1797

Integratio succincta formulae integralis maxime memorabilis $\int \partial z/((3 \pm zz) \cdot (1 \pm 3zz)^{1/3})$

Leonhard Euler

INTEGRATIO SVCCINCTA FORMVLAE INTEGRALIS MAXIME MEMORABILIS

$$\int \frac{\partial z}{(3 \pm zz)\sqrt[3]{(1 \pm 3zz)}}.$$

Auctore
L. EVLERO.

Conuentui exhib. die 28 April. 1777.

§. 1.

Valeant primo signa superiora, sitque

$$\partial V = \frac{\partial z}{(3 + zz)\sqrt[3]{(1 + 3zz)}};$$

ac posito $\sqrt[3]{(1 + 3zz)} = v$, vt fit $1 + 3zz = v^3$, erit $z\partial z = \frac{1}{2} v v \partial v$, ideoque $\partial z = \frac{v v \partial v}{2z}$, vnde fit $\partial V = \frac{v \partial v}{2z(3 + zz)}$.

§. 2. Statuatur nunc $p = \frac{1+z}{v}$ et $q = \frac{1-z}{v}$, eritque $p^3 + q^3 = 2$ et $p^3 - q^3 = \frac{6z + 2z^3}{v^3}$, vnde fit $\partial V = \frac{\partial v}{v v (p^3 - q^3)}$. Cum porro fit $p + q = \frac{2}{v}$, erit $\partial p + \partial q = -\frac{2 \partial v}{v v}$, ideoque

$$\partial V = -\frac{(\partial p + \partial q)}{2(p^3 - q^3)}.$$

§. 3.

§. 3. Discerpatur iam haec formula in duas partes, ponendo $\frac{\partial p}{p^3 - q^3} = \partial P$ et $\frac{\partial q}{p^3 - q^3} = \partial Q$, vt fit $\partial V = -\frac{1}{2}\partial P - \frac{1}{2}\partial Q$, et quia $q^3 = 2 - p^3$, erit $\partial P = -\frac{\partial p}{2(1 - p^3)}$; tum vero ob $p^3 = 2 - q^3$, erit $\partial Q = +\frac{\partial q}{2(1 - q^3)}$, sicque habebimus

$$4\partial V = +\frac{\partial p}{1 - p^3} - \frac{\partial q}{1 - q^3}.$$

§. 4. Cum nunc constet esse

$$\int \frac{\partial p}{1 - p^3} = \frac{1}{3} l \frac{\sqrt{(1 + p + pp)}}{1 - p} + \frac{1}{\sqrt{3}} \text{A tang.} \frac{p\sqrt{3}}{2 + p},$$

ob $1 + p + pp = \frac{1 - p^3}{1 - p} = \frac{1 - p^3}{(1 - p)^3}$, erit

$$\int \frac{\partial p}{1 - p^3} = \frac{1}{6} l \frac{1 - p^3}{(1 - p)^3} + \frac{1}{\sqrt{3}} \text{A tang.} \frac{p\sqrt{3}}{2 + p}.$$

§. 5. Cum igitur simili modo fit

$$\int \frac{\partial q}{1 - q^3} = \frac{1}{6} l \frac{1 - q^3}{(1 - q)^3} + \frac{1}{\sqrt{3}} \text{A tang.} \frac{q\sqrt{3}}{2 + q},$$

erit integrale quaesitum quater sumtum

$$4V = \frac{1}{6} l \frac{1 - p^3}{(1 - p)^3} - \frac{1}{6} l \frac{1 - q^3}{(1 - q)^3} + \frac{1}{\sqrt{3}} \text{A tang.} \frac{p\sqrt{3}}{2 + p} - \frac{1}{\sqrt{3}} \text{A tang.} \frac{q\sqrt{3}}{2 + q}.$$

§. 9. Quod si iam logarithmi hoc modo contrahantur, vt fiant $\frac{1}{6} l \frac{1 - p^3}{1 - q^3} + \frac{1}{6} l \frac{(1 - q)^3}{(1 - p)^3}$, haec expressio, ob $1 - p^3 = -(1 - q^3)$, praebet $\frac{1}{6} l - 1 + \frac{1}{2} l \frac{1 - q}{1 - p}$, vbi pars prior, quia est constans, omitti potest, ita vt logarithmi iunctim sumti faciant $\frac{1}{2} l \frac{1 - q}{1 - p}$, ideoque habeatur:

$$4V = \frac{1}{2} l \frac{1 - q}{1 - p} + \frac{1}{\sqrt{3}} \text{A tang.} \frac{p\sqrt{3}}{2 + p} - \frac{1}{\sqrt{3}} \text{A tang.} \frac{q\sqrt{3}}{2 + q}.$$

