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Integratio formulae differentialis maxime irrationalis, quam tamen per logarithmos et arcus circulares expedire licet

Leonhard Euler

INTEGRATIO FORMVLAE DIFFERENTIALIS MAXIME IRRATIONALIS, QVAM TAMEN PER LOGARITHMOS ET ARCVS CIRCVLARES EXPEDIRE LICET.

Auctore
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Problema.

Proposita hac formula differentiali:

$$\partial V = \frac{\partial z (1 - zz)^2}{(1 + zz) \sqrt[4]{(1 + 6zz + z^4)^3}},$$

eius integrale, per logarithmos et arcus circulares expressum, inuenire.

Solutio.

§. 1. Ponatur breuitatis gratia $\sqrt[4]{(1 + 6zz + z^4)} = v$, vt formula proposita fit $\partial V = \frac{\partial z (1 - zz)^2}{(1 + zz) v^3}$; et nunc loco z binae variabiles p et q in calculum introducantur, ponendo $p = \frac{1+z}{v}$ et $q = \frac{1-z}{v}$, eritque $p^4 + q^4 = 2$ ideoque $p^3 \partial p + q^3 \partial q = 0$.

Porro

Porro vero erit $p + q = \frac{2}{v}$ et $p - q = \frac{2z}{v}$, hincque fiet $\frac{p-q}{p+q} = z$, existente $v = \frac{2}{p+q}$.

§. 2. Cum igitur sit $z = \frac{p-q}{p+q}$, erit $\partial z = \frac{2(q\partial p - p\partial q)}{(p+q)^2}$, vbi ob $p + q = \frac{2}{v}$ erit $\partial z = \frac{1}{2} vv(q\partial p - p\partial q)$. Deinde vero, quia est $\partial p = -\frac{q^3 \partial q}{p^3}$, erit $q\partial p - p\partial q = -\frac{2\partial q}{p^3}$. Simili modo, posito $\partial q = -\frac{p^3 \partial p}{q^3}$, erit $q\partial p - p\partial q = \frac{2\partial p}{q^3}$, propterea quod $p^4 + q^4 = 2$; sicque elementum ∂z duplici modo, scilicet per ∂p et per ∂q habebimus expressum, eritque primo $\partial z = -\frac{vv\partial q}{p^3}$, tum vero $\partial z = +\frac{vv\partial p}{q^3}$, quam duplicem expressionem per $\partial z = vv\partial\omega$ repraesentemus, existente vel $\partial\omega = -\frac{\partial q}{p^3}$, vel $\partial\omega = +\frac{\partial p}{q^3}$.

§. 3. Deinde vero ob $1 + z = pv$ et $1 - z = qv$ erit $1 - zz = pqvv$, ideoque $(1 - zz)^2 = p^2 q^2 v^4$; sicque numerator nostrae formulae erit $\partial z(1 - zz)^2 = v^6 ppqq\partial\omega$. Pro denominatore autem habebimus $1 + zz = \frac{1}{2}(pp + qq)vv$, ita vt iam totus denominator fit $\frac{1}{2}v^5(pp + qq)$, quocirca ipsa formula nostra proposita ita repraesentabitur:

$$\partial V = \frac{2v\,ppqq\partial\omega}{pp+qq} = \frac{4ppqq\partial\omega}{(p+q)(pp+qq)}.$$

Multiplicemus autem porro supra et infra per $p - q$, vt prodeat ista forma:

$$\partial V = \frac{4(p-q)ppqq\partial\omega}{p^4 - q^4}.$$

§. 4. Quoniam nunc numerator ex duabus partibus constat, vtramque seorsim euoluamus. Pars igitur prior, quae est $\frac{4p^3 qq\partial\omega}{p^4 - q^4}$, si loco $\partial\omega$ valorem priorem supra datum scribamus, scilicet $\partial\omega = -\frac{\partial q}{p^3}$, erit $= -\frac{4qq\partial q}{p^4 - q^4}$, quamobrem

ſi

fi hic in denominatore pro p fcribamus eius valorem $2-q^4$, ifta pars erit per folam variabilem q ita expreffa: $-\frac{2qq\partial q}{1-q^4}$. Simili modo altera noftrae formulae pars $-\frac{4ppq^3\partial\omega}{p^4-q^4}$, fi loco $\partial\omega$ fcribamus valorem $+\frac{\partial p}{q^3}$, induet hanc formam: $-\frac{4pp\partial p}{p^4-q^4}$. Hic igitur loco q^4 fcribatur $2-p^4$, ac pars ifta iam per folam variabilem p exprimetur, fietque $=+\frac{2pp\partial p}{1-p^4}$, confequenter ipfa formula propofita reducta eft ad has partes:

$$\partial V=\frac{2pp\partial p}{1-p^4}-\frac{2qq\partial q}{1-q^4},$$

quae non folum funt rationales, fed etiam binas variabiles p et q penitus feparatas inuoluunt.

