1795

De insignibus proprietatibus formularum integralium praeter binas variabiles etiam earum differentialia cuiuscunque ordinis involventium

Leonhard Euler

DE INSIGNIBVS PROPRIETATIBVS FORMVLARVM INTEGRALIVM PRAETER BINAS VARIABILES ETIAM EARVM DIFFERENTIALIA CVIVSCVNQVE ORDINIS INVOLVENTIVM.

Auctore

L. EVLERO.

Conuentui exhibit. die 10 Mart. 1777.

§. 1.

Si Z fuerit functio quaecunque, non solum binas variabiles x et y, sed etiam earum differentialia cuiuscunque ordinis inuoluens, ea saltem a specie differentialium liberari potest ope sequentium positionum: $\partial y = p \partial x$; $\partial p = q \partial x$; $\partial q = r \partial x$; $\partial r = s \partial x$; $\partial s = t \partial x$; etc. tum enim his valoribus substitutis quantitas Z, si fuerit finita, euadet functio quantitatum finitarum $x, y, p, q, r, s, t,$ etc. Ita si fuerit

$$Z = \frac{(\partial x^2 + \partial y^2)^{\frac{3}{2}}}{\partial x \partial \partial y - \partial y \partial \partial x},$$

quae est formula notissima pro radio osculi, ob $\partial y = p \partial x$ et $\partial \partial y = p \partial \partial x + \partial p \partial x = p \partial \partial x + q \partial x^2$, primo nu- L mera-

merator hanc induet formam: $\partial x^3 (1 + p p)^{\frac{3}{2}}$, deinde vero denominator euadet $= q \partial x^3$, ficque ifta quantitas erit

$$Z = \frac{(1 + p p)^{\frac{3}{2}}}{q}.$$

§. 2. Quodfi nunc talis functio Z differentietur, eius differentiale ex tot conftabit partibus, quot in ea infunt litterarum $x, y, p, q, r, s,$ etc., ideoque tali forma exprimetur:

$$\partial Z = M \partial x + N \partial y + P \partial p + Q \partial q + R \partial r + S \partial s + \text{etc.}$$

Hic autem, ne multitudo litterarum M, N, P, Q, R, etc. in calculo moleftiam creet, eas, quoniam omnes pendent a natura functionis Z, per fequentes characteres vfu iam fatis receptos repraefentabo: $M = (\frac{\partial Z}{\partial x})$; $N = (\frac{\partial Z}{\partial y})$; $P = (\frac{\partial Z}{\partial p})$; $Q = (\frac{\partial Z}{\partial q})$; $R = (\frac{\partial Z}{\partial r})$; $S = (\frac{\partial Z}{\partial s})$; etc. hocque modo, nullas litteras peregrinas introducendo, erit

$$\partial Z = \partial x (\tfrac{\partial Z}{\partial x}) + \partial y (\tfrac{\partial Z}{\partial y}) + \partial p (\tfrac{\partial Z}{\partial p}) + \partial q (\tfrac{\partial Z}{\partial q}) + \text{etc.}$$

ac fi porro loco differentialium ∂y, ∂p, ∂q, ∂r, etc. valores fupra affignatos adhibeamus, prodibit

$$\partial Z = \partial x (\tfrac{\partial Z}{\partial x}) + p \partial x (\tfrac{\partial Z}{\partial y}) + q \partial x (\tfrac{\partial Z}{\partial p}) + r \partial x (\tfrac{\partial Z}{\partial q}) + \text{etc.}$$

§. 3. Hinc ergo fi ftatuamus:

$$V = (\tfrac{\partial Z}{\partial x}) + p (\tfrac{\partial Z}{\partial y}) + q (\tfrac{\partial Z}{\partial p}) + r (\tfrac{\partial Z}{\partial q}) + \text{etc.}$$

quae erit quantitas finita, pariterque certa functio ipfarum $x, y, p, q, r,$ etc. ab indole functionis Z pendens, erit $\partial Z = V \partial x$, ideoque integrando $Z = \int V \partial x$, in qua integratione omnes litterae $x, y, p, q, r,$ etc. tanquam variabiles infunt. Vbi probe notetur, fi V fuerit talis functio,

ctio, qualem deſcripſimus, tum formulam differentialem V ∂ x ſemper integrationem admittere, etiamſi binae variabiles x et y nullo modo a ſe inuicem pendeant, cum contra, ſi loco V alia quaecunque functio quantitatum x, y, p, q, r, etc. acciperetur, integratio locum habere non poſſet, niſi certa quaedam relatio inter binas variabiles x et y ſtatueretur.

§. 4. His conſtitutis cum ſit

$$V = \left(\frac{\partial Z}{\partial x}\right) + p\left(\frac{\partial Z}{\partial y}\right) + q\left(\frac{\partial Z}{\partial p}\right) + r\left(\frac{\partial Z}{\partial q}\right) + \text{etc.}$$

perpendamus valores differentiales ipſius V, qui oriuntur, ſi vel ſola quantitas x, vel ſola y, vel ſola p, vel ſola q, etc. pro variabili habeatur, quos valores ſimili ratione per hos characteres: $\left(\frac{\partial V}{\partial x}\right)$; $\left(\frac{\partial V}{\partial y}\right)$; $\left(\frac{\partial V}{\partial p}\right)$; $\left(\frac{\partial V}{\partial q}\right)$; etc. deſignemus. Ac primo quidem ſi ſola quantitas x vt variabilis tractetur, iisdem characteribus adhibendis reperietur:

$$\left(\frac{\partial V}{\partial x}\right) = \left(\frac{\partial \partial Z}{\partial x^2}\right) + p\left(\frac{\partial \partial Z}{\partial y \partial x}\right) + q\left(\frac{\partial \partial Z}{\partial p \partial x}\right) + r\left(\frac{\partial \partial Z}{\partial q \partial x}\right) + s\left(\frac{\partial \partial Z}{\partial r \partial x}\right) + \text{etc.}$$

vbi ſcilicet, vti iam ſatis eſt vſu receptum, formula $\left(\frac{\partial \partial Z}{\partial x^2}\right)$ indicat, functionem Z bis differentiandam eſſe, ſola x pro variabili aſſumta; at formula $\left(\frac{\partial \partial Z}{\partial x \partial y}\right)$ indicat, functionem Z etiam bis ita eſſe differentiandam, vt in altera differentiatione ſola quantitas x, in altera vero ſola y variabilis ſumatur. Demonſtratum autem eſt eundem valorem prodire, ſiue in prima operatione x, in ſecunda vero y, ſiue inuerſo modo, in prima y in altera vero x variabilis ſtatuatur; quod idem etiam de reliquis formulis duplicem differentiationem innuentibus eſt tenendum.

