1797

Ulterior disquitio de formulis integralibus imaginariis

Leonhard Euler

VLTERIOR DISQVISITIO
DE
FORMVLIS INTEGRALIBVS IMAGINARIIS.

Auctore
L. EVLERO

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§. 1.

Vniuersa Theoria Imaginariorum, vnde tot egregia incrementa nunc quidem in Analysin sunt illata, hoc potissimum nititur fundamento: quod si Z fuerit functio quaecunque ipsius z, eaque posito $z = x + y\sqrt{-1}$ abeat in hanc formam: $M + N\sqrt{-1}$, tum eadem functio Z, posito $z = x - y\sqrt{-1}$, euadat $= M - N\sqrt{-1}$; vbi quidem litterae M & N semper denotant quantitates reales. Hinc si proponatur ista formula differentialis: $Z\partial z$, cuius integrale sit $\int Z\partial z = V$, in eaque ponatur $z = x + y\sqrt{-1}$, vnde prodeat $Z = M + N\sqrt{-1}$, ipsum integrale erit huius formae: $V = P + Q\sqrt{-1}$. Cum enim sit

$$\partial V = Z\partial z = M\partial x - N\partial y + (N\partial x + M\partial y)\sqrt{-1},$$

 inte-

integrale erit

$$\int(M\partial x - N\partial y) + \sqrt{-1}\int(N\partial x + M\partial y) = P + Q\sqrt{-1}.$$

Necesse igitur est, vt posito $z = x - y\sqrt{-1}$ fiat

$$P - Q\sqrt{-1} = \int(M\partial x - N\partial y) - \sqrt{-1}\int(N\partial x + M\partial y).$$

Hinc autem manifestum est fore

$$P = \int(M\partial x - N\partial y) \text{ et } Q = \int(N\partial x + M\partial y).$$

Ex quo intelligitur, in huiusmodi substitutionibus semper partes reales et imaginarias seorsim inter se aequari debere.

§. 2. Haec euolutio nobis iam suppeditat insignes proprietates, quae inter quantitates M, N, P et Q intercedunt. Primo enim cum sit $P = \int(M\partial x - N\partial y)$, quoniam haec formula semper integrationem admittit, erit per criterium huiusmodi formularum generale $\left(\frac{\partial M}{\partial y}\right) = -\left(\frac{\partial N}{\partial x}\right)$. Eodem autem modo, quia habemus $Q = \int(N\partial x + M\partial y)$, ob integrabilitatem huius formulae erit $\left(\frac{\partial M}{\partial x}\right) = \left(\frac{\partial N}{\partial y}\right)$. Ecce ergo per talem substitutionem semper inueniuntur eiusmodi duae functionos M et N binarum variabilium x et y, his insignibus proprietatibus praeditae, vt sit tam $\left(\frac{\partial M}{\partial y}\right) = -\left(\frac{\partial N}{\partial x}\right)$ quam $\left(\frac{\partial M}{\partial x}\right) = \left(\frac{\partial N}{\partial y}\right)$.

§. 3. Similis proprietas etiam conuenit quantitatibus P et Q. Cum enim sit $\partial P = M\partial x - N\partial y$ et $\partial Q = M\partial y + N\partial x$, erit per similes characteres

$$\left(\frac{\partial P}{\partial x}\right) = M \text{ et } \left(\frac{\partial P}{\partial y}\right) = -N,$$

$$\left(\frac{\partial Q}{\partial x}\right) = N \text{ et } \left(\frac{\partial Q}{\partial y}\right) = M,$$

vnde

vnde manifeſtum eſt fore

$$\left(\frac{\partial P}{\partial x}\right) = \left(\frac{\partial Q}{\partial y}\right) \text{ et } \left(\frac{\partial P}{\partial y}\right) = -\left(\frac{\partial Q}{\partial x}\right).$$

Tales autem relationes eo magis ſunt notatu dignae, quod earum ratio minus perſpicitur, ſimulac pro Z functiones aliquanto magis complicatae accipiuntur.

