1797

Variae speculationes super area triangulorum sphaericorum

Leonhard Euler

VARIAE SPECVLATIONES SVPER AREA TRIANGVLORVM SPHAERICORVM.

Auctore

L. EVLERO.

Conuentui exhib. die 29 Ianuar. 1778.

§. 1.

Primus, qui aream trianguli ſphaerici definire docuit, erat, teſte Wallifio, Albertus Girard, qui demonſtrauit, aream trianguli ſphaerici ſemper proportionalem eſſe exceſſui ſummae ternorum angulorum ſuper duobus rectis, atque adeo ipſam aream inueniri, ſi iſte exceſſus, in arcum circuli maximi conuerſus, per radium ſphaerae multiplicetur. Quemadmodum autem area trianguli ſphaerici ex eius lateribus ſit determinanda, inueſtigationem multo difficiliorem poſtulat. Inueni autem iam olim egregium theorema, quo iſta determinatio facile inſtitui poteſt, quod ita ſe habet: *Si latera trianguli ſphaerici denotentur litteris* a, b, c, *area vero eiusdem trianguli ponatur* $=\Delta$, *tum ſemper erit*

$$\text{coſ.}\,\tfrac{1}{2}\Delta = \frac{1+\text{coſ.}\,a+\text{coſ.}\,b+\text{coſ.}\,c}{4\,\text{coſ.}\,\tfrac{1}{2}a\,\text{coſ.}\,\tfrac{1}{2}b\,\text{coſ.}\,\tfrac{1}{2}c},$$

cuius veritas non niſi per longas ambages, ſiue ex theoremate

mate Girardi, fiue immediate per calculum integralem oftendi poteft. Vtramque igitur demonftrationem hic in medium attuliffe operae erit pretium.

Problema.

Tab. I. Fig. 4. *Si trianguli fphaerici AZB fuper bafi AB exftructi bina latera AZ et BZ fuis differentialibus augeantur, vt inde oriatur triangulum Az B, inueftigare augmentum quod hinc areae trianguli AZB acceffit.*

Solutio.

§. 2. Pofita bafi huius trianguli $AB = a$ vocentur eius latera $AZ = x$ et $BZ = y$, ita vt latera trianguli aucti futura fint $Az = x + \partial x$ et $Bz = y + \partial y$. Porro vero vocentur anguli $BAZ = \Phi$ et $ABZ = \psi$, vt prodeant anguli elementares $ZAz = \partial\Phi$ et $ZBz = \partial\psi$, quibus pofitis conftat trianguli elementaris ZAz aream effe $= \partial\Phi(1 - \mathrm{cof.}\,x)$, trianguli vero $ZBz = \partial\psi(1 - \mathrm{cof.}\,y)$. Quoniam igitur haec duo triangula elementaria exhibent augmentum areae trianguli Δ, habebimus hanc aequationem:

$$\partial\Delta = \partial\Phi(1 - \mathrm{cof.}\,x) + \partial\psi(1 - \mathrm{cof.}\,y).$$

§. 3. Nunc igitur angulos Φ et ψ ex calculo eliminemus, eorumque loco ipfa latera x et y introducamus ope praeceptorum Trigonometricorum, quae nobis praebent

$$\mathrm{cof.}\,\Phi = \frac{\mathrm{cof.}\,y - \mathrm{cof.}\,a\,\mathrm{cof.}\,x}{\mathrm{fin.}\,a\,\mathrm{fin.}\,x} \text{ et } \mathrm{cof.}\,\psi = \frac{\mathrm{cof.}\,x - \mathrm{cof.}\,a\,\mathrm{cof.}\,y}{\mathrm{fin.}\,a\,\mathrm{fin.}\,y}$$

hinc igitur per differentiationem colligimus

$$-\partial\Phi\,\mathrm{fin.}\,\Phi = \frac{\partial x\,\mathrm{cof.}\,a - \partial x\,\mathrm{cof.}\,x\,\mathrm{cof.}\,y - \partial y\,\mathrm{fin.}\,x\,\mathrm{fin.}\,y}{\mathrm{fin.}\,a\,\mathrm{fin.}\,x^2},$$

eodemque modo erit

$$-\partial\psi\,\mathrm{fin.}\,\psi = \frac{\partial y\,\mathrm{cof.}\,a - \partial y\,\mathrm{cof.}\,x\,\mathrm{cof.}\,y - \partial x\,\mathrm{fin.}\,x\,\mathrm{fin.}\,y}{\mathrm{fin.}\,a\,\mathrm{fin.}\,y^2}.$$

