1795

Specimen integrationis abstrusissimae hac formula $\int \partial x/((1+x)\cdot(2xx-1)^{1/4})$ contentae

Leonhard Euler

SPECIMEN INTEGRATIONIS ABSTRVSISSIMAE HAC FORMVLA

$$\int \frac{\partial x}{(1+x)\sqrt[4]{(2xx-1)}}$$

CONTENTAE.

Auctore

L. EVLERO.

Conuentui exhib. die 26 Mart. 1777.

§. 1.

Quamquam haec formula non adeo complicata videtur, tamen non dubito affeuerare, vix quemquam fore, qui, poftquam omni cura et fagacitate eius refolutionem fuerit aggreffus, tandem non agnofcere debeat, fe oleum et operam perdidiffe. Facile quidem foret iftam formulam a figno radicali biquadratico liberare, ponendo $2xx = \frac{(tt+1)^2}{(tt-1)^2}$, vnde fieret $\sqrt[4]{(2xx-1)} = \sqrt{\frac{2t}{tt-1}}$; tum autem foret $x = \frac{tt+1}{(tt-1)\sqrt{2}}$, ideoque $\partial x = -\frac{2t\partial t\sqrt{2}}{(tt-1)^2}$ et

$$1+x = \frac{tt(1+\sqrt{2})+1-\sqrt{2}}{(tt-1)\sqrt{2}},$$

quibus fubftitutis formula propofita abit in hanc:

$$-\frac{2\partial t}{(1+\sqrt{2})tt+1-\sqrt{2}}\sqrt{\frac{2t}{tt-1}}.$$

Haec

Haec formula autem ita eft comparata, vt dubitem, eius integrale vllo alio modo erui poffe, nifi per praecedentem formam regrediendo, atque omnes operationes inftituendo, quas hic fum expofiturus, quae tandem, praeter omnem expectationem, ad integrale per logarithmos et arcus circulares expreffum perducent.

§. 2. Praecipua fubftitutio, qua via ad refolutionem fternetur, in hoc confiftit, vt ponam $x = \frac{yy-3}{4}$; tum enim erit $2xx = \frac{y^4 - 6yy + 9}{8}$, vnde fit

$2xx - 1 = \frac{y^4 - 6yy + 1}{8}$;

deinde vero erit $1 + x = \frac{yy+1}{4}$, hincque $\frac{\partial x}{1+x} = \frac{2y\partial y}{yy+1}$. Quamobrem fi ipfam formulam propofitam per V defignemus, erit facta hac fubftitutione

$$V = 2^{\frac{7}{4}} \int \frac{y \partial y}{(yy+1)\sqrt[4]{(y^4 - 6yy + 1)}}.$$

§. 3. Verum ne haec formula tractari poteft, nifi per imaginaria tranfeundo; poni enim oportet $y = z\sqrt{-1}$, vt oriatur ifta forma:

$$V = -2^{\frac{7}{4}} \int \frac{z \partial z}{(1 - zz)\sqrt[4]{(z^4 + 6zz + 1)}},$$

quam iam fingulari illa methodo, cuius aliquot fpecimina non ita pridem dedi, tractare licebit. Pono igitur br. gr. $\sqrt[4]{(1 + 6zz + z^4)} = v$, vt formula refoluenda fit

$$\int \frac{z \partial z}{v(1 - zz)} = Z,$$

indeque porro $V = -2^{\frac{7}{4}} Z$, ad quam formulam refoluendam intro-

introduco duas nouas variabiles p et q, ſtatuendo $p = \frac{1+z}{v}$ et $q = \frac{1-z}{v}$, vnde ſtatim fit

$$p^4 + q^4 = \frac{2 + 12zz + 2z^4}{v^4} = 2.$$

§. 4. Praeterea vero ex binis formulis aſſumtis erit primo $p + q = \frac{2}{v}$ et $p - q = \frac{2z}{v}$, vnde colligitur $z = \frac{p-q}{p+q}$, hinc differentiando erit $\partial z = \frac{2(q\partial p - p\partial q)}{(p+q)^2}$, vbi loco $p+q$ ſcribamus valorem $\frac{2}{v}$, fietque

$$\partial z = \tfrac{1}{2} vv(q\partial p - p\partial q);$$

porro vero habebimus $1 - zz = pqvv$. His valoribus ſubſtitutis erit

$$Z = \tfrac{1}{2}\int \frac{(p-q)(q\partial p - p\partial q)}{pqv(p+q)},$$

quae formula, poſito loco v valore $\frac{2}{p+q}$, abit in hanc:

$$Z = \tfrac{1}{4}\int \frac{(p-q)(q\partial p - p\partial q)}{pq} = \tfrac{1}{4}\int (p-q)\left(\frac{\partial p}{p} - \frac{\partial q}{q}\right).$$

§. 5. Facta ergo euolutione habebimus:

$$4\partial Z = \partial p + \partial q - \frac{q\partial p}{p} - \frac{p\partial q}{q}.$$

Cum autem ſit $p^4 + q^4 = 2$, erit $p^4 = 2 - q^4$ et hinc $\frac{\partial p}{p} = -\frac{q^3\partial q}{2-q^4}$, ſimilique modo $\frac{\partial q}{q} = -\frac{p^3\partial p}{2-p^4}$; hincque colligitur:

$$-\frac{q\partial p}{p} = \frac{q^4\partial q}{2-q^4} \text{ et } -\frac{p\partial q}{q} = \frac{p^4\partial p}{2-p^4},$$

quocirca nanciſcimur

$$4\partial Z = \frac{2\partial p}{2-p^4} + \frac{2\partial q}{2-q^4};$$

ſicque pertigimus ad binas formulas differentiales, in quibus binae variabiles p et q a ſe inuicem ſunt ſeparatae, conſequenter pro Z habebimus ſequentem expreſſionem:

$$Z = \tfrac{1}{2}\int \frac{\partial p}{2-p^4} + \tfrac{1}{2}\int \frac{\partial q}{2-q^4},$$

vnde

vnde iam manifeftum eft valorem Z per logarithmos et arcus circulares exprimi poffe.

