x - y)^{n-1}$ scribendum erit $p^{n-1}\,(x - \frac{p}{y})^{n-1}$. Hinc ergo pro altera parte habebimus

$$Q\,y\,\partial p\,(p\,x - y)^{n-1} = Q\,p\,p.\frac{y\,\partial p}{p\,p}.\,p^{n-1}\,(x - \tfrac{y}{p})^{n-1}$$
$$= Q\,p^{n+1}.\frac{y\,\partial p}{p\,p}.\,(x - \tfrac{y}{p})^{n-1},$$

hincque per reductionem erit

$$\int Q y \,\partial p\,(p x - y)^{n-1} = \tfrac{1}{n} Q p^{n+1} (x - \tfrac{y}{p})^n$$
$$- \tfrac{1}{n} \int (x - \tfrac{y}{p})^n \,\partial\,.\, Q p^{n+1}.$$

§. 32. Nunc igitur ut formula propofita integrationem admittat, neceffe eft, ut binae partes pofteriores fummatoriae ad nihilum redigantur, unde oritur ifta aequatio:

$$(p x - y)^n \partial P + (x - \tfrac{y}{p})^n \partial\,.\, Q p^{n+1} = 0,$$

hincque dividendo per $(p x - y)^n$ erit

$$p^n \partial P + \partial\,.\, Q p^{n+1} = 0,$$

cuius evolutio praebet

$$\partial P + p \,\partial Q + (n+1) Q \,\partial p = 0,$$

qua aequatione relatio requifita inter P et Q continetur; unde ergo data altera fimul altera determinari poteft; tum autem ipfum integrale formulae propofitae erit

$$\tfrac{1}{n} P (p x - y)^n + \tfrac{1}{n} Q p^{n+1} (x - \tfrac{y}{p})^n, \text{ five}$$
$$\tfrac{1}{n} (p x - y)^n (P + Q p).$$

Problema 4.

Si M et N defignent functiones quascunque datas ipfius p, invenire eiusdem quantitatis functionem Π, ut ifta formula differentialis: $(p x - y)^{n-1} (M x + N y) \Pi \,\partial p$, *fiat integrabilis.*

Solutio.

§. 33. Solutio praecedentis problematis huc transferetur ftatuendo $P = M\Pi$ et $Q = N\Pi$, unde conditio ante inventa ad hanc aequationem perducet:

M ∂ Π

$$M\partial\Pi + \Pi\partial M + Np\partial\Pi + \Pi p\partial N + (n+1)N\Pi\partial p = 0,$$

ex qua reperitur

$$\frac{\partial\Pi}{\Pi} = -\frac{\partial M - p\partial N - (n+1)N\partial p}{M+Np},$$

quae integrata praebet

$$l\Pi = -l(M+Np) - n\int\frac{N\partial p}{M+Np}.$$

§. 34. Ponamus iam, ut fupra fecimus, $\int\frac{N\partial p}{M+Np} = lK$, atque ad numeros procedendo erit $\Pi = \frac{A}{K^n(M+Np)}$, ficque formula noftra integrabilis erit

$$\frac{(px-y)^{n-1}(Mx+Ny)\partial p}{K^n(M+Np)}.$$

Eius enim integrale erit

$$\frac{\Pi}{n}\frac{(px-y)^n(M+Np)}{K^n(M+Np)} = \frac{(px-y)^n}{nK^n},$$

unde fumto $n=1$ manifefto cafus problematis tertii exfurgit.

§. 35. Cafus hic imprimis notatu dignus occurrit, quo $n=0$; tum enim, ob $K^n=1$, formula integrabilis reddita erit $\frac{(Mx+Ny)\partial p}{(M+Np)(px-y)}$. Eius vero integrale hinc videtur fieri infinitum, cuiusmodi valores ad logarithmos revocantur: formula enim $\frac{(px-y)^0}{0}$ aequivalet $l(px-y)$. Interim tamen hoc integrale neutiquam fatisfacit, cuius rei ratio in evanefcentia numeri n latet; reperitur autem haec formula differentialis refolvi in $\frac{x\partial p}{px-y} - \frac{N\partial p}{M+Np}$, unde fi, ut fecimus, ponatur $\int\frac{N\partial p}{M+Np} = lK$, eius integrale

tegrale erit $l(px - y) - lK$, ita ut hoc casu integrale fit $l\frac{px-y}{h}$. Reliquis autem casibus integralia erunt algebraica, cuius sequentia exempla perpendamus.