At

At vero cum fit $\text{cof.}\,\Phi = \frac{\text{cof.}\,y - \text{cof.}\,a\,\text{cof.}\,x}{\text{fin.}\,a\,\text{fin.}\,x}$, erit

$$\text{fin.}\,\Phi = \frac{\sqrt{(1 - \text{cof.}\,a^2 - \text{cof.}\,x^2 - \text{cof.}\,y^2 + 2\,\text{cof.}\,a\,\text{cof.}\,x\,\text{cof.}\,y)}}{\text{fin.}\,a\,\text{fin.}\,x},$$

fimilique modo erit

$$\text{fin.}\,\psi = \frac{\sqrt{(1 - \text{cof.}\,a^2 - \text{cof.}\,x^2 - \text{cof.}\,y^2 + 2\,\text{cof.}\,a\,\text{cof.}\,x\,\text{cof.}\,y)}}{\text{fin.}\,a\,\text{fin.}\,y}.$$

Quoniam hae ambae formulae radicales funt eaedem, ponamus breuitatis gratia

$$\sqrt{(1 - \text{cof.}\,a^2 - \text{cof.}\,x^2 - \text{cof.}\,y^2 + 2\,\text{cof.}\,a\,\text{cof.}\,x\,\text{cof.}\,y)} = v,$$

vt habeamus

$$\text{fin.}\,\Phi = \frac{v}{\text{fin.}\,a\,\text{fin.}\,x} \text{ et } \text{fin.}\,\psi = \frac{v}{\text{fin.}\,a\,\text{fin.}\,y}.$$

§. 4. His igitur valoribus fubftitutis nancifcemur hos valores differentiales:

$$\partial\Phi = -\frac{\partial x\,\text{cof.}\,a + \partial x\,\text{cof.}\,x\,\text{cof.}\,y + \partial y\,\text{fin.}\,x\,\text{fin.}\,y}{v\,\text{fin.}\,x} \text{ et}$$

$$\partial\psi = -\frac{\partial y\,\text{cof.}\,a + \partial y\,\text{cof.}\,x\,\text{cof.}\,y + \partial x\,\text{fin.}\,x\,\text{fin.}\,y}{v\,\text{fin.}\,y}.$$

Hinc igitur incrementum areae quaefitum erit

$$\partial\Delta = \left\{\frac{\begin{array}{l} -\text{cof.}\,a[\partial x\,\text{fin.}\,y(1-\text{cof.}\,x) + \partial y\,\text{fin.}\,x(1-\text{cof.}\,y)] \\ +\partial x\,\text{fin.}\,y[\text{cof.}\,x\,\text{cof.}\,y(1-\text{cof.}\,x) + \text{fin.}\,x^2(1-\text{cof.}\,y)] \\ +\partial y\,\text{fin.}\,x[\text{cof.}\,x\,\text{cof.}\,y(1-\text{cof.}\,y) + \text{fin.}\,y^2(1-\text{cof.}\,x)] \end{array}}{v\,\text{fin.}\,x\,\text{fin.}\,y}\right\},$$

quod euolutum induet hanc formam:

$$v\partial\Delta = \left\{\begin{array}{l} +\partial x\,\text{fin.}\,x(1-\text{cof.}\,y) + \frac{\partial x\,\text{cof.}\,x\,\text{cof.}\,y(1-\text{cof.}\,x)}{\text{fin.}\,x} - \frac{\partial x\,\text{cof.}\,a(1-\text{cof.}\,x)}{\text{fin.}\,x} \\ +\partial y\,\text{fin.}\,y(1-\text{cof.}\,x) + \frac{\partial y\,\text{cof.}\,x\,\text{cof.}\,y(1-\text{cof.}\,y)}{\text{fin.}\,y} - \frac{\partial y\,\text{cof.}\,a(1-\text{cof.}\,y)}{\text{fin.}\,y} \end{array}\right\}.$$

Hic iam notetur effe

$$\frac{1-\text{cof.}\,x}{\text{fin.}\,x} = \text{tang.}\,\tfrac{1}{2}x \text{ et } \frac{1-\text{cof.}\,y}{\text{fin.}\,y} = \text{tang.}\,\tfrac{1}{2}y,$$

 hinc

hinc ergo termini elementum ∂x inuoluentes erunt

$$\partial x \,\text{fin.}\, x (1 - \text{cof.}\, y) + \partial x \,\text{cof.}\, x \,\text{cof.}\, y \,\text{tang.}\, \tfrac{1}{2} x - \partial x \,\text{cof.}\, a \,\text{tang.}\, \tfrac{1}{2} x.$$

Quoniam autem non folum eft $\text{tang.}\, \frac{1}{2} x = \frac{1 - \text{cof.}\, x}{\text{fin.}\, x}$, fed etiam $\text{tang.}\, \frac{1}{2} x = \frac{\text{fin.}\, x}{1 + \text{cof.}\, x}$, in primo membro loco $\text{fin.}\, x$ fcribatur $(1 + \text{cof.}\, x) \,\text{tang.}\, \frac{1}{2} x$, vt ∂x vbique multiplicatum fit per $\text{tang.}\, \frac{1}{2} x$, ficque iftud membrum reducetur ad hanc formam:

$$\partial x \,\text{tang.}\, \tfrac{1}{2} x \,(1 + \text{cof.}\, x - \text{cof.}\, y - \text{cof.}\, a).$$