§. 5. Ad integrale igitur inueniendum notetur effe $\frac{2pp}{1-p^4}=\frac{1}{1-pp}-\frac{1}{1+pp}$, vnde erit prioris partis integrale

$$\int\frac{2pp\partial p}{1-p^4}=\int\frac{\partial p}{1-pp}-\int\frac{\partial p}{1+pp}=\frac{1}{2}l\frac{1+p}{1-p}-A\,\text{tang.}\,p,$$

eodemque modo altera pars erit

$$\int\frac{2qq\partial q}{1-q^4}=\frac{1}{2}l\frac{1+q}{1-q}-A\,\text{tang.}\,q,$$

quamobrem totum integrale quaefitum erit

$$V=\frac{1}{2}l\frac{1+p}{1-p}-\frac{1}{2}l\frac{1+q}{1-q}+A\,\text{tang.}\,q-A\,\text{tang.}\,p.$$

§. 6. Reftituamus nunc loco p et q valores affumtos, fcilicet $p=\frac{1+z}{v}$ et $q=\frac{1-z}{v}$, eritque

$$V=\frac{1}{2}l\frac{v+1+z}{v-1-z}-\frac{1}{2}l\frac{v+1-z}{v-1+z}+A\,\text{tang.}\,\frac{1-z}{v}-A\,\text{tang.}\,\frac{1+z}{v},$$

vbi cum fit

$$A\,\text{tang.}\,a-A\,\text{tang.}\,b=A\,\text{tang.}\,\frac{a-b}{1+ab},\text{ erit}$$

$$A\,\text{tang.}\,\frac{1-z}{v}-A\,\text{tang.}\,\frac{1+z}{v}=-A\,\text{tang.}\,\frac{2vz}{vv+1-zz}.$$

Deinde etiam logarithmi inuicem combinari poffunt et refultabit

$$V=\frac{1}{2}l\frac{(v+1+z)(v-1+z)}{(v-1-z)(v+1-z)}-A\,\text{tang.}\,\frac{2vz}{vv+1-zz}.$$

Quin

Quin etiam vtile erit logarithmos hoc modo iterum separare, vt sit

$$V = \tfrac{1}{2}\, l\, \frac{v+1+z}{v+1-z} + \tfrac{1}{2}\, l\, \frac{v-1+z}{v-1-z} - \text{A tang.}\, \frac{2vz}{vv+1-zz}.$$

§. 7. Hactenus constantem per integrationem addendam negleximus; eam igitur nunc ita definiamus, vt posito $z = 0$ ipsum integrale V euanescat. Hunc in finem consideremus z tanquam minimum, et quia $v = (1 + 6zz + z^4)^{\frac{1}{4}}$, erit $v = 1 + \frac{3}{2}zz$; euidens autem est huius particulae minimae $\frac{3}{2}zz$ rationem tantum in posteriore logarithmo esse habendam, quoniam in eo occurrit $v - 1$. Hoc igitur valore substituto erit nunc nostrum integrale

$$V = \tfrac{1}{2}\, l\, \frac{2+z}{2-z} + \tfrac{1}{2}\, l\, \frac{\frac{3}{2}zz + z}{\frac{3}{2}zz - z} - \text{A tang.}\, \frac{2z}{2-zz}.$$

§. 8. Hic iam quia z est quantitas minima, erit $\frac{1}{2}\, l\, \frac{2+z}{2-z} = \frac{z}{2}$; alter vero logarithmus erit $\frac{1}{2}\, l\, \frac{3z+2}{3z-2}$, vbi loco constantis adiici debet $l-1$, vt haec altera pars fiat $\frac{1}{2}\, l\, \frac{2+3z}{2-3z}$, cuius valor erit $\frac{3z}{2}$, ita vt ambo logarithmi iunctim praebeant $2z$. Deinde vero ex arcu circulari fit A tang. $\frac{2z}{2-zz} =$ A tang. $z = z$, ita vt tota formula praebeat $V = 2z - z = z$, qui valor cum formula proposita egregie conspirat; posito enim z infinite paruo habetur $\partial V = \partial z$, ideoque $V = z$.

§. 9. Quoniam igitur constans addenda reperta est $l - 1$, in superiori expressione loco $l(v - 1 - z)$ scribamus $l(1 + z - v)$, vt iam totum integrale rite determinatum fit:

$$V = \tfrac{1}{2}\, l\, \frac{v+1+z}{v+1-z} + \tfrac{1}{2}\, l\, \frac{v+z-1}{1+z-v} - \text{A tang.}\, \frac{2vz}{1+vv-zz},$$

qui valor euanescit sumto $z = 0$.