Bini autem arcus circulares contrahuntur in vnum

$$\frac{1}{\sqrt{3}} \text{A tang.} \frac{(p - q)\sqrt{3}}{2 + p + q + 2pq},$$

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ſicque integrale quaeſitum erit

$$V = \frac{1}{8}\, l\, \frac{1-q}{1-p} + \frac{1}{4\sqrt{3}}\, \text{A tang.}\, \frac{(p-q)\sqrt{3}}{2+p+q+2pq}.$$

§. 7. Cum nunc poſuerimus $p = \frac{1+z}{v}$ et $q = \frac{1-z}{v}$, pars logarithmica accipiet hanc formam:

$$\frac{1}{8}\, l\, \frac{v-1+z}{v-1-z} = \frac{1}{8}\, l\, \frac{1-v-z}{1-v+z}.$$

Pro arcu circulari autem erit $p - q = \frac{2z}{v}$, tum vero ob $p + q = \frac{2}{v}$ et $pq = \frac{1-zz}{v}$, arcus fiet

$$\frac{1}{4\sqrt{3}}\, \text{A tang.}\, \frac{vz\sqrt{3}}{1+v+vv-zz},$$

ſicque adepti ſumus hanc integrationem ſatis concinnam:

$$\int \frac{\partial z}{(3+zz)\sqrt[3]{(1+3zz)}} = \frac{1}{8}\, l\, \frac{1-v-z}{1-v+z} + \frac{1}{4\sqrt{3}}\, \text{A tang.}\, \frac{vz\sqrt{3}}{1+v+vv-zz},$$

exiſtente $v = \sqrt[3]{(1+3zz)}$.

§. 8. Iam pro altero caſu, quo ſigna inferiora valent, ſtatuamus $z = y\sqrt{-1}$, vt ſit $v = \sqrt[3]{(1-3yy)}$, vnde ſit integratio ſuperior

$$\int \frac{\partial y\sqrt{-1}}{(3-yy)\sqrt[3]{(1-3yy)}} = \frac{1}{8}\, l\, \frac{1-v-y\sqrt{-1}}{1-v+y\sqrt{-1}} + \frac{1}{4\sqrt{3}}\, \text{A tag.}\, \frac{vy\sqrt{3}\sqrt{-1}}{1+v+vv+yy}$$

vbi tantum opus eſt imaginaria tollere.

§. 9. Hunc in finem, cum ſit in genere

$$\text{A tang.}\, t\sqrt{-1} = \int \frac{\partial t\sqrt{-1}}{1-tt} = \frac{\sqrt{-1}}{2}\, l\, \frac{1+t}{1-t},$$

noſtro caſu, ob $t = \frac{vy\sqrt{3}}{1+v+vv+yy}$, erit pars poſterior formulae

mulae inuentae

$$\frac{\sqrt{-1}}{8\sqrt{3}}\, l\, \frac{1+v+vv+yy+vy\sqrt{3}}{1+v+vv+yy-vy\sqrt{3}}.$$

Pro parte logarithmica in formula canonica ponatur $t = u\sqrt{-1}$, fietque

$$-\text{A tang.}\, u = \frac{\sqrt{-1}}{2}\, l\, \frac{1+u\sqrt{-1}}{1-u\sqrt{-1}},$$

ideoque

$$l\, \frac{1+u\sqrt{-1}}{1-u\sqrt{-1}} = 2\sqrt{-1}\ \text{A tang.}\, u.$$

Pro noſtro iam caſu eſt $u = -\frac{y}{1-v}$, ideoque

$$l\, \frac{1-v-y\sqrt{-1}}{1-v+y\sqrt{-1}} = 2\sqrt{-1}\ \text{A tang.} -\frac{y}{1-v};$$

quibus valoribus ſubſtitutis integrale praeſentis formulae imaginariae erit

$$-\frac{\sqrt{-1}}{4}\ \text{A tang.}\, \frac{y}{1-v} + \frac{\sqrt{-1}}{8\sqrt{3}}\, l\, \frac{1+v+vv+yy+vy\sqrt{3}}{1+v+vv+yy-vy\sqrt{3}}.$$

§. 10. Hic manifeſto omnia per $\sqrt{-1}$ ſunt diuiſibilia, ſicque ſublatis imaginariis nacti ſumus hanc alteram integrationem:

$$\int \frac{\partial y}{(3-yy)\sqrt[3]{(1-3yy)}} = \frac{1}{8\sqrt{3}}\, l\, \frac{1+v+vv+yy+vy\sqrt{3}}{1+v+vv+yy-vy\sqrt{3}} - \frac{1}{4}\ \text{A tang.}\, \frac{y}{1-v},$$

vbi notetur, ſi fractionem logarithmo adiunctam ſupra et infra per $1-v$ multiplicemus, ob $1-v^3 = 3yy$, eam fore $\frac{y(4-v)+v(1-v)\sqrt{3}}{y(4-v)-v(1-v)\sqrt{3}}$. Hocque modo noſtra integratio hanc induit formam:

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$$\int \frac{\partial y}{(3-yy)\sqrt[3]{(1-3yy)}} = \frac{1}{8\sqrt{3}} l \frac{y(4-v)+v(1-v)\sqrt{3}}{y(4-v)-v(1-v)\sqrt{3}} - \tfrac{1}{4} \text{A tang.} \frac{y}{1-v},$$

existente $v = \sqrt[3]{(1-3yy)}$.