Eodem modo alterum membrum erit

$$\partial y \,\text{tang.}\, \tfrac{1}{2} y \,(1 + \text{cof.}\, y - \text{cof.}\, x - \text{cof.}\, a),$$

ficque tota noftra aequatio ita erit expreffa:

$$v \partial \Delta = \partial x \,\text{tang.}\, \tfrac{1}{2} x \,(1 + \text{cof.}\, x - \text{cof.}\, y - \text{cof.}\, a) + \partial y \,\text{tang.}\, \tfrac{1}{2} y \,(1 + \text{cof.}\, y - \text{cof.}\, x - \text{cof.}\, a).$$

§. 5. Quodfi iam breuitatis gratia ponamus $\text{cof.}\, a + \text{cof.}\, x + \text{cof.}\, y = s$, erit

$$v \partial \Delta = \partial x \,\text{tang.}\, \tfrac{1}{2} x \,(1 - s + 2 \,\text{cof.}\, x) + \partial y \,\text{tang.}\, \tfrac{1}{2} y \,(1 - s + 2 \,\text{cof.}\, y);$$

quae aequatio hoc modo repraefentari poteft:

$$v \partial \Delta = (1 - s)(\partial x \,\text{tang.}\, \tfrac{1}{2} x + \partial y \,\text{tang.}\, \tfrac{1}{2} y) + 2 \partial x \,\text{cof.}\, x \,\text{tang.}\, \tfrac{1}{2} x + 2 \partial y \,\text{cof.}\, y \,\text{tang.}\, \tfrac{1}{2} y.$$

Cum nunc fit $\text{tang.}\, \frac{1}{2} x = \frac{1 - \text{co.}\, x}{\text{fin.}\, x}$, erit

$$\text{tang.}\, \tfrac{1}{2} x \,\text{cof.}\, x = \frac{\text{cof.}\, x - \text{cof.}\, x^2}{\text{fin.}\, x} = \frac{\text{cof.}\, x - 1 + \text{fin.}\, x^2}{\text{fin.}\, x} = \text{fin.}\, x - \text{tang.}\, \tfrac{1}{2} x.$$

Eodem modo erit $\text{tang.}\, \frac{1}{2} y \,\text{cof.}\, y = \text{fin.}\, y - \text{tang.}\, \frac{1}{2} y$, hisque valoribus fubftitutis orietur haec aequatio:

$$v \partial \Delta = -(1 + s)(\partial x \,\text{tang.}\, \tfrac{1}{2} x + \partial y \,\text{tang.}\, \tfrac{1}{2} y) + 2 \partial x \,\text{fin.}\, x + 2 \partial y \,\text{fin.}\, y.$$

§. 6.

§. 6. Haec postrema forma ideo notatu maxime est digna, quod membrum dextrum absolute sit integrabile, si diuidatur per $1+s$. Facta enim hac diuisione nostra aequatio erit

$$\frac{v\partial\Delta}{1+s} = -\partial x \text{ tang.} \tfrac{1}{2}x - \partial y \text{ tang.} \tfrac{1}{2}y + \frac{2\partial x \sin. x + 2\partial y \sin. y}{1+\cos. a+\cos. x+\cos. y},$$

vbi notetur esse $\int \partial x \text{ tang.} \tfrac{1}{2}x = -2 l \cos. \tfrac{1}{2}x$, similique modo $\int \partial y \text{ tang.} \tfrac{1}{2}y = -2 l \cos. \tfrac{1}{2}y$, ac denique

$$2\int \frac{\partial x \sin. x + \partial y \sin. y}{1+\cos. a+\cos. x+\cos. y} = -2 l (1+\cos. a+\cos. x+\cos. y)$$
$$= -2 l (1+s),$$

sic igitur per integrationem reperimus

$$\int \frac{v\partial\Delta}{1+s} = +2 l \cos. \tfrac{1}{2}x + 2 l \cos. \tfrac{1}{2}y - 2 l\, 1+s$$
$$= +2 l \frac{\cos. \tfrac{1}{2}x \cos. \tfrac{1}{2}y}{1+s},$$

at vero hoc modo membrum sinistrum non est integrabile, cui ergo sequenti modo remedium afferetur.

§. 7. Cum enim posuerimus $s = \cos. a + \cos. x + \cos. y$, statuamus insuper $\cos. \tfrac{1}{2}a \cos. \tfrac{1}{2}x \cos. \tfrac{1}{2}y = q$, vt habeamus hanc aequationem:

$$\int \frac{v\partial\Delta}{1+s} = +2 l \frac{q}{(1+s)\cos. \tfrac{1}{2}a},$$

vbi porro fiat $\frac{q}{1+s} = p$, ita vt sit

$$\int \frac{v\partial\Delta}{1+s} = +2 l \frac{p}{\cos. \tfrac{1}{2}a},$$

quae aequatio denuo differentiata praebet $\frac{v\partial\Delta}{1+s} = +\frac{2\partial p}{p}$, vnde conficitur $\partial\Delta = +\frac{2\partial p(1+s)}{pv}$, quae ergo formula integrationem admittet, si modo fuerit $\frac{v}{1+s}$ functio quaedam ipsius p, id quod

quod iam certo affeuerare possumus, propterea quod $\partial\Delta$ designat differentiale ipsius areae trianguli.