Problema 2.

Proposita hac formula differentiali:

$$\partial V = \frac{\partial z (1 + zz)^2}{(1 - zz) \sqrt[4]{(1 - \beta zz + z^4)^3}},$$

eius integrale per logarithmos et arcus circulares inuestigare.

Solutio.

§. 10. Solutio huius problematis vix aliter erui posse videtur, nisi ex praecedente solutione deriuetur. Consideremus igitur formulam prioris problematis hac ratione repraesentatam: $\int \frac{\partial y (1 - yy)^2}{(1 + yy) \sqrt[4]{(1 + 6yy + y^4)^3}} = U$, et facta debita immutatione positoque $\sqrt[4]{(1 + 6yy + y^4)} = u$, integrale ita erit expressum:

$$U = \tfrac{1}{2} l \frac{u + 1 + y}{u + 1 - y} + \tfrac{1}{2} l \frac{u - 1 + y}{u - 1 - y} - \text{A tang.} \frac{2uy}{uu + 1 - yy}.$$

§. 11. In hac forma ponamus $y = z\sqrt{-1}$, et statuamus formulam radicalem hinc natam

$$\sqrt[4]{(1 - 6zz + z^4)} = v,$$

quo facto erit

$$U = \int \frac{\partial z \sqrt{-1}\,(1 + zz)^2}{(1 - zz) \sqrt[4]{(1 - 6zz + z^4)^3}},$$

vnde patet esse $U = V\sqrt{-1}$, ita vt inuento valore U eruatur valor quaesitus $V = \frac{U}{\sqrt{-1}} = -U \cdot \sqrt{-1}$. Posito autem $y = z\sqrt{-1}$ et loco u scripto v, integrale U accipiet hanc formam:

U =

$$U = \frac{1}{2} l \frac{v+1+z\sqrt{-1}}{v+1-z\sqrt{-1}} + \frac{1}{2} l \frac{v-1+z\sqrt{-1}}{v-1-z\sqrt{-1}} - \text{A tang.} \frac{2vz\sqrt{-1}}{vv+1+zz}.$$

Totum igitur negotium huc redit, vt isti logarithmi imaginarii cum arcu imaginario ad realitatem reducantur, id quod sequenti modo commodissime perficietur.

§. 12. In subsidium vocetur istud Lemma satis notum:

$$l\, a + b\sqrt{-1} = \frac{1}{2} l (aa + bb) + \sqrt{-1}\ \text{A tang.} \frac{b}{a},$$

vbi ergo, si loco b scribamus $-b$, erit

$$l\, a - b\sqrt{-1} = \frac{1}{2} l (aa + bb) - \sqrt{-1}\ \text{A tang.} \frac{b}{a},$$

quae forma a praecedente subtracta nobis dat

$$l \frac{a+b\sqrt{-1}}{a-b\sqrt{-1}} = 2\sqrt{-1}\ \text{A tang.} \frac{b}{a},$$

atque si hic faciamus $b = c\sqrt{-1}$, erit

$$l \frac{a-c}{a+c} = 2\sqrt{-1}\ \text{A tang.} \frac{c\sqrt{-1}}{a},$$

hincque vicissim

$$\text{A tang.} \frac{c\sqrt{-1}}{a} = \frac{1}{2\sqrt{-1}} l \frac{a-c}{a+c} = \frac{1}{2}\sqrt{-1}\ l \frac{a+c}{a-c}.$$

§. 13. Iam pro priore logarithmo imaginario erit $a = v + 1$ et $b = z$, vnde habebimus:

$$l \frac{v+1+z\sqrt{-1}}{v+1-z\sqrt{-1}} = 2\sqrt{-1}\ \text{A tang.} \frac{z}{v+1}.$$

Pro altero vero logarithmo erit $a = v - 1$ et $b = z$, hincque

$$l \frac{v-1+z\sqrt{-1}}{v-1-z\sqrt{-1}} = 2\sqrt{-1}\ \text{A tang.} \frac{z}{v-1}.$$

Denique pro arcu erit $c = 2vz$ et $a = vv + 1 + zz$, vnde colligitur:

$$\text{A tang.} \frac{2vz\sqrt{-1}}{vv+1+zz} = \frac{1}{2}\sqrt{-1}\ l \frac{(v+z)^2+1}{(v-z)^2+1},$$

quibus valoribus substitutis erit

 U =

$$U = \sqrt{-1}\,\text{A tang.}\,\frac{z}{v+1} + \sqrt{-1}\,\text{A tang.}\,\frac{z}{v-1}$$
$$-\tfrac{1}{2}\sqrt{-1}\; l\,\frac{1+(v+z)^2}{1+(v-z)^2},$$

qui valor ductus in $-\sqrt{-1}$ praebet ipsum valorem quaesitum:

$$V = -\tfrac{1}{2}\, l\,\frac{1+(v+z)^2}{1+(v-z)^2} + \text{A tang.}\,\frac{z}{v+1} + \text{A tang.}\,\frac{z}{v-1},$$

quae expressio, arcubus in vnum contractis, transmutatur in hanc:

$$V = \tfrac{1}{2}\, l\,\frac{1+(v-z)^2}{1+(v+z)^2} - \text{A tang.}\,\frac{2vz}{1+zz-vv}.$$