§. 5. Si iam in hac poſtrema expreſſione valorem $\left(\frac{\partial Z}{\partial x}\right)$ littera T deſignemus, hinc fiet $\left(\frac{\partial \partial Z}{\partial x^2}\right) = \frac{\partial T}{\partial x}$; tum vero ro $\left(\frac{\partial \partial Z}{\partial x \partial y}\right) = \frac{\partial T}{\partial y}$; $\left(\frac{\partial \partial Z}{\partial x \partial p}\right) = \frac{\partial T}{\partial p}$; etc. hisque formulis introductis erit

$$\left(\frac{\partial V}{\partial x}\right) = \left(\frac{\partial T}{\partial x}\right) + p\left(\frac{\partial T}{\partial y}\right) + q\left(\frac{\partial T}{\partial p}\right) + r\left(\frac{\partial T}{\partial q}\right) + \text{etc.}$$

At vero fi quantitas ifta T per variabilitatem omnium litterarum x, y, p, q, r, etc. differentietur, erit, vt fupra iam vidimus, eius differentiale plenum:

$$\partial T = \partial x\left(\frac{\partial T}{\partial x}\right) + p\,\partial x\left(\frac{\partial T}{\partial y}\right) + q\,\partial x\left(\frac{\partial T}{\partial p}\right) + r\,\partial x\left(\frac{\partial T}{\partial q}\right) + \text{etc.}$$

vnde patet fore $\partial T = \partial x\left(\frac{\partial V}{\partial x}\right)$, ita vt integrando fit

$$T = \left(\frac{\partial Z}{\partial x}\right) = \int \partial x\left(\frac{\partial V}{\partial x}\right).$$

Hinc difcimus, fi formula $V\partial x$ integrationem admittat, femper etiam hanc formulam: $\partial x\left(\frac{\partial V}{\partial x}\right)$, integrationem effe admiffuram; quam proprietatem hoc Theoremate 1. referamus.

Si fuerit $\int V\partial x = Z$, *tum etiam femper erit* $\int \partial x\left(\frac{\partial V}{\partial x}\right) = \left(\frac{\partial Z}{\partial x}\right)$, *fiue* $\partial x\left(\frac{\partial V}{\partial x}\right) = \partial.\left(\frac{\partial Z}{\partial x}\right)$.

§. 6. Nunc quantitatis V id confideremus differentiale, quod ex fola variabili y enafcitur, ac reperietur:

$$\left(\frac{\partial V}{\partial y}\right) = \left(\frac{\partial \partial Z}{\partial x \partial y}\right) + p\left(\frac{\partial \partial Z}{\partial y^2}\right) + q\left(\frac{\partial \partial Z}{\partial p \partial y}\right) + r\left(\frac{\partial \partial Z}{\partial q \partial y}\right) + \text{etc.}$$

vnde fi hic ponamus $\left(\frac{\partial Z}{\partial y}\right) = T$, erit

$$\left(\frac{\partial V}{\partial y}\right) = \left(\frac{\partial T}{\partial x}\right) + p\left(\frac{\partial T}{\partial y}\right) + q\left(\frac{\partial T}{\partial p}\right) + r\left(\frac{\partial T}{\partial q}\right) + \text{etc.}$$

hinc igitur vt fupra patet fore

$$\partial x\left(\frac{\partial V}{\partial y}\right) = \partial T = \partial.\left(\frac{\partial Z}{\partial y}\right),$$

ex quo integrando erit

$\int \partial x$

$$\int \partial x \left(\frac{\partial V}{\partial y}\right) = T = \left(\frac{\partial Z}{\partial y}\right);$$

vnde deducitur sequens Theorema 2.

Si fuerit $\int V \partial x = Z$, *tum semper erit*

$$\int \partial x \left(\frac{\partial V}{\partial y}\right) = \left(\frac{\partial Z}{\partial y}\right), \text{ siue}$$

$$\partial x \left(\frac{\partial V}{\partial y}\right) = \partial . \left(\frac{\partial Z}{\partial y}\right).$$

§. 7. Progrediamur autem vlterius, et differentiale ipsius V, ex sola variabilitate ipsius p oriundum, contemplemur, ac reperiemus

$$\left(\frac{\partial V}{\partial p}\right) = \left(\frac{\partial \partial Z}{\partial x \partial p}\right) + p \left(\frac{\partial \partial Z}{\partial y \partial p}\right) + q \left(\frac{\partial \partial Z}{\partial p^2}\right) + r \left(\frac{\partial \partial Z}{\partial q \partial p}\right) + \text{etc.}$$
$$+ \left(\frac{\partial Z}{\partial y}\right)$$

Hinc iam si ponamus $\left(\frac{\partial Z}{\partial p}\right) = T$, erit

$$\left(\frac{\partial V}{\partial p}\right) = \left(\frac{\partial T}{\partial x}\right) + p \left(\frac{\partial T}{\partial y}\right) + q \left(\frac{\partial T}{\partial p}\right) + r \left(\frac{\partial T}{\partial q}\right) + \text{etc.} + \left(\frac{\partial Z}{\partial y}\right),$$

vnde ergo sequitur fore

$$\partial x \left(\frac{\partial V}{\partial p}\right) = \partial T + \partial x \left(\frac{\partial Z}{\partial y}\right);$$

quod nobis suppeditat istud Theorema 3.

Si fuerit $\int V . \partial x = Z$, *tum etiam semper erit*

$$\int \partial x \left(\frac{\partial V}{\partial p}\right) = \left(\frac{\partial Z}{\partial p}\right) + \int \partial x \left(\frac{\partial Z}{\partial y}\right), \text{ siue}$$

$$\partial x \left(\frac{\partial V}{\partial p}\right) - \partial x \left(\frac{\partial Z}{\partial y}\right) = \partial . \left(\frac{\partial Z}{\partial p}\right).$$