§. 6. Ponamus enim $p = r\sqrt[4]{2}$, vt fiat

$$\frac{\partial p}{2 - p^4} = \frac{1}{2^{\frac{3}{4}}} \cdot \frac{\partial r}{1 - r^4}.$$

Conftat autem effe

$$\int \frac{\partial r}{1 - r^4} = \frac{1}{4} l \frac{1 + r}{1 - r} + \frac{1}{2} \text{A tang. } r,$$

hincque adeo erit

$$\int \frac{\partial p}{2 - p^4} = \frac{1}{4 \cdot 2^{\frac{3}{4}}} l \frac{\sqrt[4]{2} + p}{\sqrt[4]{2} - p} + \frac{1}{2 \cdot 2^{\frac{3}{4}}} \text{A tang. } \frac{p}{\sqrt[4]{2}};$$

quod cum fimili modo fe habeat cum altera parte $\int \frac{\partial q}{2 - q^4}$, reperimus tandem

$$Z = \frac{1}{8 \cdot 2^{\frac{3}{4}}} l \frac{\sqrt[4]{2} + p}{\sqrt[4]{2} - p} + \frac{1}{4 \cdot 2^{\frac{3}{4}}} \text{A tang. } \frac{p}{\sqrt[4]{2}}$$

$$+ \frac{1}{8 \cdot 2^{\frac{3}{4}}} l \frac{\sqrt[4]{2} + q}{\sqrt[4]{2} - q} + \frac{1}{4 \cdot 2^{\frac{3}{4}}} \text{A tang. } \frac{q}{\sqrt[4]{2}};$$

vbi tantum opus eft loco p et q valores affumtos reftituere, qui funt $p = \frac{1 + z}{v}$ et $q = \frac{1 - z}{v}$.

§. 7. Cum iam ipfum integrale quaefitum fit $V = - 2^{\frac{7}{4}} . Z$, erit nunc

$$V = -\tfrac{1}{4} l \frac{\sqrt[4]{2}+p}{\sqrt[4]{2}-p} - \text{A tang.} \frac{p}{\sqrt[4]{2}}$$

$$-\tfrac{1}{4} l \frac{\sqrt[4]{2}+q}{\sqrt[4]{2}-q} - \text{A tang.} \frac{q}{\sqrt[4]{2}},$$

atque ſi loco p et q ſcribantur valores aſſignati, prodibit

$$V = -\tfrac{1}{4} l \frac{v\sqrt[4]{2}+1+z}{v\sqrt[4]{2}-1-z} - \tfrac{1}{4} l \frac{v\sqrt[4]{2}+1-z}{v\sqrt[4]{2}-1+z}$$

$$-\tfrac{1}{2}\,\text{A tang.} \frac{1+z}{v\sqrt[4]{2}} - \tfrac{1}{2}\,\text{A tang.} \frac{1-z}{v\sqrt[4]{2}}.$$

Hic primum obſeruo, ambos arcus circulares commode in vnum contrahi poſſe ope formulae

$$\text{A tang.}\, a + \text{A tang.}\, b = \text{A tang.} \frac{a+b}{1-ab},$$

quo facto erit

$$V = -\tfrac{1}{4} l \frac{v\sqrt[4]{2}+1+z}{v\sqrt[4]{2}-1-z} - \tfrac{1}{4} l \frac{v\sqrt[4]{2}+1-z}{v\sqrt[4]{2}-1+z}$$

$$-\tfrac{1}{2}\,\text{A tang.} \frac{2v\sqrt[4]{2}}{vv\sqrt{2}-1+zz}.$$

§. 8. Simili modo etiam logarithmos tam numeratorum quam denominatorum in vnum contrahere licet, eritque.

$$l[v\sqrt[4]{2}+1+z] + l[v\sqrt[4]{2}+1-z] = l[(v\sqrt[4]{2}+1)^2 - zz],$$

$$l[v\sqrt[4]{2}-1-z] + l[v\sqrt[4]{2}-1+z] = l[(v\sqrt[4]{2}-1)^2 - zz],$$

hinc-

hincque habebimus sequentem formam:

$$V = -\tfrac{1}{4} l \frac{(v\sqrt[4]{2}+1)^2 - zz}{(v\sqrt[4]{2}-1)^2 - zz} - \tfrac{1}{2} \text{A tang.} \frac{2v\sqrt[4]{2}}{vv\sqrt{2}-1+zz}.$$

Vbi notetur esse $v = \sqrt[4]{(1+6zz+z^4)}$.