§. 4. Non ita pridem contemplatus ſum, ſecundum haec principia, formulam integralem $\int \frac{z^{m-1}\partial z}{1 \pm z^n}$, vnde plures huiusmodi relationes non contemnendas ſum adeptus. Deinde etiam hanc ſpeculationem extendi ad formulam $\int \frac{\partial z}{\sqrt[n]{(1+z^n)}}$, cuius integrale cum ſemper per logarithmos et arcus circulares exprimere liceat, inde etiam pro litteris P et Q eiusmodi formulae prodire debent, quae ſimilem integrationem admittant, etiamſi vix vlla via pateat iſtam integrationem exſequendi. Hocque modo deductus ſum ad Theorema quoddam maxime memorabile, cuius demonſtratio propemodum vires Analyſeos ſuperare videbatur; interim tamen deinceps eius demonſtrationem elicui; quam ob rem conſtitui iſtud argumentum aliquanto generalius retractare.

§. 5. Conſiderabo igitur hic iſtam formulam integralem multo latius patentem: $\int \frac{z^{m-1}\partial z}{(a+bz^n)^\lambda} = V$, vbi, vt calculus commodius ſuccedat, loco z ſubſtituo iſtam formulam imaginariam: $z = v(\text{coſ.}\,\theta + \sqrt{-1}\,\text{ſin.}\,\theta)$, quippe quae omnia Imaginaria in ſe complectitur; tum vero hic angulum θ pro conſtante ſum habiturus, ita vt ſola v nobis ſit variabilis

lis, vnde ergo ſtatim fit $\frac{\partial z}{z} = \frac{\partial v}{v}$. Cum igitur ſit

$$z^m = v^m (\text{coſ.}\, m\,\theta + \sqrt{-1}\, \text{ſin.}\, m\,\theta)$$

numerator huius formulae ſtatim fit

$$z^{m-1}\partial z = v^{m-1} (\text{coſ.}\, m\,\theta + \sqrt{-1}\, \text{ſin.}\, m\,\theta)\,\partial v.$$

Ex hac autem ſubſtitutione ſumamus prodire iſtum valorem integralem: $V = P + Q\sqrt{-1}$.

§. 6. Pro denominatore autem obtinebimus

$$a + b z^n = a + b v^n (\text{coſ.}\, n\,\theta + \sqrt{-1}\, \text{ſin.}\, n\,\theta)$$

cuius ergo pars realis eſt $a + b v^n \text{coſ.}\, n\,\theta$, pars imaginaria vero $b v^n \sqrt{-1}\, \text{ſin.}\, n\,\theta$; vnde ſi exponens eſſet numerus integer, imaginaria facile ex denominatore in numeratorem transferri poſſent, dum ſcilicet ſupra et infra multiplicaremus per

$$[a + b v^n (\text{coſ.}\, n\,\theta - \sqrt{-1}\, \text{ſin.}\, n\,\theta)]^\lambda.$$

Verum quia hi caſus nulla laborant difficultate, calculum potiſſimum ad exponentes fractos pro λ accipiendos accommodari conuenit.

§. 7. Hunc in finem loco variabilis v aliam in calculum introducamus s, cum certo angulo ϕ, ita vt ſit

$$a + b v^n \text{coſ.}\, n\,\theta = s\, \text{coſ.}\, \phi \text{ et}$$

$$b v^n \text{ſin.}\, n\,\theta = s\, \text{ſin.}\, \phi$$

vnde ergo certa relatio inter hanc nouam variabilem s et angulum ϕ definitur, ita vt vel ſola littera s vel ſolus angulus ϕ in calculum introduci queat. Euidens autem eſt, has duas quantitates per variabilem v ita definiri, vt ſit

$$\text{I}^{\circ}.\ ss = aa + 2abv^n \operatorname{cof.} n\theta + bbv^{2n},$$

$$\text{II}^{\circ}.\ \text{tang.}\ \Phi = \frac{bv^n \operatorname{fin.} n\theta}{a + bv^n \operatorname{cof.} n\theta}.$$

§. 8. Hic autem ftatim intelligitur, ipfam quantitatem s loco ipfius v non commode in calculum introduci poffe, quandoquidem angulum Φ, cuius varia multipla occurrunt, nullo modo ex calculo eliminare liceret, vel faltem formulae inextricabiles in calculum implicarentur. Quamobrem conueniet totum calculum ad folam variabilem Φ reuocare, ita vt nobis incumbat, ambas quantitates v et s per iftam nouam variabilem Φ determinare.