§. 8. Sumta nunc sola quantitate q pro variabili simili modo orietur

$$\left(\frac{\partial V}{\partial q}\right) = \left(\frac{\partial \partial Z}{\partial x \partial q}\right) + p \left(\frac{\partial \partial Z}{\partial y \partial q}\right) + q \left(\frac{\partial \partial Z}{\partial p \partial q}\right) + r \left(\frac{\partial \partial Z}{\partial q^2}\right) + \text{etc.}$$
$$+ \left(\frac{\partial Z}{\partial p}\right)$$

vnde

vnde ſi hic ponatur $\left(\frac{\partial Z}{\partial q}\right) = T$, erit

$$\left(\frac{\partial V}{\partial q}\right) = \left(\frac{\partial T}{\partial x}\right) + p\left(\frac{\partial T}{\partial y}\right) + q\left(\frac{\partial T}{\partial p}\right) + r\left(\frac{\partial T}{\partial q}\right) + \text{etc.} + \left(\frac{\partial Z}{\partial p}\right).$$

ſicque per ∂x multiplicando fiet

$$\partial x\left(\frac{\partial V}{\partial q}\right) = \partial T + \partial x\left(\frac{\partial Z}{\partial p}\right).$$

Hinc orietur iſtud Theorema quartum:

Si fuerit $\int V\partial x = Z$, *tum ſemper erit*

$$\int\partial x\left(\frac{\partial V}{\partial q}\right) = \left(\frac{\partial Z}{\partial q}\right) + \int\partial x\left(\frac{\partial Z}{\partial p}\right), \textit{ ſiue}$$

$$\partial x\left(\frac{\partial V}{\partial q}\right) - \partial x\left(\frac{\partial Z}{\partial p}\right) = \partial.\left(\frac{\partial Z}{\partial q}\right).$$

§. 9. Sumatur iam ſola quantitas r pro variabili ac prodibit

$$\left(\frac{\partial V}{\partial r}\right) = \left(\frac{\partial\partial Z}{\partial x\,\partial r}\right) + p\left(\frac{\partial\partial Z}{\partial y\,\partial r}\right) + q\left(\frac{\partial\partial Z}{\partial p\,\partial r}\right)$$
$$+ r\left(\frac{\partial\partial Z}{\partial q\,\partial r}\right) + \text{etc.} + \left(\frac{\partial Z}{\partial q}\right),$$

vnde ſi ponatur $\left(\frac{\partial Z}{\partial r}\right) = T$, erit

$$\left(\frac{\partial V}{\partial r}\right) = \left(\frac{\partial T}{\partial x}\right) + p\left(\frac{\partial Z}{\partial y}\right) + q\left(\frac{\partial T}{\partial p}\right) + r\left(\frac{\partial T}{\partial q}\right) + \text{etc.} + \left(\frac{\partial Z}{\partial q}\right).$$

Hinc igitur vt ſupra patet fore

$$\partial x\left(\frac{\partial V}{\partial r}\right) = \partial T + \partial x\left(\frac{\partial Z}{\partial q}\right),$$

ſicque orietur ſequens Theorema quintum:

Si fuerit $\int V\partial x = Z$, *tum etiam ſemper erit*

$$\int\partial x\left(\frac{\partial V}{\partial r}\right) = \left(\frac{\partial Z}{\partial r}\right) + \int\partial x\left(\frac{\partial Z}{\partial q}\right), \textit{ ſiue}$$

$$\partial x\left(\frac{\partial V}{\partial r}\right) - \partial x\left(\frac{\partial Z}{\partial q}\right) = \partial.\left(\frac{\partial Z}{\partial r}\right).$$

§. 10. Haec iam ita ſunt manifeſta, vt ſuperfluum foret iſta theoremata vlterius proſequi. Ante autem quam repetitas differentiationes proſequamur, haec theoremata nobis

bis inseruire possunt, ad criterium illud generale demonstrandum, quo primus ostendi formulam $\int V \partial x$ semper admittere integrationem, quoties fuerit:

$$0 = \left(\frac{\partial V}{\partial y}\right) - \frac{1}{\partial x} \partial . \left(\frac{\partial V}{\partial p}\right) + \frac{1}{\partial x^2} \partial \partial . \left(\frac{\partial V}{\partial q}\right) - \frac{1}{\partial x^3} \partial^3 . \left(\frac{\partial V}{\partial r}\right) + \frac{1}{\partial x^4} \partial^4 . \left(\frac{\partial V}{\partial s}\right) - \text{etc.}$$

§. 11. Ad hanc autem regulam demonstrandam, posito $\int V \partial x = Z$, per gradus progrediamur, prouti functio Z continuo plures continet litterarum x, y, p, q, r, etc. Ac primo quidem contineat functio Z tantum binas variabiles x et y, exclusis omnibus differentialibus, ita vt fit

$$\left(\frac{\partial Z}{\partial p}\right) = 0; \left(\frac{\partial Z}{\partial q}\right) = 0; \left(\frac{\partial Z}{\partial r}\right) = V, \text{ etc.}$$

Hinc iam ex theoremate tertio erit $\left(\frac{\partial V}{\partial p}\right) = \left(\frac{\partial Z}{\partial y}\right)$, theorema autem secundum nobis praebet $\partial x \left(\frac{\partial V}{\partial y}\right) = \partial . \left(\frac{\partial Z}{\partial y}\right)$. Cum igitur inde fit $\left(\frac{\partial Z}{\partial y}\right) = \left(\frac{\partial V}{\partial p}\right)$, hoc valore substituto fiet

$$\partial x \left(\frac{\partial V}{\partial y}\right) = \partial . \left(\frac{\partial V}{\partial p}\right),$$

ideoque per ∂x diuidendo orietur haec aequatio:

$$0 = \left(\frac{\partial V}{\partial y}\right) - \frac{1}{\partial x} \partial . \left(\frac{\partial V}{\partial p}\right),$$

prorsus vt criterium meum postulat. Quoniam enim ex Z litterae p, q, r, etc. excluduntur, ob $\partial Z = V \partial x$ functio V neque litteram q, neque r, neque s, etc. continere potest, vnde etiam formulae $\left(\frac{\partial V}{\partial q}\right)$; $\left(\frac{\partial V}{\partial r}\right)$; etc. euanescunt.