§. 9. Nunc vlterius regrediamur, et quia posuimus $y = z\sqrt{-1}$, fiet nunc $zz = -yy$, et iam erit

$$v = \sqrt[4]{(1-6yy+y^4)},$$

hincque per y integrale quaesitum hoc modo exprimetur:

$$V = -\tfrac{1}{4} l \frac{(v\sqrt[4]{2}+1)^2 + yy}{(v\sqrt[4]{2}-1)^2 + yy} - \tfrac{1}{2} \text{A tang.} \frac{2v\sqrt[4]{2}}{vv\sqrt{2}-1-yy}.$$

§. 10. Quoniam igitur posueramus $x = \frac{yy-3}{4}$, erit $yy = 4x+3$, eratque

$$\sqrt[4]{(2xx-1)} = \sqrt[4]{\frac{1-6yy+y^4}{8}} = \frac{v}{\sqrt[4]{8}} = \frac{v\sqrt[4]{2}}{2},$$

vnde fit

$$v\sqrt[4]{2} = 2\sqrt[4]{(2xx-1)},$$

quibus valoribus substitutis integrale quaesitum erit

$$V = -\tfrac{1}{4} l \frac{[2\sqrt[4]{(2xx-1)}+1]^2 + 4x+3}{[2\sqrt[4]{(2xx-1)}-1]^2 + 4x+3}$$

$$-\tfrac{1}{2} \text{A tang.} \frac{4\sqrt[4]{(2xx-1)}}{4\sqrt{(2xx-1)}-4x-4}.$$

Facta

Facta autem euolutione reperietur:

$$V = -\tfrac{1}{4} l \frac{1 + x + \sqrt{(2xx-1)} + \sqrt[4]{(2xx-1)}}{1 + x + \sqrt{(2xx-1)} - \sqrt[4]{(2xx-1)}}$$

$$-\tfrac{1}{2} \text{A tang.} \frac{\sqrt[4]{(2xx-1)}}{\sqrt{(2xx-1)} - x - 1},$$

quae expressio etiam ita referri potest:

$$V = +\tfrac{1}{4} l \frac{1 + x + \sqrt{(2xx-1)} - \sqrt[4]{(2xx-1)}}{1 + x + \sqrt{(2xx-1)} + \sqrt[4]{(2xx-1)}}$$

$$+\tfrac{1}{2} \text{A tang.} \frac{\sqrt[4]{(2xx-1)}}{1 + x - \sqrt{(2xx-1)}}.$$

Hunc igitur valorem operae pretium erit per sequens theorema in medium proferre.

Theorema.

Proposita hac formula differentiali:

$$\partial V = \frac{\partial x}{(1+x)\sqrt[4]{(2xx-1)}},$$

eius integrale sequenti modo per logarithmos et arcus circulares exprimetur:

$$V = +\tfrac{1}{4} l \frac{1 + x + \sqrt{(2xx-1)} - \sqrt[4]{(2xx-1)}}{1 + x + \sqrt{(2xx-1)} + \sqrt[4]{(2xx-1)}}$$

$$+\tfrac{1}{2} \text{A tang.} \frac{\sqrt[4]{(2xx-1)}}{1 + x - \sqrt{(2xx-1)}}.$$

Corol-

Corollarium 1.

Hinc ergo si loco x scribamus $-x$, erit

$$\int \frac{\partial x}{(x-1)\sqrt[4]{(2xx-1)}} = \frac{1}{4} l \frac{1-x+\sqrt{(2xx-1)}-\sqrt[4]{(2xx-1)}}{1-x+\sqrt{(2xx-1)}+\sqrt[4]{(2xx-1)}}$$

$$+ \frac{1}{2} \text{A tang.} \frac{\sqrt[4]{(2xx-1)}}{1+x-\sqrt{(2xx-1)}}.$$

Corollarium 2.

Hae integrationes eo magis sunt notatu dignae, quod formulam differentialem non generaliorem admittant. Ita haec formula differentialis: $\frac{\partial x}{(1+\alpha x)\sqrt[4]{(\beta xx-1)}}$ integrationem haud admittit, nisi casibus $\alpha = \pm 1$ et $\beta = 2$, vel generalius, nisi fuerit $\beta = 2\alpha\alpha$.

§. 11. Ipsam autem formulam nostram integralem pluribus modis transformare licet, vt signum radicale biquadraticum elidatur. Commodissime hoc praestabitur, ponendo $\sqrt[4]{(2xx-1)} = s$, vnde fit $2xx = 1+s^4$, consequenter $x = \sqrt{\frac{1+s^4}{2}}$, hincque $\partial x = \frac{s^3 \partial s \sqrt{2}}{\sqrt{1+s^4}}$, quo valore substituto erit

$$V = \int \frac{2ss\partial s}{[\sqrt{2}+\sqrt{(1+s^4)}]\sqrt{(1+s^4)}},$$

cuius ergo formulae integrale erit

$$V = \frac{1}{4} l \frac{\sqrt{2}+\sqrt{(1+s^4)}+ss\sqrt{2}-s\sqrt{2}}{\sqrt{2}+\sqrt{(1+s^4)}+ss\sqrt{2}+s\sqrt{2}}$$

$$+ \frac{1}{2} \text{A tang.} \frac{s\sqrt{2}}{\sqrt{2}+\sqrt{(1+s^4)}-ss\sqrt{2}}.$$

§. 12. Haec autem formula si supra et infra multiplicetur per $\sqrt{2} - \sqrt{(1+s^4)}$, ita in duas partes discerpetur, vt fit

$$V = \int \frac{2ss\partial s\sqrt{2}}{(1-s^4)\sqrt{(1+s^4)}} - \int \frac{2ss\partial s}{1-s^4}.$$