§. 14. Haec solutio eo magis est notatu digna, quod per imaginaria est traducta, atque adeo nulla via patere videtur eam directe inueniendi. Fortasse autem si forma integralis inuenta probe perpendatur, inde methodus excogitare poterit, cuius ope sine subsidio imaginariorum ista solutio directe elici queat, hocque argumentum vtique dignum videtur, in quo Geometrae sagacitatem suam exerceant.

§. 15. At vero, quoniam nulla via patet, integrale posterioris formulae directe inueniendi, operae pretium erit rursus ex aequatione integrali differentialem propositam elicere, visuri, num forsan haec operatio nobis inseruire possit, aliam resolutionem detegendi, quam per imaginaria progrediendo, quem in finem sequens problema coronidis loco adiungamus.

Problema.

Inuenire differentiale huius expressionis:

$$V = \tfrac{1}{2}\, l\,\frac{1+zz+vv-2vz}{1+zz+vv+2vz} - \text{A tang.}\,\frac{2vz}{1+zz-vv},$$

existente $v = \sqrt[4]{(1-6zz-z^4)}$.

Solu-

Solutio.

§. 16. Ponatur

$$\frac{2vz}{1+zz+vv} = p \text{ et } \frac{2vz}{1+zz-vv} = q,$$

eritque

$$V = \frac{1}{2} l \frac{1-p}{1+p} - A \text{ tang. } q,$$

vnde fit differentiando:

$$\partial V = -\frac{\partial p}{1-pp} - \frac{\partial q}{1+qq}.$$

Eſt vero

$$\partial p = \frac{2v\partial z(1-zz+vv)+2z\partial v(1+zz-vv)}{(1+zz+vv)^2} \text{ et}$$

$$\partial q = \frac{2v\partial z(1-zz-vv)+2z\partial v(1+zz+vv)}{(1+2zz-vv)^2}.$$

Deinde $1-pp = \frac{1+2zz+2vv+z^4-2vvzz+v^4}{(1+zz+vv)^2}$,

ſiue ob $v^4 = 1-6zz+z^4$, erit

$$1-pp = \frac{2(1-zz)(1-zz+vv)}{(1+zz+vv)^2},$$

eodemque modo

$$1+qq = \frac{2(1-zz)(1-zz-vv)}{(1+zz-vv)^2},$$

quibus ſubſtitutis fit

$$\partial V = -\frac{v\partial z(1-zz+vv)-z\partial v(1+zz-vv)}{(1-zz)(1-zz+vv)}$$
$$-\frac{v\partial z(1-zz-vv)-z\partial v(1+zz+vv)}{(1-zz)(1-zz-vv)}.$$

§. 17. Reducatur haec expreſſio ad eundem denominatorem:

$$(1-zz)[(1-zz)^2-v^4] = 4zz(1-zz),$$

fietque

$$\partial V = -\frac{2v\partial z[(1-zz)^2-v^4]-z\partial v[(1-vv)^2+(1+vv)^2-2z^4]}{4zz(1-zz)}.$$

Haec forma porro reducitur ad hanc:

$$\partial V = -\frac{2vz\partial z-\partial v(1-3zz)}{z(1-zz)},$$

quae ſi ſupra et infra per v^3 multiplicetur, ob

 $v^4 =$

$$v^4 = 1 - 6zz + z^4 \text{ et}$$

$$v^3 \partial v = -3z\partial z + z^3 \partial z = z\partial z(zz - 3),$$

abibit in sequentem formam:

$$\partial V = -\frac{2\partial z(1 - 6zz + z^4) - \partial z(zz - 3)(1 - 3zz)}{(1 - zz)v^3},$$

qua euoluta prodit denique

$$\partial V = \frac{\partial z(1 + zz)^2}{(1 - zz)\sqrt[4]{(1 - 6zz + z^4)^3}}.$$

Haec igitur formula cum proposita formula differentiali superioris problematis prorsus conuenit, ita vt certi simus huius formulae integrale reuera esse

$$V = \frac{1}{2} l \frac{1 + (v - z)^2}{1 + (v + z)^2} - \text{A tang.} \frac{2xz}{1 + zz - vv},$$

etiamsi non pateat, quomodo hoc integrale methodo directa elici queat.

EVO-