§. 12. Contineat nunc functio Z, praeter litteras x et y, etiam p, vnde ob $\partial Z = V \partial x$ et $\partial p = q \partial x$, quantitas V etiamnunc q inuoluet, sequentes vero litterae r, s, t, etc. excludentur. Cum igitur iam fit $\left(\frac{\partial Z}{\partial q}\right) = 0$ multoque magis $\left(\frac{\partial Z}{\partial r}\right) = 0$; $\left(\frac{\partial Z}{\partial s}\right) = 0$; etc. theorema quartum nobis

nobis praebebit:

$$\partial x \left(\frac{\partial V}{\partial q}\right) - \partial x \left(\frac{\partial Z}{\partial p}\right) = 0,$$

vnde fit $\left(\frac{\partial Z}{\partial p}\right) = \left(\frac{\partial V}{\partial q}\right)$, qui valor in tertio theoremate substitutus dat

$$\partial x \left(\frac{\partial V}{\partial p}\right) - \partial x \left(\frac{\partial Z}{\partial y}\right) = \partial . \left(\frac{\partial V}{\partial q}\right),$$

vnde ergo erit

$$\frac{\partial Z}{\partial y} = \left(\frac{\partial V}{\partial p}\right) - \frac{1}{\partial x} \partial . \left(\frac{\partial V}{\partial q}\right),$$

qui valor in theoremate fecundo fubftitutus praebet:

$$\partial x \left(\frac{\partial V}{\partial y}\right) = \partial . \left(\frac{\partial V}{\partial p}\right) - \frac{1}{\partial x} \partial \partial . \left(\frac{\partial V}{\partial q}\right),$$

vnde fequitur, prorfus vt noftrum criterium poftulat,

$$0 = \left(\frac{\partial V}{\partial y}\right) - \frac{1}{\partial x} \partial . \left(\frac{\partial V}{\partial p}\right) + \frac{1}{\partial x^2} \partial \partial . \left(\frac{\partial V}{\partial q}\right).$$

§. 13. Inuoluat nunc functio Z etiam litteram q, et quantitas V etiamnunc continebit litteram r, ob $\partial q = r \partial x$: fequentes vero inde excludentur. Cum igitur fit $\left(\frac{\partial Z}{\partial r}\right) = 0$, theorema quintum nobis praebet

$$\partial x \left(\frac{\partial V}{\partial r}\right) - \partial x \left(\frac{\partial Z}{\partial q}\right) = 0,$$

vnde fit $\left(\frac{\partial Z}{\partial q}\right) = \left(\frac{\partial V}{\partial r}\right)$, qui valor in theoremate quarto fubftitutus fuppeditat hanc aequationem:

$$\partial x \left(\frac{\partial V}{\partial q}\right) - \partial x \left(\frac{\partial Z}{\partial p}\right) = \partial . \left(\frac{\partial V}{\partial r}\right),$$

vnde colligitur:

$$\left(\frac{\partial Z}{\partial p}\right) = \left(\frac{\partial V}{\partial q}\right) - \frac{1}{\partial x} \partial . \left(\frac{\partial V}{\partial r}\right).$$

Subftituatur hic valor in theoremate tertio, fietque

$$\partial x \left(\frac{\partial V}{\partial p}\right) - \partial x \left(\frac{\partial Z}{\partial y}\right) = \partial . \left(\frac{\partial V}{\partial q}\right) - \frac{1}{\partial x} \partial \partial . \left(\frac{\partial V}{\partial r}\right),$$

vnde fit

$$\left(\frac{\partial Z}{\partial y}\right) = \left(\frac{\partial V}{\partial p}\right) - \frac{1}{\partial x} \partial . \left(\frac{\partial V}{\partial q}\right) + \frac{1}{\partial x^2} \partial^2 . \left(\frac{\partial V}{\partial r}\right),$$

qui

qui valor in ſecundo theoremate ſubſtitutus praebet hanc aequationem:

$$0 = \left(\frac{\partial V}{\partial y}\right) - \frac{1}{\partial x}\,\partial.\left(\frac{\partial V}{\partial p}\right) + \frac{1}{\partial x^2}\,\partial\partial.\left(\frac{\partial V}{\partial q}\right) - \frac{1}{\partial x^3}\,\partial^3.\left(\frac{\partial V}{\partial r}\right).$$

§. 14. Hoc igitur modo criterium ſupra memoratum, quod primum ex contemplatione maximorum et minimorum, via maxime indirecta, concluſeram, omni rigore eſt demonſtratum; atque haec demonſtratio non multum diſcrepat ab ea, quam ſagaciſſimus noſter Profeſſor Lexell exhibuit, (Novor. Commentar. Acad. Scientiar. Petropol. Tomo XV. pag. 127). Nunc igitur formulas differentiales ſupra ex functione Z deductas per vlteriores differentiationes euoluamus, quandoquidem hinc innumerabilia alia theoremata, iis quae dedimus ſimilia, deriuari poſſunt.

Euolutio formulae

$$\left(\frac{\partial V}{\partial x}\right) = \left(\frac{\partial\partial Z}{\partial x^2}\right) + p\left(\frac{\partial\partial Z}{\partial y\,\partial x}\right) + q\left(\frac{\partial\partial Z}{\partial p\,\partial x}\right) + r\left(\frac{\partial\partial Z}{\partial q\,\partial x}\right) + \text{etc.}$$

per vlteriorem differentiationem.

§. 15. Sumamus primo ſolam x pro variabili, ac facta differentiatione prodibit

$$\left(\frac{\partial\partial V}{\partial x^2}\right) = \left(\frac{\partial^3 Z}{\partial x^3}\right) + p\left(\frac{\partial^3 Z}{\partial y\,\partial x^2}\right) + q\left(\frac{\partial^3 Z}{\partial p\,\partial x^2}\right) + r\left(\frac{\partial^3 Z}{\partial q\,\partial x^2}\right) + \text{etc.}$$

vbi ſi ſtatuamus $\left(\frac{\partial\partial Z}{\partial x^2}\right) = T$, erit

$$\left(\frac{\partial\partial V}{\partial x^2}\right) = \left(\frac{\partial T}{\partial x}\right) + p\left(\frac{\partial T}{\partial y}\right) + q\left(\frac{\partial T}{\partial p}\right) + r\left(\frac{\partial T}{\partial q}\right) + \text{etc.}$$

vnde manifeſto erit

$$\partial x\left(\frac{\partial\partial V}{\partial x^2}\right) = \partial T = \partial.\left(\frac{\partial\partial Z}{\partial x^2}\right),$$

atque hinc naſcitur ſequens theorema:

Si fuerit $\int V\,\partial x = Z$, *tum ſemper erit*

$$\int\partial x\left(\frac{\partial\partial V}{\partial x^2}\right) = \left(\frac{\partial\partial Z}{\partial x^2}\right), \textit{ ſiue } \partial x\left(\frac{\partial\partial V}{\partial x^2}\right) = \partial.\left(\frac{\partial\partial Z}{\partial x^2}\right).$$