Cum igitur sit

$$\frac{2ss}{1-s^4} = \frac{1}{1-ss} - \frac{1}{1+ss}, \text{ erit}$$

$$\int \frac{2ss\partial s}{1-s^4} = \frac{1}{2} l \frac{1+s}{1-s} - \text{A tang. } s,$$

sicque prodibit ista aequatio memorabilis:

$$\int \frac{2ss\partial s\sqrt{2}}{(1-s^4)\sqrt{(1+s^4)}} = \frac{1}{2} l \frac{1+s}{1-s} - \text{A tang. } s$$
$$+ \frac{1}{4} l \frac{\sqrt{2}+\sqrt{(1+s^4)}+(s-1)s\sqrt{2}}{\sqrt{2}+\sqrt{(1+s^4)}+(s+1)s\sqrt{2}} + \frac{1}{2} \text{A tang. } \frac{s\sqrt{2}}{\sqrt{2}+\sqrt{(1+s^4)}-ss\sqrt{2}},$$

vbi notasse iuuabit esse

$$\text{A tang. } s = 2 \text{ A tang. } \frac{2s}{1-ss}.$$

Verum si has partes coniungere vellemus, in formulas fere inextricabiles illaberemur. Olim autem, cum huiusmodi formulas tractassem, iam incidi in hanc integrationem:

$$\int \frac{ss\partial s}{(1-s^4)\sqrt{1+s^4}} = \frac{1}{4\sqrt{2}} l \frac{s\sqrt{2}+\sqrt{(1+s^4)}}{1-ss} - \frac{1}{4\sqrt{2}} \text{A tang. } \frac{s\sqrt{2}}{\sqrt{(1+s^4)}},$$

vnde pro nostro casu fit

$$\int \frac{2ss\partial s\sqrt{2}}{(1-s^4)\sqrt{(1+s^4)}} = \frac{1}{2} l \frac{s\sqrt{2}+\sqrt{(1+s^4)}}{1-ss} - \frac{1}{2} \text{A tang. } \frac{s\sqrt{2}}{\sqrt{(1+s^4)}},$$

cuius expressionis consensus cum ante inuenta, propter radicalium complicationem, minus facile perspici potest.

Alia Resolutio.

Formulae propositae $V = \int \frac{\partial x}{(1+x)\sqrt[4]{(2xx-1)}}$.

§. 13. Vtamur hic substitutione modo memorata $\sqrt[4]{(2xx-1)} = s$, vt fit $x = \sqrt{\frac{1+s^4}{2}}$, atque iam vidimus for-

formulam nostram hoc modo exprimi:

$$V = \int \frac{2ss\partial s}{[\sqrt{2}+\sqrt{(1+s^4)}]\sqrt{(1+s^4)}} = \int \frac{2ss\partial s\sqrt{2}}{(1-s^4)[\sqrt{(1+s^4)}]} - \int \frac{2ss\partial s}{1-s^4}.$$

Cum igitur istud integrale duabus constet partibus, id hoc modo repraesentemus: $V = 2\sqrt{2}.M - 2N$, ita vt sit

$$M = \int \frac{ss\partial s}{(1-s^4)\sqrt{(1+s^4)}} \text{ et } N = \int \frac{ss\partial s}{1-s^4},$$

vbi posterior pars nullam habet difficultatem. Cum enim sit

$$\frac{ss}{1-s^4} = \frac{1}{2}.\frac{1}{1-ss} - \frac{1}{2}.\frac{1}{1+ss}, \text{ erit}$$

$$N = \frac{1}{4} l \frac{1+s}{1-s} - \frac{1}{2} \text{A tang. } s,$$

§. 14. Pro priore vero parte, quae exigit maiorem sagacitatem, ponamus $\frac{\sqrt{1+s^4}}{s} = t\sqrt{2}$, eritque differentiando

$$-\frac{\partial s(1-s^4)}{ss\sqrt{(1+s^4)}} = \partial t\sqrt{2}, \text{ vnde sit}$$

$$\partial s = -\frac{ss\partial t\sqrt{2}\sqrt{(1+s^4)}}{1-s^4},$$

quo valore substituto erit

$$\partial M = -\frac{s^4\partial t\sqrt{2}}{(1-s^4)^2}.$$

Cum iam sit $\sqrt{(1+s^4)} = st\sqrt{2}$, erit $1+s^4 = 2sstt$ et denuo quadrando $1+2s^4+s^8 = 4s^4t^4$, hinc aufferatur vtrinque $4s^4$ fietque

$$1-2s^4+s^8 = 4s^4(t^4-1) = (1-s^4)^2,$$

quo valore substituto erit

$$\partial M = \frac{-\partial t}{2\sqrt{2}.(t^4-1)} = \frac{\partial t}{2(1-t^4)\sqrt{2}},$$

ideoque tota pars prior $2\sqrt{2}.M = \frac{\partial t}{1-t^4}$.