§. 9. Ante autem quam hoc exfequamur, obferuemus, denominatorem noftrae formulae per binas variabiles affumtas s & Φ ita concinne expreffum iri, vt fiat

$$a + bz^n = s(\operatorname{cof.}\Phi + \sqrt{-1}\operatorname{fin.}\Phi)$$

hincque totus denominator

$$(a + bz^n)^\lambda = s^\lambda(\operatorname{cof.}\lambda\Phi + \sqrt{-1}\operatorname{fin.}\lambda\Phi).$$

Quodfi igitur fupra et infra per $\operatorname{cof.}\lambda\Phi - \sqrt{-1}\operatorname{fin.}\lambda\Phi$ multiplicemus, formula noftra propofita, retento adhuc numeratore, fequentem accipiet formam:

$$\int \frac{v^{m-1}\partial v(\operatorname{cof.} m\theta + \sqrt{-1}\operatorname{fin.} m\theta)(\operatorname{cof.}\lambda\Phi - \sqrt{-1}\operatorname{fin.}\lambda\Phi)}{s^\lambda} = W$$

quae contrahitur in hanc formam fatis fimplicem:

$$\int \frac{v^{m-1}\partial v}{s^\lambda}[\operatorname{cof.}(m\theta - \lambda\Phi) + \sqrt{-1}\operatorname{fin.}(m\theta - \lambda\Phi)],$$

cuius valor cum pofitus fit $= P + Q\sqrt{-1}$, realia ab

ima-

imaginariis ſeparando erit

$$P = \int \frac{v^{m-1}\, \partial v \,\text{coſ.}\, (m\theta - \lambda\Phi)}{s^{\lambda}} \quad \text{et}$$

$$Q = \int \frac{v^{m-1}\, \partial v \,\text{ſin.}\, (m\theta - \lambda\Phi)}{s^{\lambda}}.$$

§. 10. Vt nunc hinc binas litteras v et s abigamus, recurramus ad binas poſitiones ante ſtabilitas:

$$\text{I.}\quad a + b v^{n} \,\text{coſ.}\, n\theta = s \,\text{coſ.}\, \Phi,$$

$$\text{II.}\quad b v^{n} \,\text{ſin.}\, n\theta = s \,\text{ſin.}\, \Phi.$$

Hic primo quantitas s eliminabitur per hanc combinationem: I. ſin. Φ — II. coſ. Φ, vnde fit $a \,\text{ſin.}\, \Phi = b v^{n} \,\text{ſin.}\, (n\theta - \Phi)$, ideoque $v^{n} = \frac{a \,\text{ſin.}\, \Phi}{b \,\text{ſin.}\, (n\theta - \Phi)}$ ſicque iam valorem litterae v per angulum Φ ſumus adepti. Porro vero haec combinatio I. ſin. $n\theta$ — II. coſ. $n\theta$ praebet $a \,\text{ſin.}\, n\theta = s \,\text{ſin.}\, (n\theta - \Phi)$, vnde fit $s = \frac{a \,\text{ſin.}\, n\theta}{\text{ſin.}\, (n\theta - \Phi)}$, ex quo valore nanciſcimur

$$P = \frac{1}{a^{\lambda} \,\text{ſin.}\, n\theta^{\lambda}} \int v^{m-1}\, \partial v \,\text{ſin.}\, (n\theta - \Phi)^{\lambda} \,\text{coſ.}\, (m\theta - \lambda\Phi) \quad \text{et}$$

$$Q = \frac{1}{a^{\lambda} \,\text{ſin.}\, n\theta^{\lambda}} \int v^{m-1}\, \partial v \,\text{ſin.}\, (n\theta - \Phi)^{\lambda} \,\text{ſin.}\, (m\theta - \lambda\Phi).$$

§. 11. Quoniam porro inuenimus $v^{n} = \frac{a \,\text{ſin.}\, \Phi}{b \,\text{ſin.}\, (n\theta - \Phi)}$, ſumtis logarithmis habebimus

$$n\, l\, v = l\, a \,\text{ſin.}\, \Phi - l\, b \,\text{ſin.}\, (n\theta - \Phi)$$

hincque differentiando

$$\frac{n\,\partial v}{v} = \frac{\partial\Phi \,\text{coſ.}\, \Phi}{\text{ſin.}\, \Phi} + \frac{\partial\Phi \,\text{coſ.}\, (n\theta - \Phi)}{\text{ſin.}\, (n\theta - \Phi)} = \frac{\partial\Phi \,\text{ſin.}\, n\theta}{\text{ſin.}\, \Phi \,\text{ſin.}\, (n\theta - \Phi)}.$$