§. 16. Sumatur nunc pro eadem formula ſola y pro variabili, ac reperietur

$$\left(\frac{\partial\partial V}{\partial x\,\partial y}\right) = \left(\frac{\partial^3 Z}{\partial y\,\partial x^2}\right) + p\left(\frac{\partial^3 Z}{\partial y^2\,\partial x}\right) + q\left(\frac{\partial^3 Z}{\partial p\,\partial x\,\partial y}\right) + r\left(\frac{\partial^3 Z}{\partial q\,\partial x\,\partial y}\right) + \text{etc.}$$

vbi ſi ſtatuamus $\left(\frac{\partial\partial Z}{\partial x\,\partial y}\right) = T$, erit

$$\left(\frac{\partial\partial V}{\partial x\,\partial y}\right) = \left(\frac{\partial T}{\partial x}\right) + p\left(\frac{\partial T}{\partial y}\right) + q\left(\frac{\partial T}{\partial p}\right) + r\left(\frac{\partial T}{\partial q}\right) + \text{etc.}$$

vnde manifeſto erit

$$\partial x\left(\frac{\partial\partial V}{\partial x\,\partial y}\right) = \partial T = \partial.\left(\frac{\partial\partial Z}{\partial x\,\partial y}\right),$$

ſicque adepti ſumus ſequens theorema:

Si fuerit $\int V\partial x = Z$, *tum ſemper erit*
$\int\partial x\left(\frac{\partial\partial V}{\partial x\,\partial y}\right) = \left(\frac{\partial\partial Z}{\partial x\,\partial y}\right)$, *ſiue* $\partial x\left(\frac{\partial\partial V}{\partial x\,\partial y}\right) = \partial.\left(\frac{\partial\partial Z}{\partial x\,\partial y}\right)$.

§. 17. At ſi ſola p variabilis capiatur, tum erit

$$\left(\frac{\partial\partial V}{\partial x\,\partial p}\right) = \left(\frac{\partial^3 Z}{\partial x^2\,\partial p}\right) + p\left(\frac{\partial^3 Z}{\partial x\,\partial y\,\partial p}\right) + q\left(\frac{\partial^3 Z}{\partial x\,\partial p^2}\right) + r\left(\frac{\partial^3 Z}{\partial q\,\partial x\,\partial p}\right) + \text{etc.} + \left(\frac{\partial\partial Z}{\partial x\,\partial y}\right).$$

Hinc ergo ſi ponatur $\left(\frac{\partial\partial Z}{\partial x\,\partial p}\right) = T$, erit

$$\partial x\left(\frac{\partial\partial V}{\partial x\,\partial p}\right) = \partial T + \partial x\left(\frac{\partial\partial Z}{\partial x\,\partial y}\right),$$

hincque formatur ſequens theorema:

Si fuerit $\int V\partial x = Z$, *tum ſemper erit*
$\int\partial x\left(\frac{\partial\partial V}{\partial x\,\partial p}\right) = \left(\frac{\partial\partial Z}{\partial x\,\partial p}\right) + \int\partial x\left(\frac{\partial\partial Z}{\partial x\,\partial y}\right)$, *ſiue*
$\partial x\left(\frac{\partial\partial V}{\partial x\,\partial p}\right) - \partial x\left(\frac{\partial\partial Z}{\partial x\,\partial y}\right) = \partial.\left(\frac{\partial\partial Z}{\partial x\,\partial p}\right)$.

§. 18. Sit nunc ſola littera q variabilis, eritque

$$\left(\frac{\partial\partial V}{\partial x\,\partial q}\right) = \left(\frac{\partial^3 Z}{\partial x^2\,\partial q}\right) + p\left(\frac{\partial^3 Z}{\partial x\,\partial y\,\partial q}\right) + q\left(\frac{\partial^3 Z}{\partial x\,\partial p\,\partial q}\right) + r\left(\frac{\partial^3 Z}{\partial x\,\partial q^2}\right) + \text{etc.} + \left(\frac{\partial\partial Z}{\partial p\,\partial x}\right)$$

hinc ergo ſi ponatur $\left(\frac{\partial\partial Z}{\partial x\,\partial q}\right) = T$, erit

$$\partial x\left(\frac{\partial\partial V}{\partial x\,\partial q}\right) = \partial T + \partial x\left(\frac{\partial\partial Z}{\partial p\,\partial x}\right),$$

hincque formatur ſequens theorema:

Si

Si fuerit $\int V \partial x = Z$, *tum semper erit*

$$\int \partial x \left(\frac{\partial \partial V}{\partial x \partial q}\right) = \left(\frac{\partial \partial Z}{\partial x \partial q}\right) + \int \partial x \left(\frac{\partial \partial Z}{\partial p \partial x}\right), \text{ siue}$$

$$\partial x \left(\frac{\partial \partial V}{\partial x \partial q}\right) - \partial x \left(\frac{\partial \partial Z}{\partial p \partial x}\right) = \partial . \frac{\partial \partial Z}{\partial x \partial q}.$$

Euolutio formulae

$$\left(\frac{\partial V}{\partial y}\right) = \left(\frac{\partial \partial Z}{\partial x \partial y}\right) + p\left(\frac{\partial \partial Z}{\partial y^2}\right) + q\left(\frac{\partial \partial Z}{\partial p \partial y}\right) + r\left(\frac{\partial \partial Z}{\partial q \partial y}\right) + \text{etc.}$$

per vlteriorem differentiationem.

§. 19. Hanc euolutionem iam multo concinnius absoluere licebit. Cum enim forma proposita ita repraesentari possit, vt sit $\partial x \left(\frac{\partial V}{\partial y}\right) = \partial . \left(\frac{\partial Z}{\partial y}\right)$, singulas differentiationes in hac forma instituere poterimus. Ita si sola x variabilis sumatur, erit $\partial x \left(\frac{\partial \partial V}{\partial x \partial y}\right) = \partial . \left(\frac{\partial \partial Z}{\partial x \partial y}\right)$, quod iam est theorema 5. praecedentis euolutionis. Simili modo si sola y variabilis sumatur, prodibit $\partial x \left(\frac{\partial \partial V}{\partial y^2}\right) = \partial . \left(\frac{\partial \partial Z}{\partial y^2}\right)$, quod est nouum theorema ad hanc euolutionem pertinens, vnde fit $\int \partial x \left(\frac{\partial \partial V}{\partial y^2}\right) = \left(\frac{\partial \partial Z}{\partial y^2}\right)$. Hinc patet si fuerit $\int V \partial x = Z$, tum semper fore $\int \partial x \left(\frac{\partial \partial V}{\partial y^2}\right) = \left(\frac{\partial \partial Z}{\partial y^2}\right)$. At si sola p variabilis accipiatur, tum quadam circumspectione opus est, quoniam hoc casu non erit $\partial x \left(\frac{\partial \partial V}{\partial y \partial p}\right) = \partial . \left(\frac{\partial \partial Z}{\partial x \partial y \partial p}\right)$, sed insuper aliquod membrum accedet. Quoniam enim formula $\partial . \left(\frac{\partial Z}{\partial y}\right)$ euoluta continet partem $p \partial x \left(\frac{\partial \partial Z}{\partial y^2}\right)$, huius differentiatio praebet $\partial x \left(\frac{\partial \partial Z}{\partial y^2}\right)$, quod ergo insuper adiici oportet, ita vt sit