§. 15. Cum nunc sit

$$\frac{1}{1-t^4} = \frac{1}{2}.\frac{1}{1-tt} + \frac{1}{2}.\frac{1}{1+tt},$$

erit ista pars

$$2\sqrt{2}.\,M = \frac{1}{4}\,l\,\frac{1+t}{1-t} + \frac{1}{2}\,A\,\text{tang.}\,t,$$

hincque pro t reſtituto valore $t = \frac{\sqrt{(1+s^4)}}{s\sqrt{2}}$, erit

$$2\sqrt{2}.\,M = \frac{1}{4}\,l\,\frac{s\sqrt{2}+\sqrt{(1+s^4)}}{s\sqrt{2}-\sqrt{(1+s^4)}} + \frac{1}{2}\,A\,\text{tang.}\,\frac{\sqrt{(1+s^4)}}{s\sqrt{2}},$$

conſequenter valor integralis quaeſitus erit

$$V = \frac{1}{4}\,l\,\frac{s\sqrt{2}+\sqrt{(1+s^4)}}{s\sqrt{2}-\sqrt{(1+s^4)}} + \frac{1}{2}\,A\,\text{tang.}\,\frac{\sqrt{(1+s^4)}}{s\sqrt{2}}$$
$$- \frac{1}{2}\,l\,\frac{1+s}{1-s} + A\,\text{tang.}\,s.$$

§. 16. Eſt vero

$$\sqrt{[s\sqrt{2}+\sqrt{(1+s^4)}]} = \sqrt{\frac{s\sqrt{2}+(1-ss)\sqrt{-1}}{2}} + \sqrt{\frac{s\sqrt{2}-(1-ss)\sqrt{-1}}{2}};$$

eodem modo:

$$\sqrt{[s\sqrt{2}-\sqrt{(1+s^4)}]} = \sqrt{\frac{s\sqrt{2}+(1-ss)\sqrt{-1}}{2}} - \sqrt{\frac{s\sqrt{2}-(1-ss)\sqrt{-1}}{2}},$$

hincque ergo prior logarithmus:

$$\frac{1}{4}\,l\,\frac{s\sqrt{2}+\sqrt{(1+s^4)}}{s\sqrt{2}-\sqrt{(1+s^4)}},$$

transmutatur in hanc formam:

$$\frac{1}{2}\,l\,\frac{\sqrt{[s\sqrt{2}+(1-ss)\sqrt{-1}]}+\sqrt{[s\sqrt{2}-(1-ss)\sqrt{-1}]}}{\sqrt{[s\sqrt{2}+(1-ss)\sqrt{-1}]}-\sqrt{[s\sqrt{2}-(1-ss)\sqrt{-1}]}},$$

quae forma porro reducitur ad hanc:

$$\frac{1}{2}\,l\,\frac{s\sqrt{2}+\sqrt{(1+s^4)}}{(1-ss)\sqrt{-1}},$$

vbi imaginarium in denominatore non turbat, quoniam addita conſtante $l\sqrt{-1}$ tollitur, ita vt habeamus iſtam partem logarithmicam:

$$= \frac{1}{2}\,l\,\frac{s\sqrt{2}+\sqrt{(1+s^4)}}{1-ss} - \frac{1}{2}\,l\,\frac{1+s}{1-s} = \frac{1}{2}\,l\,\frac{s\sqrt{2}+\sqrt{(1+s^4)}}{(1+s)^2}$$

§. 17. Pari modo etiam ambos arcus circulares in vnum contrahere licebit, hoc modo: Ponatur

$$A\,\text{tang.}\,\frac{\sqrt{(1+s^4)}}{s\sqrt{2}} = A\,\text{tang.}\,\frac{2u}{1-uu},\ \text{eritque}$$

$\frac{1}{2}$ A

$$\tfrac{1}{2}\,\text{A tang.}\,\frac{\sqrt{(1+s^4)}}{s\sqrt{2}} = \text{A tang.}\,u.$$

Erit igitur $\frac{\sqrt{(1+s^4)}}{s\sqrt{2}} = \frac{2u}{1-uu}$, vnde colligitur

$$u = \frac{1-s\sqrt{2}+ss}{\sqrt{(1+s^4)}},$$

ficque ambo arcus erunt:

$$\text{A tang.}\,\frac{1-s\sqrt{2}+ss}{\sqrt{(1+s^4)}} + \text{A tang.}\,s = \text{A tang.}\,\frac{1-s\sqrt{2}+ss+s\sqrt{(1+s^4)}}{\sqrt{(1+s^4)}-s+ss\sqrt{2}-s^3}.$$

§. 18. Etfi autem talibus reductionibus calculus irrationalium non mediocriter illuftratur, tamen formulae non euadunt fimpliciores; ideoque iis, quas immediate invenimus, vtamur, vbi, quia pofuimus $\sqrt[4]{(2xx-1)} = s$, commode litteram s loco huius formulae in calculo retinere poterimus. Tantum igitur loco $\sqrt{(1+s^4)}$ eius valorem, qui eft $x\sqrt{2}$, fcribamus, vnde fiet integrale quaefitum:

$$V = \tfrac{1}{4}\,l\,\frac{s+x}{s-x} - \tfrac{1}{2}\,l\,\frac{1+s}{1-s} + \tfrac{1}{2}\,\text{A tang.}\,\frac{x}{s} + \text{A tang.}\,s.$$

§. 19. Quoniam vero quantitatem conftantem quamcunque adiicere licet, loco $l(s-x)$ fcribamus $l(x-s)$, et quia A tang. $\frac{x}{s} = 90° -$ A tang. $\frac{s}{x}$, habebimus:

$$V = \tfrac{1}{4}\,l\,\frac{x+s}{x-s} - \tfrac{1}{2}\,l\,\frac{1+s}{1-s} - \tfrac{1}{2}\,\text{A tang.}\,\frac{s}{x} + \text{A tang.}\,s.$$

Quod fi vero in forma, quam prior folutio fuppeditauerat, etiam loco $\sqrt[4]{(2xx-1)}$ fcribamus s, ea erit:

$$V = \tfrac{1}{4}\,l\,\frac{1+x+ss-s}{1+x+ss+s} + \tfrac{1}{2}\,\text{A tang.}\,\frac{s}{1+x-ss},$$

quae forma maxime a praecedente difcrepare videtur, quia nulli adeo communes factores deprehenduntur. Interim tamen egregie inter fe conueniunt, ad quod oftendendum fingularis dexteritas in calculo irrationalium requiritur.