§. 8. Ad hoc ostendendum obseruasse iuuabit esse

$$vv+(1+s)^2 = 2(1+\cos a+\cos x+\cos y+\cos a\cos x + \cos a\cos y+\cos x\cos y+\cos a\cos x\cos y) = 2(1+\cos a)(1+\cos x)(1+\cos y).$$

Constat autem esse $1+\cos a = 2\cos \tfrac{1}{2}a^2$; $1+\cos x = 2\cos\tfrac{1}{2}x^2$ et $1+\cos y = 2\cos\tfrac{1}{2}y^2$, quare cum posuerimus $q = \cos\tfrac{1}{2}a\cos\tfrac{1}{2}x\cos\tfrac{1}{2}y$, erit

$$vv+(1+s)^2 = 16qq \text{ ideoque } v = \sqrt{[16qq-(1+s)^2]}$$

hincque porro

$$\frac{v}{(1+s)} = \sqrt{\left(\frac{16qq}{(1+s)^2}-1\right)}.$$

§. 9. Cum igitur aequatio nostra differentialis fuisset $\frac{v\partial\Delta}{1+s} = +\frac{2\partial p}{p}$, ob $p = \frac{q}{1+s}$ ea induet hanc formam:

$$\partial\Delta\sqrt{(16pp-1)} = \frac{2\partial p}{p}, \text{ ideoque } \partial\Delta = \frac{2\partial p}{p\sqrt{(16pp-1)}}.$$

Fiat iam $p = \frac{1}{r}$, vt habeatur $\partial\Delta = -\frac{2\partial r}{\sqrt{(16-rr)}}$, vnde integrando colligimus $\Delta = C + 2\,\text{Arc.}\cos\frac{r}{4}$ et loco r valore substituto, qui est

$$r = \frac{1}{p} = \frac{1+s}{q} = \frac{1+\cos a+\cos x+\cos y}{\cos\frac{1}{2}a\cos\frac{1}{2}x\cos\frac{1}{2}y}$$

nostra aequatio integralis erit

$$\Delta = C + 2\,\text{Arc.}\cos\frac{1+\cos a+\cos x+\cos y}{4\cos\frac{1}{2}a\cos\frac{1}{2}x\cos\frac{1}{2}y}.$$

§. 10.

§. 10. Nunc ergo totum negotium eo redit, vt valor conſtantis per integrationem ingreſſae C indagetur, quem ſcilicet ex caſu quodam cognito erui oportet; manifeſtum autem eſt aream trianguli euaneſcere debere, quando alterum binorum crurum x vel y euaneſcit. Ponamus igitur eſſe $y = 0$, tum vero neceſſe eſt, vt fiat $x = 0$, hoc ergo caſu conſtituto noſtra aequatio erit

$$0 = C + 2 \text{ Arc. coſ. } \frac{2 + 2 \text{ coſ. } a}{4 \text{ coſ. } \frac{1}{2} a^2}.$$

Quoniam vero $4 \text{ coſ. } \frac{1}{2} a^2 = 2 + 2 \text{ coſ. } \frac{1}{2} a$ et Arc. coſ. $1 = 0$, euidens eſt ſtatui debere $C = 0$, ita vt habeamus

$$\Delta = 2 \text{ Arc. coſ. } \frac{1 + \text{coſ. } a + \text{coſ. } x + \text{coſ. } y}{4 \text{ coſ. } \frac{1}{2} a \text{ coſ. } \frac{1}{2} x \text{ coſ. } \frac{1}{2} y},$$

vnde concluditur

$$\text{coſ. } \tfrac{1}{2} \Delta = \frac{1 + \text{coſ. } a + \text{coſ. } x + \text{coſ. } y}{4 \text{ coſ. } \frac{1}{2} a \text{ coſ. } \frac{1}{2} x \text{ coſ. } \frac{1}{2} y},$$

quae ipſa expreſſio cum theoremate ſupra memorato egregie conuenit, ſi modo loco x et y ſcribantur litterae b et c.

Alia demonſtratio Geometrica theorematis initio allati.