$$\partial x \left(\frac{\partial \partial V}{\partial y \partial p}\right) = \partial . \left(\frac{\partial \partial Z}{\partial y \partial p}\right) + \partial x \left(\frac{\partial \partial Z}{\partial y^2}\right),$$

consequenter sumtis integralibus erit

$$\int \partial x \left(\frac{\partial \partial V}{\partial y \partial p}\right) = \left(\frac{\partial \partial Z}{\partial y \partial p}\right) + \int \partial x \left(\frac{\partial \partial Z}{\partial y^2}\right),$$

sicque integratio formulae $\int \partial x \left(\frac{\partial \partial V}{\partial y \partial p}\right)$ insuper inuoluit formulam integralem $\int \partial x \left(\frac{\partial \partial Z}{\partial y^2}\right)$.

§. 20. Sumatur nunc ſola q pro variabili, et quia formula $\partial.(\frac{\partial z}{\partial y})$ continet terminum $q\,\partial x(\frac{\partial z}{\partial y \partial p})$, variabilitas ipſius q producet terminum $\partial x(\frac{\partial \partial z}{\partial y \partial p})$, ſicque orietur iſta aequatio:

$$\partial x(\frac{\partial \partial v}{\partial y \partial q}) = \partial.(\frac{\partial \partial z}{\partial y \partial q}) + \partial x(\frac{\partial \partial z}{\partial y \partial p}).$$

Eodem modo patet ſi ſola littera r variabilis accipiatur, tum fore

$$\partial x(\frac{\partial \partial v}{\partial y \partial r}) = \partial.(\frac{\partial \partial z}{\partial y \partial r}) + \partial x(\frac{\partial \partial z}{\partial y \partial q}).$$

Ac ſi ſola s variabilis accipiatur, tum erit

$$\partial x(\frac{\partial \partial v}{\partial y \partial s}) = \partial.(\frac{\partial \partial z}{\partial y \partial s}) + \partial x(\frac{\partial \partial z}{\partial y \partial r}),$$

ſicque porro

Euolutio formulae

$$(\frac{\partial v}{\partial p}) = (\frac{\partial \partial z}{\partial x \partial p}) + p(\frac{\partial \partial z}{\partial y \partial p}) + q(\frac{\partial \partial z}{\partial p^2}) + r(\frac{\partial \partial z}{\partial q \partial p}) + \text{etc.} + (\frac{\partial z}{\partial y})$$

quae reducitur ad hanc formam:

$$\partial x(\frac{\partial v}{\partial p}) = \partial.(\frac{\partial z}{\partial p}) + \partial x(\frac{\partial z}{\partial y}).$$

§. 21. Quodſi hic vel ſola x vel ſola y variabilis accipiatur, haec forma ſimpliciter differentiata ad quaeſitum perducit: priori ſcilicet caſu prodit

$$\partial x(\frac{\partial \partial v}{\partial x \partial p}) = \partial.(\frac{\partial \partial z}{\partial x \partial p}) + \partial x(\frac{\partial \partial z}{\partial x \partial y});$$

poſteriore vero erit

$$\partial x(\frac{\partial \partial v}{\partial y \partial p}) = \partial.(\frac{\partial \partial z}{\partial y \partial p}) + \partial x(\frac{\partial \partial z}{\partial y^2}),$$

haecque duae formulae iam ante prodierunt.

§. 22. Sin autem littera p variabilis ſtatuatur, quoniam formula $\partial.(\frac{\partial z}{\partial p})$ continet partem $p\,\partial x(\frac{\partial \partial z}{\partial y \partial p})$, huius differentiatio praebet $\partial x(\frac{\partial \partial z}{\partial y \partial p})$, quod ergo ad differentialia, ex reliquis membris oriunda, inſuper addi debet; hoc ergo modo prodibit iſta aequalitas:

∂x

$$\partial x \left(\frac{\partial \partial v}{\partial p^2}\right) = \partial \cdot \left(\frac{\partial \partial z}{\partial p^2}\right) + 2\, \partial x \left(\frac{\partial \partial z}{\partial y\, \partial p}\right),$$

vnde intelligitur ob ∂p^2, quod in prima formula occurrit, postremum terminum duplicari debere. Theorematibus autem hinc deducendis non immoramur, quandoquidem deinceps theoremata multo generaliora producere licebit.

§. 23. Sumamus nunc folam q variabilem, et quoniam in formula $\partial \cdot \left(\frac{\partial z}{\partial p}\right)$ euoluta occurrit terminus $q\, \partial x \left(\frac{\partial \partial z}{\partial p^2}\right)$, ex hoc per differentiationem nafcitur terminus $\partial x \left(\frac{\partial \partial z}{\partial p^2}\right)$. Hinc ergo facta tota differentiatione perueniemus ad hanc aequationem:

$$\partial x \left(\frac{\partial \partial v}{\partial p\, \partial q}\right) = \partial \cdot \left(\frac{\partial \partial z}{\partial p\, \partial q}\right) + \partial x \left(\frac{\partial \partial z}{\partial y\, \partial q}\right) + \partial x \left(\frac{\partial \partial z}{\partial p^2}\right),$$

vbi patet, ob bina elementa ∂p et ∂q, infuper duos terminos adiici oportere, id quod etiam eueniet, fi fola r pro variabili fumatur; nam quia formula $\partial \cdot \left(\frac{\partial z}{\partial p}\right)$ continet terminum $r\, \partial x \left(\frac{\partial \partial z}{\partial p\, \partial q}\right)$, ex hoc per differentiationem prodibit $\partial x \left(\frac{\partial \partial z}{\partial p\, \partial q}\right)$, quem ad reliquas partes infuper adiici oportet; hocque modo impetrabimus hanc aequationem:

$$\partial x \left(\frac{\partial \partial v}{\partial p\, \partial r}\right) = \partial \cdot \left(\frac{\partial \partial z}{\partial p\, \partial r}\right) + \partial x \left(\frac{\partial \partial z}{\partial y\, \partial r}\right) + \partial x \left(\frac{\partial \partial z}{\partial p\, \partial q}\right),$$

vbi iterum ob elementa ∂p et ∂r duo membra acceſſerunt.