Demonſtratio conſenſus harum duarum formularum:

$$V = \tfrac{1}{4} l \frac{1+x+ss-s}{1+x+ss+s} + \tfrac{1}{2} \text{A tang.} \frac{s}{1+x-ss} \quad \text{et}$$

$$V = \tfrac{1}{4} l \frac{x+s}{x-s} - \tfrac{1}{2} l \frac{1+s}{1-s} - \tfrac{1}{2} \text{A tang.} \frac{s}{x} + \text{tang.}\, s.$$

§. 20. Quoniam logarithmi et arcus circulares nullo modo inter ſe comparari patiuntur, neceſſe eſt, vt vtrinque tam logarithmi quam arcus inter ſe ſeorſim aequentur. Incipiamus igitur a logarithmis, et oſtendendum eſt fore:

$$l \frac{1+x+ss-s}{1+x+ss+s} = l \frac{x+s}{x-s} - 2 l \frac{1+s}{1-s},$$

ſiue

$$l \frac{(x-s)(1+x+ss-s)}{(x+s)(1+x+ss+s)} = + 2 l \frac{1-s}{1+s}.$$

§. 21. Euoluamus nunc tam numeratorem quam denominatorem prioris fractionis, ac numerator abibit in hanc formam:

$$-s(1+ss) - 2sx + ss + x(1+ss) + xx,$$

quae porro, ob $xx = \frac{1+s^4}{2}$, abit in hanc:

$$-s(1+ss) - 2sx + ss + x(1+ss) + \tfrac{1}{2}(1+s^4),$$

vbi termini ſolam s continentes ſunt:

$$-s(1+ss) + ss + \tfrac{1}{2}(1+s)^4$$
$$= -s(1+ss) + \tfrac{1}{2}(1+ss)^2,$$
$$= + \tfrac{1}{2}(1+ss)(1-s)^2;$$

termini vero litteram x continentes ſunt:

$$-2sx + x(1+ss) = x(1-s)^2,$$

ſicque numerator ad hanc formam eſt reductus:

$$\tfrac{1}{2}(1-s)^2(2x+1+ss).$$

§. 22.

§. 22. Simili modo denominatorem tractemus, eritque facta euolutione

$$s(1+ss)+2sx+ss+x(1+ss)+xx,$$

vbi termini solam s continentes sunt

$$s(1+ss)+ss+\tfrac{1}{2}(1+s^4)=s(1+ss)+\tfrac{1}{2}(1+ss)^2$$
$$=\tfrac{1}{2}(1+ss)(1+s)^2;$$

termini vero litteram x continentes erunt

$$2sx+x(1+ss)=x(1+s)^2,$$

ideoque denominator hanc induit formam: $\frac{1}{2}(1+s)^2(2x+1+ss)$. Cum igitur numerator et denominator habeat communem factorem $\frac{1}{2}(2x+1+ss)$, pars finistra nostrae aequationis fit $l\frac{(1-s)^2}{(1+s)^2}=2\,l\frac{1-s}{1+s}$, vti postulabatur.

§. 23. Superest igitur, vt etiam aequalitatem inter arcus circulares demonstremus, hoc est vt fit

$$\tfrac{1}{2}\,\text{A tang.}\,\frac{s}{1+x-ss}=\text{A tang.}\,s-\tfrac{1}{2}\,\text{A tang.}\,\frac{s}{x}.$$

Transferamus hunc in finem A tang. $\frac{s}{x}$ in alteram partem, et cum fit

$$\text{A tang.}\,a+\text{A tang.}\,b=\text{A tang.}\,\frac{a+b}{1-ab},$$

haec aequatio proueniet:

$$\text{A tang.}\,\frac{2sx+s-s^3}{x+xx-ssx-ss}=2\,\text{A tang.}\,s.$$

At vero, si loco xx scribatur valor $\frac{1}{2}(1+s^4)$, denominator euadet $(1-ss)[\frac{1}{2}(1-ss)+x]$, numerator vero:

$$s(2x+1-ss)=2s[\tfrac{1}{2}(1-ss)+x],$$

sicque adest factor communis $\frac{1}{2}(1-ss)+x$, quo sublato fiet A tang. $\frac{2s}{1-ss}=2$ A tang. s. Sicque perfecta aequalitas rigide est

eſt demonſtrata, quia notum eſt reuera eſſe 2 A tang. $s =$ A tang. $\frac{2s}{1 - ss}$.

§. 24. Manifeſtum eſt, in vniuerſa hac tractatione plura occurrere artificia analytica minime obuia et comunia, quam ob cauſſam confido iſtam ſpeculationem Geometris non fore ingratam. Imprimis autem mihi maxime notatu dignum videtur, quod ſimili modo, quo poſteriorem reſolutionem adornauimus, etiam iſta formula differentialis multo latius patens:

$$\partial V = \frac{\partial x (1 - x^{n-1})}{(1 - x^n) \sqrt[2n]{(2x^n - 1)}},$$

ad integrabilitatem per logarithmos et arcus circulares reduci poteſt, ad quod oſtendendum ſequens problema adiungamus.