Deinde

Deinde vero erit $v^m = \left(\frac{a\,\text{ſin.}\,\Phi}{b\,\text{ſin.}\,(n\theta - \Phi)}\right)^{\frac{m}{n}}$. His igitur valoribus ſubſtitutis ad ſequentes formulas integrales deducemur:

$$P = \frac{1}{n a^{\lambda}\,\text{ſin.}\,n\theta^{\lambda-1}} \int \left(\frac{a\,\text{ſin.}\,\Phi}{b\,\text{ſin.}\,(n\theta - \Phi)}\right)^{\frac{m}{n}} \frac{\partial\Phi\,\text{ſin.}\,(n\theta-\Phi)^{\lambda}\,\text{coſ.}\,(m\theta-\lambda\Phi)}{\text{ſin.}\,\Phi\,\text{ſin.}\,(n\theta-\Phi)}$$

$$Q = \frac{1}{n a^{\lambda}\,\text{ſin.}\,n\theta^{\lambda-1}} \int \left(\frac{a\,\text{ſin.}\,\Phi}{b\,\text{ſin.}\,(n\theta - \Phi)}\right)^{\frac{m}{n}} \frac{\partial\Phi\,\text{ſin.}\,(n\theta-\Phi)^{\lambda}\,\text{ſin.}\,(m\theta-\lambda\Phi)}{\text{ſin.}\,\Phi\,\text{ſin.}\,(n\theta-\Phi)}$$

Quod ſi iam breuitatis gratia ponamus $n\theta - \Phi = \psi$, vt ſit $\Phi + \psi = n\theta$, ideoque $\partial\Phi + \partial\psi = 0$, ambae formulae concinnius ſequenti modo repraeſentari poterunt:

$$P = \frac{a^{\frac{m}{n}-\lambda}}{n\,b^{\frac{m}{n}}\,\text{ſin.}\,n\theta^{\lambda-1}} \int \partial\Phi\,\text{ſin.}\,\Phi^{\frac{m-n}{n}}\,\text{ſin.}\,\psi^{\lambda-\frac{m}{n}-1}\,\text{coſ.}\,(m\theta-\lambda\Phi),$$

$$Q = \frac{a^{\frac{m}{n}-\lambda}}{n\,b^{\frac{m}{n}}\,\text{ſin.}\,n\theta^{\lambda-1}} \int \partial\Phi\,\text{ſin.}\,\Phi^{\frac{m-n}{n}}\,\text{ſin.}\,\psi^{\lambda-\frac{m}{n}-1}\,\text{ſin.}\,(m\theta-\lambda\Phi);$$

§. 12. En ergo deducti ſumus ad binas formulas integrales, quarum integratio, quantumuis, ob exponentem fractum $\frac{m}{n}$, videatur difficilis, tamen ſemper pendet a formula principali propoſita $\int \frac{z^{m-1}\,\partial z}{(a \pm b z^n)^{\lambda}}$, cuius ergo integrale, ſi vel algebraice, vel ſaltem per logarithmos et arcus circulares aſſignari queat, etiam certo affirmare poterimus, ambas formulas hic inuentas ſecundum eandem legem integrari poſſe. Hic quidem primo ſe offert caſus $m = n$, quo

quo adeo integrale algebraice exhiberi poteſt; verum quia hoc caſu $\frac{m}{n}$ non amplius eſt fractio, eum praetereamus.

§. 13. Imprimis autem hic occurrit caſus maxime memorabilis, quo $\lambda = \frac{m}{n}$, quippe quo integrationem per logarithmos & arcus circulares expedire licet. Si enim pro noſtra formula integrali $\int \frac{\partial z}{z} \cdot \frac{z^m}{(a + b z^n)^{\frac{m}{n}}}$ ſtatuamus $\frac{z}{(a + b z^n)^{\frac{1}{n}}} = t$, vt formula integranda ſit $\int t^m \frac{\partial z}{z}$, erit $t^n = \frac{z^n}{a + b z^n}$, vnde colligitur $z^n = \frac{a t^n}{b t^n - 1}$, hincque differentiando ſumtis logarithmis, erit

$$\frac{\partial z}{z} = \frac{\partial t}{t} - \frac{b t^{n-1} \partial t}{b t^n - 1} = \frac{-\partial t}{t (b t^n - 1)},$$

ita vt formula noſtra integranda ſit $-\int \frac{t^{m-1} \partial t}{b t^n - 1}$, quae cum ſit rationalis, ſemper per logarithmos & arcus circulares integrari poteſt, quod ergo etiam de binis noſtris formulis P et Q erit tenendum.