Euolutio formulae

$$\left(\frac{\partial v}{\partial q}\right) = \left(\frac{\partial \partial z}{\partial x\, \partial q}\right) + p \left(\frac{\partial \partial z}{\partial y\, \partial q}\right) + q \left(\frac{\partial \partial z}{\partial p\, \partial q}\right) + r \left(\frac{\partial \partial z}{\partial q^2}\right) + \text{etc.} + \left(\frac{\partial z}{\partial p}\right),$$

quae reducta eſt ad hanc:

$$\partial x \left(\frac{\partial v}{\partial q}\right) = \partial \cdot \left(\frac{\partial z}{\partial q}\right) + \partial x \left(\frac{\partial z}{\partial p}\right).$$

§. 24. Si hic vel x vel y folum variabile capiatur, nihil in differentiatione de nouo accedit, eritque cafu priore

$$\partial x \left(\frac{\partial \partial v}{\partial x\, \partial q}\right) = \partial \cdot \left(\frac{\partial \partial z}{\partial x\, \partial q}\right) + \partial x \left(\frac{\partial \partial z}{\partial x\, \partial p}\right),$$

 po-

posteriore vero casu

$$\partial x\left(\frac{\partial\partial V}{\partial y\partial q}\right)=\partial.\left(\frac{\partial\partial Z}{\partial y\partial q}\right)+\partial x\left(\frac{\partial\partial Z}{\partial y\partial p}\right).$$

In reliquis autem differentiationibus elementum ∂p suppeditat, praeter differentiationem solitam, insuper membrum $\partial x\left(\frac{\partial\partial Z}{\partial y\partial q}\right)$, at vero elementum ∂q producit $\partial x\left(\frac{\partial\partial Z}{\partial p\partial q}\right)$; elementum porro ∂r producit $\partial x\left(\frac{\partial\partial Z}{\partial q^2}\right)$, elementum ∂s vero praebet $\partial x\left(\frac{\partial\partial Z}{\partial q\partial r}\right)$ etc. quibus obseruatis obtinebuntur sequentes aequationes:

$$\text{I. } \partial x\left(\frac{\partial\partial V}{\partial p\partial q}\right)=\partial.\left(\frac{\partial\partial Z}{\partial p\partial q}\right)+\partial x\left(\frac{\partial\partial Z}{\partial p^2}\right)+\partial x\left(\frac{\partial\partial Z}{\partial y\partial q}\right),$$

$$\text{II. } \partial x\left(\frac{\partial\partial V}{\partial q^2}\right)=\partial.\left(\frac{\partial\partial Z}{\partial q^2}\right)+2\,\partial x\left(\frac{\partial\partial Z}{\partial p\partial q}\right),$$

$$\text{III. } \partial x\left(\frac{\partial\partial V}{\partial q\partial r}\right)=\partial.\left(\frac{\partial\partial Z}{\partial q\partial r}\right)+\partial x\left(\frac{\partial\partial Z}{\partial p\partial r}\right)+\partial x\left(\frac{\partial\partial Z}{\partial q^2}\right),$$

$$\text{IV. } \partial x\left(\frac{\partial\partial V}{\partial q\partial s}\right)=\partial.\left(\frac{\partial\partial Z}{\partial q\partial s}\right)+\partial x\left(\frac{\partial\partial Z}{\partial p\partial s}\right)+\partial x\left(\frac{\partial\partial Z}{\partial q\partial r}\right).$$

§. 25. Ex his iam abunde perspicitur, perpetuo, quoties vel sola x, vel sola y variabilis accipitur, differentiationem more consueto institui debere, nihilque insuper esse adiiciendum; sin autem reliquae litterae p, q, r, s, etc. variabiles accipiantur, tum pro quolibet elemento siue ∂p, siue ∂q, siue ∂r, etc. praeterea vnum nouum terminum accedere debere. Hinc igitur pro solis elementis ∂x et ∂y iam sequens theorema latissime patens constitui potest:

Theorema generale. I.

Si fuerit $\int V\partial x = Z$, tum semper erit

$$\int\partial x\left(\frac{\partial^{\alpha+\beta}V}{\partial x^\alpha\,\partial y^\beta}\right)=\left(\frac{\partial^{\alpha+\beta}Z}{\partial x^\alpha\,\partial y^\beta}\right).$$

Euo-

Euolutio harum formularum, ſi ſola p pro variabili accipiatur.

§. 26. Quemadmodum iam vidimus, cum ſit

$$\partial x\left(\frac{\partial V}{\partial p}\right)=\partial.\left(\frac{\partial Z}{\partial p}\right)+\partial x\left(\frac{\partial Z}{\partial y}\right),$$

tum fore

$$\partial x\left(\frac{\partial\partial V}{\partial p^2}\right)=\partial.\left(\frac{\partial\partial Z}{\partial p^2}\right)+2\,\partial x\left(\frac{\partial\partial Z}{\partial y\,\partial p}\right);$$

ita ſi porro differentiemus, ex ſola variabilitate ipſius p prodibit:

$$\text{I}^{\circ}.\ \partial x\left(\frac{\partial^3 V}{\partial p^3}\right)=\partial.\left(\frac{\partial^3 Z}{\partial p^3}\right)+3\,\partial x\left(\frac{\partial^3 Z}{\partial y\,\partial p^2}\right),$$

$$\text{II}^{\circ}.\ \partial x\left(\frac{\partial^4 V}{\partial p^4}\right)=\partial.\left(\frac{\partial^4 Z}{\partial p^4}\right)+4\,\partial x\left(\frac{\partial^4 Z}{\partial y\,\partial p^3}\right),$$

$$\text{III}^{\circ}.\ \partial x\left(\frac{\partial^5 V}{\partial p^5}\right)=\partial.\left(\frac{\partial^5 Z}{\partial p^5}\right)+5\,\partial x\left(\frac{\partial^5 Z}{\partial y\,\partial p^4}\right),$$

vnde generaliter habebimus

$$\partial x\left(\frac{\partial^\gamma V}{\partial p^\gamma}\right)=\partial.\left(\frac{\partial^\gamma Z}{\partial p^\gamma}\right)+\gamma\,\partial x\left(\frac{\partial^\gamma Z}{\partial y\,\partial p^{\gamma-1}}\right);$$

hincque deducimus ſequens

Theorema generale 2.