Resolutio magis naturalis formulae differentialis propositae.

§. 11. Quanquam solutio superior totum negotium pulcherrime conficit, tamen id in ea desiderari potest, quod nulla ratio patet, quae substitutiones ibi adhibitas suadere potuerit; quam ob rem haud ingratum erit aliam solutionem subiungere, cuius ratio quodammodo clarius perspici queat.

§. 12. Considerantem autem formulam priorem

$$\partial V = \frac{\partial z}{(3+zz)\sqrt[3]{(1+3zz)}}$$

expressiones $1+3zz$ et $3z+z^3$ admonere possunt, huiusmodi substitutionem $z = \frac{1+x}{1-x}$ haud sine successu in vsum vocari posse, cum altera superiorum expressionum sit summa duorum cuborum, altera differentia. Hinc autem fit $\partial z = \frac{2\partial x}{(1-\alpha)^2}$, tum vero

$$3+zz = \frac{4-4x+4xx}{(1-x)^2} = \frac{4(1+x^3)}{(1+x)(1-x)^2}$$

denique erit

$$1+3zz = \frac{4+4x+4xx}{(1-x)^2} = \frac{4(1-x^3)}{(1-x)^3}$$

vnde fit

$$\sqrt[3]{(1+3zz)} = \frac{\sqrt[3]{4(1-x^3)}}{1-x}$$

quibus fubftitutis prodit

$$\partial V = \frac{1}{2\sqrt[3]{4}} \times \frac{(1-xx)\partial x}{(1+x^3)\sqrt[3]{(1-x^3)}}.$$

§. 13. Hoc modo formula inuenta vltro in duas partes discerpitur, atque integratio hoc modo repraefentari poteft:

$$2V\sqrt[3]{4} = \int \frac{\partial x}{(1+x^3)\sqrt[3]{(1-x^3)}} - \int \frac{xx\partial x}{(1+x^3)\sqrt[3]{(1-x^3)}},$$

quarum formularum prior ad rationalitatem perduci poteft, ponendo $\frac{x}{\sqrt[3]{(1-x^3)}} = t$, ita vt pars prior fit $\int \frac{t\partial x}{x(1+x^3)}$; tum autem erit $x^3 = t^3 - t^3 x^3$, ideoque $x^3 = \frac{t^3}{1+t^3}$, vnde ftatim fit $1 + x^3 = \frac{1+2t^3}{1+t^3}$. Sumtis autem logarithmis differentiando colligitur $\frac{\partial x}{x} = \frac{\partial t}{t(1+t^3)}$ ficque pars ifta prior euadet $\int \frac{\partial t}{1+2t^3}$, cuius integratio eft in promtu.

§. 14. Partis pofterioris tractatio adhuc magis eft obuia. Pofito enim $\sqrt[3]{(1-x^3)} = u$, fit $x^3 = 1 - u^3$, tum vero $xx\partial x = -uu\partial u$ et $1 + x^3 = 2 - u^3$; hoc ergo modo habebitur

$$\int \frac{xx\partial x}{(1+x^3)\sqrt[3]{(1-x^3)}} = -\int \frac{u\partial u}{2-u^3}.$$

Totum igitur integrale quaesitum erit

$$2V\sqrt[3]{4} = \int \frac{\partial t}{1+2t^3} + \int \frac{u\,\partial u}{2-u^3}.$$

§. 15. Hoc igitur modo formulam propositam etiam transformauimus in duas alias formulas mere rationales, quarum ergo integratio per regulas cognitas facile expeditur, vnde idcirco idem integrale resultare debet, quod prior methodus suppeditauit, si modo debitae reductiones rite instituantur. Facile autem patet, priore methodo formulam finalem multo facilius obtineri, quam si has postremas formulas euoluere vellemus, atque ob hanc ipsam causam methodus ante tradita huic palmam praeripere est censenda.

§. 16. Si alteram formulam $\frac{\partial z}{(3-zz)\sqrt[3]{(1-3zz)}}$ simili modo tractare velimus, statui oportebit $z = \frac{1+x}{1-x}\sqrt{-1}$ ita vt ista resolutio non aliter nisi per Imaginaria institui possit, vnde paradoxon iam ante allatum multo magis confirmatur, quo eiusmodi formulas differentiales exhiberi posse affirmaueram, quarum integratio nonnisi per Imaginaria procedendo perfici queat, ex quo summus vsus calculi Imaginariorum in Analysi multo magis perspicitur.