1795

Evolutio formulae integralis $\int \partial z(3+zz)/((1+zz)\cdot(1+6zz+z^4)^{1/4})$ per logarithmos et arcus circulares

Leonhard Euler

EVOLVTIO FORMVLAE INTEGRALIS

$$\int \frac{\partial z (3 + zz)}{(1 + zz)\sqrt[4]{(1 + 6zz + z^4)}}.$$

PER LOGARITHMOS ET ARCVS CIRCVLARES.

Auctore

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Conuentui exhib. die 26 Mart. 1777.

§. 1.

Videtur hoc nullo alio modo fieri posse, nisi statuatur $z = \frac{1+x}{1-x}$. Hinc autem fiet

$$\partial z = \frac{2\partial x}{(1-x)^2} \text{ et } \frac{3+zz}{1+zz} = \frac{2(1-x+xx)}{1+xx} = \frac{2(1+x^3)}{(1+x)(1+xx)},$$

tum vero

$$1 + 6zz + z^4 = \frac{8(1+x^4)}{(1-x)^4}, \text{ ideoque}$$

$$\sqrt[4]{(1 + 6zz + z^4)} = 2^{\frac{3}{4}} \frac{\sqrt[4]{1 + x^4}}{(1 - x)},$$

quibus substitutis formula proposita induet hanc formam:

$$2^{\frac{5}{4}} \int \frac{\partial x (1 + x^3)}{(1 - x^4)\sqrt[4]{(1 + x^4)}}.$$

§. 2.

§. 2. Discerpamus istam formam in has duas partes: $2^{\frac{5}{4}}(P + Q)$, ita vt sit:

$$P = \int \frac{\partial x}{(1 - x^4)\sqrt[4]{(1 + x^4)}} \text{ et } Q = \int \frac{x^3 \partial x}{(1 - x^4)\sqrt[4]{(1 + x^4)}},$$

quas seorsim euoluamus. Pro priore quidem parte statuamus $\frac{x}{\sqrt[4]{(1 + x^4)}} = t$, vt sit $P = \int \frac{t \partial x}{x(1 - x^4)}$, tum autem fiet $\frac{x^4}{1 + x^4} = t^4$, hincque $x^4 = \frac{t^4}{1 - t^4}$, ergo $1 - x^4 = \frac{1 - 2t^4}{1 - t^4}$; deinde ob $4lx = 4lt - l(1 - t^4)$, erit $\frac{\partial x}{x} = \frac{\partial t}{t(1 - t^4)}$, hocque ergo modo probibit

$$P = \int \frac{\partial t}{1 - 2t^4}.$$

At pro altera parte Q ponatur $1 + x^4 = u^4$, vt fiat $\sqrt{(1 + x^4)} = u$ et $x^3 \partial x = u^3 \partial u$; vnde deducitur:

$$Q = \int \frac{u u \partial u}{2 - u^4},$$

sicque totum negotium ad formulas rationales est reductum.

§. 3. Quo nunc has formulas commodius tractare queamus, pro priore ponamus $t = \frac{p}{\sqrt[4]{2}}$, hocque modo erit

$P = \frac{1}{\sqrt[4]{2}} \int \frac{\partial p}{1 - p^4}$. Nunc vero est

$$\frac{1}{1 - p^4} = \frac{1}{2} \cdot \frac{1}{1 - pp} + \frac{1}{2} \cdot \frac{1}{1 + pp},$$

vnde fit

$$P = \frac{1}{2\sqrt[4]{2}} \int \frac{\partial p}{1 - pp} + \frac{1}{2\sqrt[4]{2}} \int \frac{\partial p}{1 + pp},$$

ideoque integrando:

P =

$$P = \frac{1}{4\sqrt[4]{2}}\, l\, \frac{1+p}{1-p} + \frac{1}{2\sqrt[4]{2}}\, \text{A tang. } p,$$

sicque erit pars prior

$$2^{\frac{5}{4}} P = \tfrac{1}{2}\, l\, \frac{1+p}{1-p} + \text{A tang. } p.$$

Vbi notetur esse $p = t\sqrt[4]{2}$, porro vero $t = \frac{x}{\sqrt[4]{(1+x^4)}}$; et quoniam posuimus $z = \frac{1+x}{1-x}$, erit $x = \frac{z-1}{z+1}$, sicque tota haec pars prior integralis quaesiti per z exprimi poterit.