§. 14. Statuamus igitur in noſtris formulis ſupra inuentis $\lambda = \frac{m}{n}$, eaeque transmutabuntur in ſequentes:

$$P = \frac{1}{n b^{\frac{m}{n}} \text{ſin. } n \theta^{\lambda - 1}} \cdot \int \frac{\partial \Phi \text{ ſin. } \Phi^{\frac{m}{n} - 1} \text{ coſ. } (m \theta - \frac{m}{n} \Phi)}{\text{ſin. } \psi},$$

Q =

$$Q = \frac{1}{n\, b^{\frac{m}{n}} \text{ fin. } n\, \theta^{\lambda - 1}} \cdot \int \frac{\partial \Phi \text{ fin. } \Phi^{\frac{m}{n} - 1} \text{ fin. } (m\,\theta - \frac{m}{n}\Phi)}{\text{fin. } \psi},$$

vbi breuitatis gratia loco coëfficientis conſtantis ſcribatur C, et cum ex indole formulae propoſitae ſemper ſit $m < n$, has formulas ita ſuccinctius exhibere licet:

$$P = C \int \frac{\partial \Phi \text{ cof. } (m\,\theta - \frac{m}{n}\Phi)}{\text{fin. } \psi \text{ fin. } \Phi^{\frac{n-m}{n}}} \text{ et}$$

$$Q = C \int \frac{\partial \Phi \text{ fin. } (m\,\theta - \frac{m}{n}\Phi)}{\text{fin. } \psi \text{ fin. } \Phi^{\frac{n-m}{n}}},$$

quae ergo formulae, quicunque numeri pro m et n accipiantur, ſemper a logarithmis et arcubus circularibus pendere ſunt cenſendae.

§. 15. Quodſi binae formulae cof. $(m\,\theta - \frac{m}{n}\Phi)$ et fin. $(m\,\theta - \frac{m}{n}\Phi)$, euoluantur, ambae formulae integrales inuentae commode in vnam contrahi poterunt, quae hanc habebit formam:

$$C \int \partial \Phi \frac{\alpha \text{ fin. } \frac{m}{n}\Phi + \beta \text{ cof. } \frac{m}{n}\Phi}{\text{fin. } \psi \text{ fin. } \Phi^{\frac{n-m}{n}}},$$

quae praefata lege integrationem admittet, quicunque valores litteris α et β tribuantur. Deinde quia $\psi = n\,\theta - \Phi$, facta euolutione loco fin. ψ, eiusue multipli cuiusque, ſcribi poterit γ fin. $\Phi + \delta$ cof. Φ, ſicque nunc formula noſtra erit

$$\int \frac{\partial \Phi}{\text{fin. } \Phi^{\frac{n-m}{n}}} \cdot \frac{\alpha \text{ fin. } \frac{m}{n}\Phi + \beta \text{ cof. } \frac{m}{n}\Phi}{\gamma \text{ fin. } \Phi + \delta \text{ cof. } \Phi},$$

vbi litterae α, β, γ, δ pro lubitu accipi possunt; quamobrem si ad fractiones tollendas statuamus $\Phi = n\,\omega$, vt fit $\frac{m}{n}\Phi = m\,\omega$, ad sequens perducimur theorema notatu dignissimum.

Theorema.

Si litterae m *et* n *denotent numeros integros quoscunque, integratio huius formulae:*

$$\int \frac{\partial\,\omega}{(\text{fin.}\,n\,\omega)^{\frac{n-m}{n}}} \cdot \frac{\alpha\,\text{fin.}\,m\,\omega + \beta\,\text{cof.}\,m\,\omega}{\gamma\,\text{fin.}\,n\,\omega + \delta\,\text{cof.}\,n\,\omega},$$

semper ad logarithmos et arcus circulares reduci potest, quicunque etiam valores litteris α, β, γ, δ *tribuantur.*