Si fuerit $\int V\partial x=Z$, tum ſemper erit

$$\int\partial x\left(\frac{\partial^{\alpha+\beta+\gamma}V}{\partial x^\alpha\partial y^\beta\partial p^\gamma}\right)=\left(\frac{\partial^{\alpha+\beta+\gamma}Z}{\partial x^\alpha\partial y^\beta\partial p^\gamma}\right)+\gamma\int\partial x\left(\frac{\partial^{\alpha+\beta+\gamma}Z}{\partial x^\alpha\partial y^{\beta+1}\partial p^{\gamma-1}}\right).$$

§. 27. Quodſi iam vlterius quantitas Q pro variabili ſumatur, et differentiatio continuo repetatur, inueſtigationem ſequenti modo ſuſcipiamus. Quoniam elementa ∂x et ∂y nihil turbant, proficiſcamur a formula ſupra inuenta:

$$\partial x\left(\frac{\partial^\gamma V}{\partial p^\gamma}\right)=\partial.\left(\frac{\partial^\gamma Z}{\partial p^\gamma}\right)+\gamma\,\partial x\left(\frac{\partial^\gamma Z}{\partial y\,\partial p^{\gamma-1}}\right),$$

vnde

vnde variabilitas solius q primo dabit:

$$\partial x\left(\frac{\partial^{\gamma+1}V}{\partial p^{\gamma}\partial q}\right)=\partial\cdot\left(\frac{\partial^{\gamma+1}Z}{\partial p^{\gamma}\partial q}\right)+\gamma\,\partial x\left(\frac{\partial^{\gamma+1}Z}{\partial y\,\partial p^{\gamma-1}\partial q}\right)$$
$$+\partial x\left(\frac{\partial^{\gamma+1}Z}{\partial p^{\gamma+1}}\right).$$

§. 28. Quodsi iam hanc formam vlterius secundum ∂q differentiemus, perueniemus ad hanc aequationem:

$$\partial x\left(\frac{\partial^{\gamma+2}V}{\partial p^{\gamma}\partial q^{2}}\right)=\partial\cdot\left(\frac{\partial^{\gamma+2}Z}{\partial p^{\gamma}\partial q^{2}}\right)+\gamma\,\partial x\left(\frac{\partial^{\gamma+2}Z}{\partial y\,\partial p^{\gamma-1}\partial q^{2}}\right)$$
$$+2\,\partial x\left(\frac{\partial^{\gamma+2}Z}{\partial p^{\gamma+1}\partial q}\right),$$

et denuo differentiando prodibit:

$$\partial x\left(\frac{\partial^{\gamma+3}V}{\partial p^{\gamma}\partial q^{3}}\right)=\partial\cdot\left(\frac{\partial^{\gamma+3}Z}{\partial p^{\gamma}\partial q^{3}}\right)+\gamma\,\partial x\left(\frac{\partial^{\gamma+3}Z}{\partial y\,\partial p^{\gamma-1}\partial q^{3}}\right)$$
$$+3\,\partial x\left(\frac{\partial^{\gamma+3}Z}{\partial p^{\gamma+1}\partial q^{2}}\right),$$

haecque sufficiunt ad constituendum sequens

Theorema generale 3.

Si fuerit $\int V\partial x=Z$, tum semper erit:

$$\int\partial x\left(\frac{\partial^{\alpha+\beta+\gamma+\delta}V}{\partial x^{\alpha}\partial y^{\beta}\partial p^{\gamma}\partial q^{\delta}}\right)=\left(\frac{\partial^{\alpha+\beta+\gamma+\delta}Z}{\partial x^{\alpha}\partial y^{\beta}\partial p^{\gamma}\partial q^{\delta}}\right)$$
$$+\gamma\int\partial x\left(\frac{\partial^{\alpha+\beta+\gamma+\delta}Z}{\partial x^{\alpha}\partial y^{\beta+1}\partial p^{\gamma-1}\partial q^{\delta}}\right)$$
$$+\delta\int\partial x\left(\frac{\partial^{\alpha+\beta+\gamma+\delta}Z}{\partial x^{\alpha}\partial y^{\beta}\partial p^{\gamma+1}\partial q^{\delta-1}}\right).$$

§. 29.

§. 29. Iam pluribus ambagibus opus non erit ad sequens theorema generalissimum constituendum:

Theorema generalissimum.

Si fuerit $\int V \partial x = Z$, *tum semper erit*

$$\int \partial x \left(\frac{\partial^{\alpha+\beta+\gamma+\delta+\varepsilon+\zeta} V}{\partial x^{\alpha} \partial y^{\beta} \partial p^{\gamma} \partial q^{\delta} \partial r^{\varepsilon} \partial s^{\zeta}} \right) = \left(\frac{\partial^{\alpha+\beta+\gamma+\delta+\varepsilon+\zeta} Z}{\partial x^{\alpha} \partial y^{\beta} \partial p^{\gamma} \partial q^{\delta} \partial r^{\varepsilon} \partial s^{\zeta}} \right)$$

$$+ \gamma \int \partial x \left(\frac{\partial^{\alpha+\beta+\gamma+\delta+\varepsilon+\zeta} Z}{\partial x^{\alpha} \partial y^{\beta+1} \partial p^{\gamma-1} \partial q^{\delta} \partial r^{\varepsilon} \partial s^{\zeta}} \right)$$

$$+ \delta \int \partial x \left(\frac{\partial^{\alpha+\beta+\gamma+\delta+\varepsilon+\zeta} Z}{\partial x^{\alpha} \partial y^{\beta} \partial p^{\gamma+1} \partial q^{\delta-1} \partial r^{\varepsilon} \partial s^{\zeta}} \right)$$

$$+ \varepsilon \int \partial x \left(\frac{\partial^{\alpha+\beta+\gamma+\delta+\varepsilon+\zeta} Z}{\partial x^{\alpha} \partial y^{\beta} \partial p^{\gamma} \partial q^{\delta+1} \partial r^{\varepsilon-1} \partial s^{\zeta}} \right)$$

$$+ \zeta \int \partial x \left(\frac{\partial^{\alpha+\beta+\gamma+\delta+\varepsilon+\zeta} Z}{\partial x^{\alpha} \partial y^{\beta} \partial p^{\gamma} \partial q^{\delta} \partial r^{\varepsilon+1} \partial s^{\zeta-1}} \right).$$