Problema.

Hanc formulam differentialem:

$$\partial V = \frac{\partial x (1 - x^{n-1})}{(1 - x^n) \sqrt[2n]{(2x^n - 1)}}$$

ad rationalitatem perducere.

Solutio.

§. 25. Ponatur breuitatis gratia $\sqrt[2n]{(2x^n - 1)} = s$, vt ſit

$$2x^n = 1 + s^{2n} \text{ et}$$

$$1 - x^n = x^n - s^{2n} = \frac{1 - s^{2n}}{2},$$

vnde

vnde forma propofita hoc modo repraefentari poteft:

$$\partial V = \frac{\partial x (1 - x^{n-1})}{s (x^n - s^{2n})},$$

quae in has partes discerpatur:

$$\frac{\partial x}{s (x^n - s^{2n})} = \partial M \text{ et } \frac{x^{n-1} \partial x}{s (x^n - s^{2n})} = \partial N,$$

ita vt fit $\partial V = \partial M - \partial N$. Cum autem fit $2 x^n = 1 + s^{2n}$, erit differentiando $x^{n-1} \partial x = s^{2n-1} \partial s$, ideoque

$$\partial x = \frac{s^{2n-1} \partial s}{x^{n-1}},$$

quibus fubftitutis erit

$$\partial M = \frac{s^{2n-2} \partial s}{x^{n-1} (x^n - s^{2n})} \text{ et}$$

$$\partial N = \frac{s^{2n-2} \partial s}{x^n - s^{2n}} = \frac{2 s^{2n-2} \partial s}{1 - s^{2n}},$$

quae pofterior forma iam eft rationalis, ita vt fola ∂M nobis tractanda relinquatur.

§. 26. Ponatur igitur $x = s t$, eritque differentiando $\partial x = s \partial t + t \partial s$. Cum autem fupra inuenerimus

$$\partial x = \frac{s^{2n-1} \partial s}{x^{n-1}}, \text{ erit}$$

$$x^{n-1} (s \partial t + t \partial s) = s^{2n-1} \partial s = s^{n-1} t^{n-1} (s \partial t + t \partial s),$$

vnde fit

$$\partial s = \frac{t^{n-1} s \partial t}{s^n - t^n},$$

quo valore fubftituto fit

$$\partial M = \frac{s^{2n-1}\, t^{n-1}\, \partial t}{x^{n-1}\, (x^n - s^{2n})\, (s^n - t^n)},$$

quae forma porro reducetur ad hanc:

$$\partial M = -\frac{\partial t}{(t^n - s^n)^2}.$$

Cum autem fit

$2x^n = 2s^n t^n = 1 + s^{2n}$, erit $(t^n - s^n)^2 = t^{2n} - 1$, vnde fit

$$\partial M = + \frac{\partial t}{1 - t^{2n}}, \text{ ideoque}$$

$$\partial V = \frac{\partial t}{1 - t^{2n}} - \frac{2 s^{2n-2}\, \partial s}{1 - s^{2n}}.$$

§. 27. Hoc igitur modo formulam propofitam ad duas alias ab irrationalitate prorfus liberatas perduximus, quarum integratio nulla amplius laborat difficultate et manifefto per logarithmos et arcus circulares abfolui poteft. Quo autem haec integrationis operatio, fi inftituere lubet, facilius et vno quafi ictu perfici queat, priorem partem $\frac{\partial t}{1 - t^{2n}}$, ad pofteriorem formam reducemus, quod fit ope huius fubftitutionis: $t = \frac{1}{u}$; fit enim hinc

$$\partial t = -\frac{\partial u}{uu} \text{ et } 1 - t^{2n} = \frac{u^{2n} - 1}{u^{2n}},$$

ideoque

$$\frac{\partial t}{1 - t^{2n}} = + \frac{u^{2n-2}\, \partial u}{1 - u^{2n}},$$

quo

quo notato erit

$$V = \int \frac{u^{2n-2}\,\partial u}{1-u^{2n}} + 2\int \frac{s^{2n-2}\,\partial s}{1-s^{2n}}.$$

§. 28. Simili modo etiam tractari possunt sequentes formulae integrales multo latius patentes:

$$\text{I. } \int \frac{\partial x}{(a+bx^n)\sqrt[2n]{(a+2bx^n)}};$$

$$\text{II. } \int \frac{\partial x}{(a+bx^n)\sqrt[3n]{(aa+3abx^n+3bbx^{2n})}};$$

$$\text{III. } \int \frac{\partial x}{(a+bx^n)\sqrt[4n]{(a^3+4aabx^n+6abbx^{2n}+4b^3x^{3n})}};$$

$$\text{IV. } \int \frac{\partial x}{(a+bx^n)\sqrt[5n]{(a^4+5a^3bx^n+10aabbx^{2n}+10ab^3x^{3n}+5b^4x^{4n})}};$$

quae omnes, ponendo quantitatem irrationalem denominatoris $= s$, tum vero $x = st$, ad rationalitatem perducuntur ideoque integrationem per logarithmos et arcus circulares admittunt.