§. 16. Quam ardua huius Theorematis demonstratio fit, clarius intelligemus, si hanc formulam ab angulis ad quantitates ordinarias revocemus. Ponamus igitur tang. $\omega = t$, erit $\partial\,\omega = \frac{\partial t}{1 + t\,t}$; deinde si brev. gr. vncias potestatum binomii hoc modo designemus, vt fit

$$(1 + x)^{\lambda} = 1 + \left(\tfrac{\lambda}{1}\right) x + \left(\tfrac{\lambda}{2}\right) x^2 + \left(\tfrac{\lambda}{3}\right) x^3 + \text{etc.}$$

sinus et cosinus angulorum multiplorum ipsius ω sequenti modo per t exprimentur:

$$\text{fin.}\,m\,\omega = \frac{\left(\frac{m}{1}\right) t - \left(\frac{m}{3}\right) t^3 + \left(\frac{m}{5}\right) t^5 - \left(\frac{m}{7}\right) t^7 + \text{etc.}}{(1 + t\,t)^{\frac{m}{2}}}$$

$$\text{cof.}\,m\,\omega = \frac{1 - \left(\frac{m}{2}\right) t^2 + \left(\frac{m}{4}\right) t^4 - \left(\frac{m}{6}\right) t^6 + \text{etc.}}{(1 + t\,t)^{\frac{m}{2}}}$$

Ponamus autem porro breuitatis ergo

$$\left(\tfrac{m}{1}\right) t - \left(\tfrac{m}{3}\right) t^3 + \left(\tfrac{m}{5}\right) t^5 - \text{etc.} = \text{M}$$

$$1 - \left(\tfrac{m}{2}\right) t^2 + \left(\tfrac{m}{4}\right) t^4 - \text{etc.} = \mathfrak{M}$$

fimili

ſimilique modo etiam ponamus:

$$\left(\tfrac{n}{1}\right)t - \left(\tfrac{n}{3}\right)t^3 + \left(\tfrac{n}{5}\right)t^5 - \text{etc.} = \mathrm{N}$$
$$1 - \left(\tfrac{n}{2}\right)t^2 + \left(\tfrac{n}{4}\right)t^4 - \text{etc.} = \mathfrak{N}$$

vt habeamus

$$\text{ſin.}\, m\,\omega = \frac{\mathrm{M}}{(1+tt)^{\frac{m}{2}}},\quad \text{coſ.}\, m\,\omega = \frac{\mathfrak{M}}{(1+tt)^{\frac{m}{2}}},$$
$$\text{ſin.}\, n\,\omega = \frac{\mathrm{N}}{(1+tt)^{\frac{n}{2}}},\quad \text{coſ.}\, n\,\omega = \frac{\mathfrak{N}}{(1+tt)^{\frac{m}{2}}},$$

quibus valoribus ſubſtitutis formula noſtra integralis ſequentem induet formam:

$$\int \frac{(1+tt)^{n-m-1}\,\partial t\,(\alpha\mathrm{M}+\beta\mathfrak{M})}{\mathrm{N}^{\frac{n-m}{n}}\,(\gamma\mathrm{N}+\delta\mathfrak{N})}.$$

Vbi omnia quidem ſunt rationalia, praeter formulam $\mathrm{N}^{\frac{m}{n}}$, quae autem, quia abit in $\left[\left(\tfrac{n}{1}\right)t - \left(\tfrac{n}{3}\right)t^3 + \left(\tfrac{n}{5}\right)t^5 - \text{etc.}\right]^{\frac{m}{n}}$, ſtatim atque n binarium ſuperat, tantopere ſit irrationalis, vt nulla plane via pateat irrationalitatem tollendi, ſi tantum fuerit $n = 3$; multo minus, ſi exponens n magis increſcat, vllo modo reductionem ad rationalitatem ſperare licebit. Interim tamen ſequentem demonſtrationem mihi eruere contigit.

Demonſtratio ſuperioris Theorematis.

§. 17. Ante omnia hic in ſubſidium vocari convenit formulas illas imaginarias, quibus iam ſaepius cum egregio ſucceſſu ſum vſus, quibus pono

$$\cos.\,\omega + \sqrt{-1}\,\sin.\,\omega = p \text{ et } \cos.\,\omega - \sqrt{-1}\,\sin.\,\omega = q,$$

eritque $p\,q = 1$ et $\frac{\partial p}{p} = \partial\,\omega\,\sqrt{-1}$. Deinde vero hinc pro ſinibus et coſinibus angulorum multiplorum habebimus:

$$\sin.\,m\,\omega = \frac{p^m - q^m}{2\sqrt{-1}} = \frac{p^{2m} - 1}{2\,p^m\sqrt{-1}} \text{ et}$$

$$\cos.\,m\,\omega = \frac{p^m + q^m}{2} = \frac{p^{2m} + 1}{2\,p^m},$$

ſimilique modo

$$\sin.\,n\,\omega = \frac{p^{2n} - 1}{2\,p^n\sqrt{-1}} \text{ et } \cos.\,n\,\omega = \frac{p^{2n} + 1}{2\,p^n}.$$