§. 11. Sit igitur ABC triangulum ſphaericum propoſitum, cuius latera vocentur a, b, c, et anguli iis oppoſiti α, β, γ, area vero, quam quaerimus, deſignemus charactere Δ. Cum igitur ex theoremate Girardi ſit Tab. I. Fig. 5.

$$\Delta = \alpha + \beta + \gamma - 180^0, \text{ erit coſ. } \Delta = -\text{coſ. } (\alpha + \beta + \gamma).$$

Nunc vero ex compoſitione angulorum conſtat eſſe

$$\text{ſin. } (\alpha + \beta) = \text{ſin. } \alpha \text{ coſ. } \beta + \text{coſ. } \alpha \text{ ſin. } \beta \text{ et}$$

$$\text{cof.}\,(\alpha+\beta) = \text{cof.}\,\alpha\,\text{cof.}\,\beta - \text{fin.}\,\alpha\,\text{fin.}\,\beta,$$

vnde colligitur

$$\begin{aligned}\text{cof.}\,(\alpha+\beta+\gamma) &= \text{cof.}\,(\alpha+\beta)\,\text{cof.}\,\gamma - \text{fin.}\,(\alpha+\beta)\,\text{fin.}\,\gamma\\ &= \text{cof.}\,\alpha\,\text{cof.}\,\beta\,\text{cof.}\,\gamma - \text{cof.}\,\alpha\,\text{fin.}\,\beta\,\text{fin.}\,\gamma\\ &\quad - \text{cof.}\,\beta\,\text{fin.}\,\alpha\,\text{fin.}\,\gamma - \text{cof.}\,\gamma\,\text{fin.}\,\alpha\,\text{fin.}\,\beta,\end{aligned}$$

confequenter habebimus

$$\text{cof.}\,\Delta = +\,\text{cof.}\,\alpha\,\text{fin.}\,\beta\,\text{fin.}\,\gamma + \text{cof.}\,\beta\,\text{fin.}\,\alpha\,\text{fin.}\,\gamma + \text{cof.}\,\gamma\,\text{fin.}\,\alpha\,\text{fin.}\,\beta - \text{cof.}\,\alpha\,\text{cof.}\,\beta\,\text{cof.}\,\gamma.$$

§. 12. Ex Trigonometria fphaerica autem nouimus effe

$$\text{cof.}\,\alpha = \frac{\text{cof.}\,a - \text{cof.}\,b\,\text{cof.}\,c}{\text{fin.}\,b\,\text{fin.}\,c},$$
$$\text{cof.}\,\beta = \frac{\text{cof.}\,b - \text{cof.}\,a\,\text{cof.}\,c}{\text{fin.}\,a\,\text{fin.}\,c} \text{ et}$$
$$\text{cof.}\,\gamma = \frac{\text{cof.}\,c - \text{cof.}\,a\,\text{cof.}\,b}{\text{fin.}\,a\,\text{fin.}\,b},$$

hincque colligimus porro

$$\text{fin.}\,\alpha = \frac{\sqrt{(1 - \text{cof.}\,a^2 - \text{cof.}\,b^2 - \text{cof.}\,c^2 + 2\,\text{cof.}\,a\,\text{cof.}\,b\,\text{cof.}\,c)}}{\text{fin.}\,b\,\text{fin.}\,c};$$
$$\text{fin.}\,\beta = \frac{\sqrt{(1 - \text{cof.}\,a^2 - \text{cof.}\,b^2 - \text{cof.}\,c^2 + 2\,\text{cof.}\,a\,\text{cof.}\,b\,\text{cof.}\,c)}}{\text{fin.}\,a\,\text{fin.}\,c} \text{ et}$$
$$\text{fin.}\,\gamma = \frac{\sqrt{(1 - \text{cof.}\,a^2 - \text{cof.}\,b^2 - \text{cof.}\,c^2 + 2\,\text{cof.}\,a\,\text{cof.}\,b\,\text{cof.}\,c)}}{\text{fin.}\,a\,\text{fin.}\,b}.$$

Ponamus igitur breuitatis gratia

$$\sqrt{(1 - \text{cof.}\,a^2 - \text{cof.}\,b^2 - \text{cof.}\,c^2 + 2\,\text{cof.}\,a\,\text{cof.}\,b\,\text{cof.}\,c)} = v,$$

ita vt fit

$$\text{fin.}\,\alpha = \frac{v}{\text{fin.}\,b\,\text{fin.}\,c};\ \text{fin.}\,\beta = \frac{v}{\text{fin.}\,a\,\text{fin.}\,c} \text{ et fin.}\,\gamma = \frac{v}{\text{fin.}\,a\,\text{fin.}\,b};$$