§. 4. Pro altera parte Q ponatur $u = q\sqrt[4]{2}$, fietque $Q = \frac{1}{\sqrt[4]{2}} \int \frac{qq\,\partial q}{1-q^4}$. Nunc vero est

$$\frac{qq}{1-q^4} = \tfrac{1}{2} \cdot \frac{1}{1-qq} - \tfrac{1}{2} \cdot \frac{1}{1+qq},$$

vnde fiet

$$\int \frac{qq\,\partial q}{1-q^4} = \tfrac{1}{2}\int \frac{\partial q}{1-qq} - \tfrac{1}{2}\int \frac{\partial q}{1+qq} = \tfrac{1}{2}\, l\, \frac{1+q}{1-q} - \tfrac{1}{2}\, \text{A tang. } q.$$

Hoc ergo modo prodit

$$Q = \frac{1}{4\sqrt[4]{2}}\, l\, \frac{1+q}{1-q} - \frac{1}{2\sqrt[4]{2}}\, \text{A tang. } q,$$

consequenter ipsa altera pars integralis erit

$$2^{\frac{5}{4}} Q = \tfrac{1}{2}\, l\, \frac{1+q}{1-q} - \text{A tang. } q.$$

Vbi est $q = \frac{u}{\sqrt[4]{2}}$, porro vero $u = \sqrt[4]{(1+x^4)}$, denique vero, vt vidimus, est $x = \frac{z-1}{z+1}$.

§. 5. Quoniam igitur omnes isti valores sunt cogniti, formulae propositae integrale quaesitum erit

$$\int\frac{\partial z\,(3+zz)}{(1+zz)\sqrt[4]{(1+6zz+z^4)}}+\tfrac{1}{2}\,l\,\frac{1+p}{1-p}+\tfrac{1}{2}\,l\,\frac{1+q}{1-q}$$

$$+\,A\ \text{tang.}\,p-A\ \text{tang.}\,q,$$

vbi notetur esse

$$p=\frac{z-1}{\sqrt[4]{(1+6zz+z^4)}}\ \text{et}\ q=\sqrt[4]{(1+6zz+z^4)}.$$

§. 6. His ergo valoribus substitutis nostrum integrale erit

$$\tfrac{1}{2}\,l\,\frac{\sqrt[4]{(1+6zz+z^4)}+z-1}{\sqrt[4]{(1+6zz+z^4)}-z+1}+\tfrac{1}{2}\,l\,\frac{1+\sqrt[4]{(1+6zz+z^4)}}{1-\sqrt[4]{(1+6zz+z^4)}}$$

$$+\,A\ \text{tang.}\,\frac{z-1}{\sqrt[4]{(1+6zz+z^4)}}-A\ \text{tang.}\,\sqrt[4]{(1+6zz+z^4)}.$$

Vbi notetur ambos arcus circulares ita in vnum colligi posse, vt prodeat

$$A\ \text{tang.}\,\frac{z-1-\sqrt{(1+6zz+z^4)}}{z\sqrt[4]{(1+6zz+z^4)}}.$$

Ambo autem logarithmi ita in vnum colligi poterunt:

$$\tfrac{1}{2}\,l\,\frac{z\sqrt[4]{(1+6zz+z^4)}+z-1+\sqrt{(1+6zz+z^4)}}{z\sqrt[4]{(1+6zz+z^4)}-z+1-\sqrt{(1+6zz+z^4)}}.$$

§. 7.

§. 7. Haec adhuc commodius exprimi poterunt. Si enim breuitatis gratia ponamus $\sqrt[4]{(1+6zz+z^4)}=v$, pars logarithmica noſtri integralis erit

$$\frac{1}{2}\,l\,\frac{vz+z-1+vv}{vz-z+1-vv}=\frac{1}{2}\,l\,\frac{(1+v)(z-1+v)}{(v-1)(z-1-v)};$$

altera vero pars circularis eſt

$$\text{A tang.}\,\frac{z-1-vv}{vz}.$$