§. 18. Quo ſubſtitutio horum valorum magis ſublevetur, obſeruaſſe juuabit, coëfficientes conſtantes nihil ad integrabilitatem conferre, ideoque vel omitti, vel ſub alia forma referri poſſe. Hanc ob rem ſtatuemus

$$\partial\,\omega = \frac{\partial\,p}{p} \text{ et } \sin.\,n\,\omega = \frac{p^{2n} - 1}{p^n};$$

deinde vero, mutata conſtantium forma, poni poterit

$$\alpha \sin.\,m\,\omega + \beta \cos.\,m\,\omega = \frac{\alpha'\,p^{2m} + \beta'}{p^m},$$

eodemque modo

$$\gamma \sin.\,n\,\omega + \delta \cos.\,n\,\omega = \frac{\gamma'\,p^{2n} + \delta'}{p^n}.$$

His igitur valoribus ſubſtitutis formula noſtra hanc induet formam:

$$\int \frac{p^{2n-2m-1}\,\partial p}{(p^{2n}-1)^{\frac{n-m}{n}}} \cdot \frac{\alpha' p^{2m}+\beta'}{\gamma' p^{2n}+\delta'}.$$

§. 19. Haec iam formula vltro ſe ſcindit in duas partes, quas ita ſeorſim repraeſentemus:

$$\alpha \int \frac{p^{2n-1}\,\partial p}{(p^{2n}-1)^{\frac{n-m}{n}}(\gamma' p^{2n}+\delta')} + \beta' \int \frac{p^{2n-2m-1}\,\partial p}{(p^{2n}-1)^{\frac{n-m}{n}}(\gamma' p^{2n}+\delta')}$$

et nunc non amplius difficile erit, vtramque harum formularum ſeorſim ad rationalitatem reducere. In priore enim tantum opus eſt ſtatuere $p^{2n}-1=x^{2n}$; tum enim erit $p^{2n-1}\,\partial p = x^{2n-1}\,\partial x$ et $(p^{2n}-1)^{\frac{n-m}{n}} = x^{2n-2m}$, ſicque formula prior accipiet hanc formam: $\alpha \int \frac{x^{2m-1}\,\partial x}{\gamma' x^{2n}+\gamma'+\delta'}$, cuius ergo integrale per logarithmos et arcus circulares exhibere licet.

§. 20. Quod vero ad alteram formulam attinet reductio etiam ſe facile offeret, ſi ipſa formula hoc modo repraeſentetur:

$$\beta' \int \frac{\partial p}{p} \cdot \frac{p^{2n-2m}}{(p^{2n}-1)^{\frac{n-m}{n}}(\gamma' p^{2n}+\delta')}, \text{ ſiue}$$

$$\beta' \int \frac{\partial p}{p} \left(\frac{p^2}{(p^{2n}-1)^{\frac{1}{n}}}\right)^{n-m} \cdot \frac{1}{\gamma' p^{2n}+\delta'}.$$

Si enim hic ponatur $\frac{p^2}{(p^{2n}-1)^{\frac{1}{n}}} = y^2$, fiet $\frac{p^{2n}}{p^{2n}-1} = y^{2n}$,

hinc-

hincque $p^{2n} = \frac{y^{2n}}{y^{2n} - 1}$, vnde fumtis logarithmis et differentiando prodit $\frac{\partial p}{p} = \frac{-\partial y}{y(y^{2n} - 1)}$, quibus valoribus fubftitutis ifta formula euadet

$$-\beta' \int \frac{y^{2n-2m-1}\partial y}{(\gamma' + \delta')y^{2n} - \delta'},$$

quae ergo pariter eft rationalis.