quibus valoribus fubftitutis fiet

$$\text{cof.}\,\Delta = \frac{v\,v\,(\text{cof.}\,a - \text{cof.}\,b\,\text{cof.}\,c)}{\text{fin.}\,a^2\,\text{fin.}\,b^2\,\text{fin.}\,c^2} + \frac{v\,v\,(\text{cof.}\,b - \text{cof.}\,a\,\text{cof.}\,c)}{\text{fin.}\,a^2\,\text{fin.}\,b^2\,\text{fin.}\,c^2}$$
$$+ \frac{v\,v\,(\text{cof.}\,c - \text{cef.}\,a\,\text{cof.}\,b)}{\text{fin.}\,a^2\,\text{fin.}\,b^2\,\text{fin.}\,c^2} - \frac{(\text{cof.}\,a - \text{cof.}\,b\,\text{cof.}\,c)(\text{cof.}\,b - \text{cof.}\,a\,\text{cof.}\,c)(\text{cof.}\,c - \text{cof.}\,a\,\text{cof.}\,b)}{\text{fin.}\,a^2\,\text{fin.}\,b^2\,\text{fin.}\,c^2},$$

ficſicque erit

$$\sin. a^2 \sin. b^2 \sin. c^2 \cos. \Delta$$
$$= vv(\cos. a + \cos. b + \cos. c - \cos. a \cos. b - \cos. a \cos. c - \cos. b \cos. c)$$
$$- \cos. a \cos. b \cos. c + \cos. a^2 \cos. b^2 + \cos. a^2 \cos. c^2 + \cos. b^2 \cos. c^2$$
$$- \cos. a \cos. b \cos. c (\cos. a^2 + \cos. b^2 + \cos. c^2) + \cos. a^2 \cos. b^2 \cos. c^2.$$

§. 13. Quo nunc has formulas non parum complicatas commodius tractare liceat, ponamus primo breuitatis gratia $\cos. a = A$; $\cos. b = B$; $\cos. c = C$, vt habeamus

$$(1 - A^2)(1 - B^2)(1 - C^2)\cos. \Delta = vv(A + B + C - AB - AC - BC)$$
$$- ABC + AABB + AACC + BBCC$$
$$- ABC(AA + BB + CC) + AABBCC,$$

vbi iam erit

$$vv = 1 - A^2 - B^2 - C^2 + 2ABC.$$

§. 14. Quoniam hic ternae litterae A, B, C aequaliter in calculum ingrediuntur, ita vt tanquam radices cuiuspiam aequationis cubicae ſpectari queant; ad calculum contrahendum non parum conferet ſtatui

$$A + B + C = P$$
$$AB + AC + BC = Q$$
$$ABC = R,$$

hincque facile colligitur fore

$$AA + BB + CC = PP - 2Q$$

ideoque

$$vv = 1 - PP + 2Q + 2R.$$

Dein-

Deinde notetur formulam $(1-A^2)(1-B^2)(1-C^2)$ esse productum ex his duabus formulis:

$$(1+A)(1+B)(1+C) = 1+P+Q+R,$$

et ex

$$(1-A)(1-B)(1-C) = 1-P+Q-R,$$

sicque nostra aequatio hanc induet formam:

$$(1+P+Q+R)(1-P+Q-R)\cos.\Delta$$
$$= (1-PP+2Q+2R)(P-Q)-R+QQ-2PR-R(PP-2Q)+RR$$

cuius membrum dextrum euolutum dat

$$P-Q-R-QQ+2PQ+RR-P^3+PPQ-PPR,$$

quod per $1-P+Q-R$ diuisum praebet quotientem $P-Q-R+PP$, consequenter nostra aequatio hanc induet formam:

$$(1+P+Q+R)\cos.\Delta = P-Q-R+PP.$$

§. 15. Hactenus igitur deducti sumus ad hanc aequationem: $\cos.\Delta = \frac{P-Q-R+PP}{2+P+Q+R}$, vnde porro colligimus

$$1+\cos.\Delta = \frac{(1+P)^2}{1+P+Q+R} = 2\cos.\tfrac{1}{2}\Delta^2,$$

consequenter habebimus

$$\cos.\tfrac{1}{2}\Delta = \frac{1+P}{\sqrt{2(1+P+Q+R)}}.$$

Cum igitur sit

$$1+P+Q+R = (1+A)(1+B)(1+C)$$
$$= (1+\cos.a)(1+\cos.b)(1+\cos.c),$$

angulis dimidiis introductis erit

1 + P

$$1 + P + Q + R = 8 \text{ cof. } \tfrac{1}{2} a \text{ cof. } \tfrac{1}{2} b \text{ cof. } \tfrac{1}{2} c.$$

Quare cum fit

$$1 + P = 1 + \text{cof. } a + \text{cof. } b + \text{cof. } c;$$

hinc tandem impetramus iftum valorem:

$$\text{cof. } \tfrac{1}{2} \Delta = \frac{1 + \text{cof. } a + \text{cof. } b + \text{cof. } c}{4 \text{ cof. } \tfrac{1}{2} a \cdot \text{cof. } \tfrac{1}{2} b \cdot \text{cof. } \tfrac{1}{2} c},$$

quae eft altera demonftratio theorematis initio commemorati.