Exemplum 1.

§. 36. Sit $M = 1$ et $N = 1$, eritque ut ante $lK = \int \frac{\partial p}{1+p} = l(1+p)$, ideoque $K = 1 + p$, hincque $\Pi = \frac{A}{(1+p)^{n+1}}$, unde formula nostra integrabilis iam erit

$$\frac{(px-y)^{n-1}(x+y)\partial p}{(1+p)^{n+1}},$$

cuius integrale est $\frac{(px-y)^n}{n(1+p)^n}$.

Exemplum 2.

§. 37. Ponamus nunc $M = \alpha$ et $N = \beta$, ut formula integrabilis reddenda fit $(px-y)^{n-1}(\alpha x + \beta y)\Pi\partial p$. Hic ergo erit $lK = \int \frac{\beta\partial p}{\alpha + \beta p} = l(\alpha + \beta p)$, ideoque $K = \alpha + \beta p$, hincque $\Pi = \frac{A}{(\alpha + \beta p)^{n+1}}$, unde nostra formula integrabilis reddenda erit $\frac{(px-y)^{n-1}(\alpha x + \beta y)\partial p}{(\alpha + \beta p)^{n+1}}$, quippe cuius integrale est $\frac{(px-y)^n}{n(\alpha + \beta p)^n}$.

Exemplum 3.

§. 38. Sit nunc $M = 1$ et $N = p$, ut formula integrabilis reddenda fit $(px-y)^{n-1}(x + py)\Pi\partial p$. Hic ergo go erit

$$l\,K = \int \frac{p\,\partial p}{1 + pp} = l\,\surd(1 + pp),$$

ideoque $K = \surd(1 + pp)$, hincque $\Pi = \frac{A}{(1 + pp)^{\frac{3}{2}}}$; ſicque formula noſtra integrabilis erit $\frac{(px - y)^{n-1}(x + py)\,\partial p}{(1 + pp)^{\frac{n+2}{2}}}$; eius enim integrale erit $\frac{(px - y)^n}{n(1 + pp)^{\frac{n}{2}}}$.

Exemplum 4.

§. 39. Sit nunc $M = \alpha$ et $N = \beta p$, ut formula integrabilis reddenda ſit $(px - y)^{n-1}(\alpha x + \beta py)\,\Pi\,\partial p$. Hic igitur erit

$$l\,K = \int \frac{\beta p\,\partial p}{\alpha + \beta pp} = \tfrac{1}{2}\,l(\alpha + \beta pp),$$

ideoque $K = \surd(\alpha + \beta pp)$, unde functio quaeſita Π erit $= \frac{A}{(\alpha + \beta pp)^{\frac{n+2}{2}}}$. Hinc formula noſtra integrabilis erit

$$\frac{(px - y)^{n-1}(\alpha x + \beta py)\,\partial p}{(\alpha + \beta pp)^{\frac{n+2}{2}}},$$

quippe cuius integrale erit $\frac{(px - y)^n}{n(\alpha + \beta pp)^{\frac{n}{2}}}$.

Exemplum 5.

§. 40. Sit $M = \alpha$ et $N = \beta p^{\lambda - 1}$, ut formula integrabilis reddenda ſit

$(p\,x$

$(p\,x - y)^{n-1}(\alpha\,x + \beta\,p^{\lambda-1}y)\,\Pi\,\partial p$. Hic ergo erit

$$l\,K = \int \frac{\beta\,p^{\lambda-1}\,\partial p}{\alpha + \beta\,p^{\lambda}} = \tfrac{1}{\lambda}\,l\,(\alpha + \beta\,p^{\lambda}),$$

ideoque $K = (\alpha + \beta\,p^{\lambda})^{\frac{1}{\lambda}}$, unde functio quaesita Π erit $= \frac{A}{(\alpha + \beta\,p^{\lambda})^{\frac{n+\lambda}{\lambda}}}$, sicque formula nostra integrabilis erit

$$\frac{(p\,x - y)^{n-1}(\alpha\,x + \beta\,p^{\lambda-1}y)\,\partial p}{(\alpha + \beta\,p^{\lambda})^{\frac{n+\lambda}{\lambda}}},$$

quippe cuius integrale est $= \frac{(p\,x - y)^{n}}{n\,(\alpha + \beta\,p^{\lambda})^{\frac{n}{\lambda}}}$.

Exemplum 6.