§. 29. Quo hoc exemplo illustretur, sumatur

$$V = \int \frac{\partial x}{(a+bxx)\sqrt[8]{(x^3+4aabxx+6abbx^4+4b^3x^6)}},$$

et cum esse debeat

$$\sqrt[8]{(a^3+4aabxx+6abbx^4+4b^3x^6)} = s$$

et $x = st$, erit

$t =$

$$t = \frac{x}{\sqrt[8]{(a^3 + 4\,a\,a\,b\,x\,x + 6\,a\,b\,b\,x^4 + 4\,b^3\,x^6)}},$$

et differentiando

$$\partial t = \frac{\partial x\,(a^3 + 3\,a\,a\,b\,x\,x + 3\,a\,b\,b\,x^4 + b^3\,x^6)}{(a^3 + 4\,a\,a\,b\,x\,x + 6\,a\,b\,b\,x^4 + 4\,b^3\,x^6)^{\frac{9}{8}}}, \text{ fiue}$$

$$\partial t = \frac{\partial x\,(a + b\,x\,x)^3}{(a^3 + 4\,a\,a\,b\,x\,x + 6\,a\,b\,b\,x^4 + 4\,b^3\,x^6)^{\frac{9}{8}}}, \text{ ideoque}$$

$$\partial x = \frac{\partial t\,(a^3 + 4\,a\,a\,b\,x\,x + 6\,a\,b\,b\,x^4 + 4\,b^3\,x^6)^{\frac{9}{8}}}{(a + b\,x\,x)^3},$$

quo valore fubftituto fit

$$\partial V = \frac{\partial t\,(a^3 + 4\,a\,a\,b\,x\,x + 6\,a\,b\,b\,x^4 + 4\,b^3\,x^6)}{(a + b\,x\,x\,4} = \frac{s^8\,\partial t}{(a + b\,x\,x)^4},$$

fiue $\partial V = \frac{x^8\,\partial t}{t^8\,(a + b\,x\,x)^4}$. Eft vero

$$s^8 = \frac{x^8}{t^8} = a^3 + 4\,a\,a\,b\,x\,x + 6\,a\,b\,b\,x^4 + 4\,b^3\,x^6 = \frac{(a + b\,x\,x)^4 - b^4\,x^8}{a},$$

vnde fit $(a + b\,x\,x)^4 = \frac{a\,x^8}{t^8} + b^4\,x^8$, quo fubftituto prodit denique $\partial V = \frac{\partial t}{a + b^4\,t^8}$, quae igitur forma iam eft rationalis.

§. 30. Quin etiam haec formula adhuc generalior ope fimilium fubftitutionem ad rationalitatem perduci ideoque integrari poteft:

$$V = \int \frac{x^{m-1}\,\partial x}{(a + b\,x^n)\,[(a + b\,x)^\lambda - b^\lambda\,x^{\lambda n}]^{\frac{m}{\lambda n}}}$$

quod ita oftenditur. Ponatur $(a + b^n)^\lambda - b^\lambda\,x^{\lambda n} = s^{\lambda n}$, vt habeatur

$$\partial V = \frac{\partial x}{x} \cdot \frac{x^m}{(a + b\,x^n)\,s^m}.$$

Pona-

Ponatur porro $\frac{x}{s} = t$ eritque

$$\partial V = \frac{\partial x}{x} \cdot \frac{t^m}{a + b x^n}.$$

Porro cum fit $\frac{\partial t}{t} = \frac{\partial x}{x} - \frac{\partial s}{s}$, ob

$$\frac{\partial s}{s} = \frac{(b x^{n-1} - b^\lambda x^{\lambda n - 1}) \partial x}{(a + b x^n)^\lambda - b^\lambda x^{\lambda n}}, \text{ erit}$$

$$\frac{\partial t}{t} = \frac{\partial x}{x} \cdot \left[1 - \frac{b x^n (a + b x^n)^{\lambda - 1} + b^\lambda x^{\lambda n}}{(a + b x^n)^\lambda - b^\lambda x^{\lambda n}} \right]$$

$$= \frac{a \partial x}{x} \cdot \frac{(a + b x^n)^{\lambda - 1}}{s^{\lambda n}}, \text{ fiue}$$

$$\frac{\partial t}{t} = \frac{a \partial x}{x} (a + b x^n)^{\lambda - 1} \cdot \frac{t^{\lambda n}}{x^{\lambda n}}, \text{ vnde fit}$$

$$\frac{\partial x}{x} = \frac{\partial t}{t} \cdot \frac{x^{\lambda n}}{a t^{\lambda n} (a + b x^n)^{\lambda - 1}},$$

quo valore fubftituto prodit

$$\partial V = \frac{\partial t}{t} \cdot \frac{x^{\lambda n} t^m}{a t^{\lambda n} (a + b x^n)^\lambda}. \text{ Eft vero}$$

$$(a + b x^n)^\lambda = s^{\lambda n} + b^\lambda x^{\lambda n} = \frac{x^{\lambda n} (1 + b^\lambda t^{\lambda n})}{t^{\lambda n}}, \text{ vnde fit}$$

$$\partial V = \frac{\partial t}{t} \cdot \frac{t^m}{a (1 + b^\lambda t^{\lambda n})} = \frac{t^{m-1} \partial t}{a (1 + b^\lambda t^{\lambda n})}.$$

Miro igitur modo etiam hanc pofteriorem formulam generaliffimam ad rationalitatem perduximus, quae reductio ideo notatu maxime digna mihi vifa eft, quod tales fubftitutiones fingularem dexteritatem et plura artificia calculi requirunt.