§. 16. Haec cogitandi occafionem mihi dedit theorema a Celeb. Profeffore Lexell in medium allatum, circa omnia triangula fphaerica eiusdem areae fuper eadem bafi exftruenda, quo acutiffime demonftrauit, omnes vertices horum triangulorum femper in quodam circulo minore fphaerae effe fitos; quae elegantiffima proprietas non nifi per plures ambages ex noftro theoremate deriuari poteft; verum fequens confideratio viam planiffimam ad hoc praeftandum aperiet.

De eximiis proprietatibus binorum circulorum parallelorum inter fe aequalium in fuperficie fphaerica.

§. 17. Sint MN et mn duo huiusmodi circuli paralleli, et quia fumuntur inter fe aequales, ab aequatore AB vtrinque aequaliter erunt remoti, perinde atque ab vtroque polo P et p. Hic prima proprietas, quae fe offert, in hoc confiftit, quod quilibet arcus circuli maximi Ee, inter hos duos parallelos interceptus, ad vtrumque aequaliter inclinetur, atque ab aequatore in O in duas par- Tab. I. Fig. 6.

 tes

tes aequales ſecetur. Si enim per punctum O ducatur meridianus O P p, binos parallelos ſecans in H et h, ob angulos H O E et h O e inter ſe aequales ambo trilinea H O E et h O e manifeſto inter ſe erunt aequalia et ſimilia, ideoque tam erit O E $=$ O e, quam angulus O E H $=$ O e h. Praeterea hic obſeruaſſe iuuabit, ſi iſte arcus E e vsque ad ſemicirculum continuetur, eum iterum in circulum minorem $m\,n$ incidere, ſcilicet in eius puncto, quod puncto E diametraliter opponitur.

Tab. I. Fig. 7. §. 18. Ducatur nunc inter eosdem parallelos inſuper alius arcus circuli maximi F f, ad vtrumque perinde inclinatus atque arcus E e, et manifeſtum eſt non ſolum hos duos arcus E e et F f inter ſe eſſe aequales, ſed etiam circulorum minorum arcus E F et $e\,f$. Quare cum in hoc quadrilineo E F $e\,f$ non ſolum latera oppoſita, ſed etiam anguli oppoſiti ſint inter ſe aequales, iſtud quadrilineum rite vocari poſſet parallelogrammum ſphaericum, propterea quod omnibus proprietatibus parallelogrammorum eſt praeditum. Euidens enim eſt, iſtud quadrilineum etiam ab vtraque diagonali E f et F e in duo trilinea aequalia ſecari, ſcilicet tam area trilinei $e\,f$ F quam E e F erit ſemiſſis areae parallelogrammi E $e\,f$ F.

Fig. 8. §. 19. Exſtrui nunc concipiatur ſuper eodem arcu E F, tanquam baſi, aliud huiusmodi parallelogrammum ſphaericum E F $\zeta\,\varepsilon$, atque facile intelligitur, areas horum duorum parallelogrammorum E F $f\,e$ et E F $\zeta\,\varepsilon$ eſſe inter ſe aequales. Hic enim prorſus eodem modo, vti in plano, ambo trilinea E $e\,\varepsilon$ et F $f\,\zeta$ inter ſe perfecte ſunt aequalia, a quibus ſi trilineum commune $o\,e\,\zeta$ auferatur, quadrilinea reſi-

residua $E o \zeta \varepsilon$ et $F o e f$ erunt inter se aequalia; quibus si addatur trilineum $E o F$, ambo parallelogramma integra erunt etiam aequalia; sicque etiam euictum est, omnia parallelogramma sphaerica, inter binos circulos parallelos et aequales, super eadem basi E F exstructa, esse inter se aequalia.

§. 20. Cum igitur talia parallelogramma sphaerica a diagonalibus in duas partes aequales diuidantur, etiam omnia trilinea super eadem basi E F exstructa, et in altero parallelo $m n$ terminata, areas habebunt inter se aequales; in hac scilicet figura quatuor habebuntur trilinea inter se aequalia, scilicet: 1. $E f F$; 2. $E e F$; 3. $E \zeta F$; 4. $E \varepsilon F$.

§. 21. Haec autem trilinea ideo non vocamus triangula, quia eorum basis EF non est arcus circuli maximi, quemadmodum in triangulis spaericis statui solet. Facile autem haec trilinea in triangula sphaerica conuertuntur, si ab E ad F ducatur arcus circuli maximi $E \alpha F$, quo praedictis trilineis, idem augmentum $E F \alpha E$ accedit, ita vt nunc etiam omnia triangula sphaerica super eadem basi $E \alpha F$ exstructa, quorum vertices in alterum parallelum $m n$ incidunt areas habeant aequales, si modo termini baseos E et F in altero parallelo illi opposito M N fuerint assumti; sicque iam clare euictum est, si super basi quacunque innumera constituantur triangula sphaerica, quorum areae sint inter se aequales, eorum vertices semper sitos esse in circulo quodam sphaerae minore. Hoc obseruato problema clarissimi Professoris Lexell sequenti modo facillime resolui poterit.