§. 41. Sit nunc $M = \alpha\,p$ et $N = \beta$, ita ut formula integrabilis reddenda sit

$$(p\,x - y)^{n-1}(\alpha\,p\,x + \beta\,y)\,\Pi\,\partial p,$$

Hic igitur erit

$$l\,K = \int \frac{\beta\,\partial p}{\alpha\,p + \beta\,p} = \frac{\beta}{\alpha + \beta}\,l\,p,$$

ideoque $K = p^{\frac{\beta}{\alpha+\beta}}$. Hinc igitur functio proposita Π erit

$$\Pi = \frac{A}{(\alpha + \beta)\,p^{\frac{\alpha + (n+1)\beta}{\alpha+\beta}}},$$

sicque formula integrabilis nunc erit

$$\frac{(p\,x - y)^{n-1}(\alpha\,p\,x + \beta\,y)\,\partial p}{(\alpha + \beta)\,p^{\frac{\alpha + (n+1)\beta}{\alpha+\beta}}},$$

cuius ergo integrale est $= \frac{(p\,x - y)^{n}}{n\,p^{\frac{\beta n}{\alpha+\beta}}}$.

Ex-

Exemplum 7.

§. 42. Sumatur nunc $M = \alpha p p$ et $N = \beta$, ut integrabilis reddi debeat haec formula:

$$(p x - y)^{n-1} (\alpha p p x + \beta y) \Pi \partial p.$$

Hic ergo erit

$$l K = \int \frac{\beta \partial p}{\alpha p p + \beta p} = l p - l (\alpha p + \beta),$$

consequenter $K = \frac{p}{\alpha p + \beta}$, hincque $\Pi = \frac{A (\alpha p + \beta)^{n-1}}{p^{n+1}}$,

sicque formula integrabilis iam erit

$$\frac{(p x - y)^{n-1} (\alpha p p x + \beta y) (\alpha p + \beta)^{n-1} \partial p}{p^{n+1}},$$

quippe cuius integrale est $= \frac{(p x - y)^n (\alpha p + \beta)^n}{n p^n}$.

Exemplum 8.

§. 43. Sit nunc $M = p^{\lambda + 1}$ et $N = 1$, ita ut formula integrabilis reddenda sit

$$(p x - y)^{n-1} (p^{\lambda+1} x + y) \Pi \partial p.$$

Hic ergo erit

$$l K = \int \frac{\partial p}{p^{\lambda+1} + p} = l p - \tfrac{1}{\lambda} l (p^\lambda + 1),$$

consequenter $K = \frac{p}{(p^\lambda + 1)^{\frac{1}{\lambda}}}$, hincque $\Pi = \frac{A (p^\lambda + 1)^{\frac{n-\lambda}{\lambda}}}{p^{n+1}}$,

unde formula integrabilis erit

$$\frac{(p x - y)^{n-1} (p^{\lambda+1} x + y) (p^\lambda + 1)^{\frac{n-\lambda}{\lambda}} \partial p}{p^{n+1}},$$

quip-

quippe cuius integrale erit

$$= \frac{(p x - y)^n (p^\lambda + 1)^{\frac{n}{\lambda}}}{n p^n}.$$

Exemplum 9.

§. 44. Sit denique $M = \alpha p^{\lambda + 1}$ et $N = \beta$, ut formula integrabilis reddenda fit

$$(p x - y)^{n-1} (\alpha p^{\lambda + 1} x + \beta y) \Pi \partial p.$$ Hic ergo erit

$$l K = \int \frac{\beta \partial p}{\alpha p^{\lambda + 1} + \beta p} = l p - \tfrac{1}{\lambda} l (\alpha p^\lambda + \beta),$$

ideoque $K = \frac{p}{(\alpha p^{\lambda} + \beta)^{\frac{1}{\lambda}}}$, hincque $\Pi = \frac{A (\alpha p^\lambda + \beta)^{\frac{n-\lambda}{\lambda}}}{p^{n+1}}$, unde formula integrabilis erit

$$\frac{(p x - y)^{n-1} (\alpha p^{\lambda + 1} x + \beta y) (\alpha p^\lambda + \beta)^{\frac{n-\lambda}{\lambda}} \partial p}{p^{n+1}},$$

cuius ergo integrale erit $\frac{(p x - y)^n (\alpha p^\lambda + \beta)^{\frac{n}{\lambda}}}{n p^n}$.

FOR-