§. 21. Hoc igitur modo veritas noftri theorematis fatis firmiter eft demonftrata, atque ifte cafus ita eft comparatus, vt tota formula ope vnius fubftitutionis nullo modo rationalis reddi queat, quae circumftantia eo magis eft notatu digna, quod vulgo ftatui folet, omnes formulas differentiales, quantumuis fuerint irrationales, fi earum integralia per logarithmos et arcus circulares exhiberi poffunt, eas femper ope certae fubftitutionis ad rationalitatem perduci poffe. Nunc igitur videmus hoc effatum ita reftringi debere, vt tantum ad fingulas partes totius formulae propofitae extendatur, quandoquidem fieri poteft, vt quaelibet pars peculiarem fubftitutionem poftulet.

§. 22. Quod fi hanc demonftrationem attentius perpendamus, facile videre licebit, eam ad formulas multo latius patentes extendi poffe. Apparebit enim fequentem formulam multo generaliorem femper ad rationalitatem perduci poffe, id quod in fequente theoremate clarius explicemus.

Theo-

Theorema maxime generale.

§. 23. *Si litterae P et Q denotent functiones quascunque rationales formae* x^n, *istius formulae:*

$$\int \frac{\partial x (P x^{m-1} + Q x^{n-1})}{(a + b x^n)^{\frac{m}{n}}},$$

integrale semper per logarithmos et arcus circulares exprimetur.

Demonstratio.

Secetur, vt supra fecimus, ista formula etiam in duas partes, quae sint

$$\int \frac{P x^{m-1} \partial x}{(a + b x^n)^{\frac{m}{n}}} \text{ et } \int \frac{Q x^{n-1} \partial x}{(a + b x^n)^{\frac{m}{n}}},$$

atque statim patet posteriorem partem rationalem reddi, ponendo $a + b x^n = r^n$; tum enim erit $(a + b x^n)^{\frac{m}{n}} = r^m$, tum vero $x^n = \frac{r^n - a}{b}$ et $x^{n-1} \partial x = \frac{r^{n-1} \partial r}{b}$. Quia nunc Q est functio rationalis ipsius x^n, facta hac substitutione prodibit certe functio rationalis ipsius r^n, sicque pars posterior accipiet hanc formam: $\frac{1}{b} \int Q r^{n-m-1} \partial r$.

Quo prior pars ad rationalitatem reuocetur, statuatur $\frac{x}{(a + b x^n)^{\frac{1}{n}}} = s$, vt fiat $\frac{x^m}{(a + b x^n)^{\frac{m}{n}}} = s^m$, tum vero erit $x^n = \frac{a s^n}{1 - b s^n}$, qui ergo valor si in P loco x^n substituatur, manifesto dabit functionem rationalem ipsius s^n; deinde vero hinc fit $\frac{\partial x}{x} = \frac{\partial s}{s(1 - b s^n)}$, ex quibus valoribus oritur pars nostra prior $= \int \frac{P s^{m-1} \partial s}{1 - b s^m}$, quae ergo formula etiam est rationalis.

Quin etiam in angulis Theorema multo generalius proponi potest, quod ita se habebit:

Theorema generale.

§. 24. *Si litterae P et Q denotent functiones quascunque rationales binarum formularum fin. $2n\omega$ et cof. $2n\omega$, sequens formula integralis semper per logarithmos et arcus circulares expediri poterit:*

$$\int (P \sin. m\omega + Q \cos. m\omega)\, \partial . (\sin. n\omega)^{\frac{m}{n}},$$

vbi notetur esse

$$\partial (\sin. n\omega)^{\frac{m}{n}} = \frac{m \partial \omega \cos. n\omega}{(\sin. n\omega)^{\frac{n-m}{n}}},$$

cuius demonstratio simili modo succedet vt supra, dum pariter ad binas partes peruenietur, quarum vtramque certa substitutione rationalem reddere licebit.

§. 25. Pluribus fortasse displicebit, quod resolutio postremae formulae per substitutiones imaginarias peragatur, cum tamen hic nobis sit propositum imaginaria a realibus separare; plurimum igitur optandum esset vt hoc negotium um fine imaginariis abfolui poffet; verum equidem fateri cogor, me neutiquam perfpicere, quomodo hoc praeftari queat. Ceterum quia reductio Imaginariorum ad realia iam fatis eft exculta, tale remedium non adeo defiderandum videtur. Quin potius hic nouus fe prodit vfus Imaginariorum in ipfa refolutione formularum integralium, dum eiusmodi formulae integrabiles exhiberi poffunt, quarum integralia fine auxilio Imaginariorum eruere non licet.