Problema.

Tab. I. Fig. 9. *In ſuperficie ſphaerica ſuper data baſi EF omnia triangula ſphaerica exſtruere, quorum area ſit data $= \Delta$, vbi quidem Δ deſignat arcum circuli maximi, qui per radium ſphaerae multiplicatus producat aream praeſcriptam.*

Solutio.

§. 22. Sit igitur EF ipſa baſis propoſita $= a$, et totum negotium huc redit, vt inueniantur poli P et p, qui quaeſito ſatisfaciant; his enim inuentis ſi ex polo p interuallo $pe =$ PE deſcribatur circulus minor ef, omnium triangulorum ſuper baſi EF exſtructorum, et in circulo minori ef terminatorum areae erunt inter ſe aequales; tantumque ſupereſt, vt ex area propoſita Δ poſitio polorum P et p determinetur.

§. 23. Cum igitur EF ſit arcus circuli maximi $= a$, ponatur EP $=$ FP $= x$ et angulus EPF $= \omega$, eritque ex Trigonometria ſphaerica $\operatorname{cof.} \omega = \frac{\operatorname{cof.} a - \operatorname{cof.} x^2}{\operatorname{ſin.} x^2}$, ideoque

$$1 - \operatorname{cof.} \omega = \frac{1 - \operatorname{cof.} a}{\operatorname{ſin.} x^2} = 2 \operatorname{ſin.} \tfrac{1}{2} \omega^2.$$

Quare cum ſit $1 - \operatorname{cof.} a = 2 \operatorname{ſin.} \frac{1}{2} a^2$, erit $\operatorname{ſin.} \frac{1}{2} \omega = \frac{\operatorname{ſin.} \frac{1}{2} a}{\operatorname{ſin.} x}$, hinque viciſſim $\operatorname{ſin.} x = \frac{\operatorname{ſin.} \frac{1}{2} a}{\operatorname{ſin.} \frac{1}{2} \omega}$. Ex cognito autem angulo ω innoteſcit tota area ſegmenti ſphaerici inter binos ſemicirculos PEp et PFp, quippe quae erit $= 2\omega$. Scilicet arcus

arcus circuli maximi $= 2\omega$ per radium ſphaerae $= 1$ multiplicatus dabit aream huius ſegmenti.

§. 24. Quaeramus nunc etiam aream trianguli EPF, quem in finem vocetur angulus $PEF = PFE = \Phi$, ita vt ſumma trium angulorum huius trianguli ſit $= \omega + 2\Phi$, vnde area huius trianguli erit $= \omega + 2\Phi - \pi$. Quare ſi etiam ab e ad f arcus circuli maximi $e\omega f$ ducatur, erit quoque area trianguli ſphaerici $pe\omega f = \omega + 2\Phi - \pi$. Hinc ergo area quadrilateri ſphaerici $EFfe$ inter arcus circulorum maximorum Ee; Ff; EF et $e\omega f$ comprehenſi erit $2\omega - 2(\omega + 2\Phi - \pi) = 2\pi - 4\Phi$, cuius ſemiſſis manifeſto praebet aream trianguli ſphaerici EFe.

§. 25. Cum igitur punctum e ſit etiam in circulo minori ef, erit triangulum EeF vnum ex illis triangulis infinitis, quae ſuper baſi EF exſtruere oportet, cuius area debet eſſe $= \Delta$, ſicque adepti ſumus hanc aequationem $\Delta = \pi - 2\Phi$, vnde colligimus angulum $\Phi = \frac{1}{2}\pi - \frac{1}{2}\Delta$. Cum igitur angulus Δ detur, ſuper baſi data EF exſtruantur vtrinque anguli aequales $FEP = EFP = 90 - \frac{1}{2}\Delta$. Sicque innoteſcet polus P, ideoque et ei oppoſitus p, ex quo ſi interuallo $pe = PE$ deſcribatur circulus minor ef, omnia triangula ſuper baſi EF exſtructa et in peripheria circuli minoris ef terminata habebunt ipſam aream propoſitam $= \Delta$.

§. 26. Quo haec conſtructio facilior reddatur, ex polo P in medium baſis Π ducatur arcus normalis $P\Pi$, et quia in triangulo $EP\Pi$ habetur latus $E\Pi = \frac{1}{2}a$, cum

 angulo

angulo $PE\Pi = 90 - \frac{1}{2}\Delta$, hinc colligitur latus $EP = x$; cuius tangens est tang. $x = \frac{\text{tang}\ \frac{1}{2}a}{\text{sin.}\ \frac{1}{2}\Delta}$. Nunc igitur inuenta quantitate arcuum EP et FP eorum intersectio dabit polum P, ex cuius opposito p circulus minor interuallo $pe = x$ descriptus praebet loca verticum omnium triangulorum super basi EF describendorum, quae constructio egregie conuenit cum ea, quam clarissimus Lexell inuenit.

VTRVM