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Dilucidationes super formulis, quibus sinus et cosinus angulorum multiplorum exprimi solent, ubi simul ingentes difficultates diluuntur

Leonhard Euler

DILVCIDATIONES SVPER FORMVLIS, QVIBVS SINVS ET COSINVS ANGVLORVM MVLTIPLORVM EXPRIMI SOLENT, VBI SIMVL INGENTES DIFFICVLTATES DILVVNTVR.

Auctore

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§. 1.

Proposito angulo quocunque ϕ, si eius cosinus vocetur $= \frac{1}{2} x$, vt sit $2 \cos. \phi = x$, constat tam sinus quam cosinus angulorum multiplorum constituere progressionem recurrentem, cuius scala relationis est $x, -1$; erit enim

$$\sin. (n + 1) \phi = x \sin. n \phi - \sin. (n - 1) \phi,$$
$$\cos. (n + 1) \phi = x \cos. n \phi - \cos. (n - 1) \phi.$$

Incipiamus a posteriori formula, et quoniam dupla cosinuum eandem legem seruant, hinc sequens tabula construatur:

2 cos.

$2 \cos. 0\,\Phi = 2,$
$2 \cos. 1\,\Phi = x,$
$2 \cos. 2\,\Phi = xx - 2,$
$2 \cos. 3\,\Phi = x^3 - 3x,$
$2 \cos. 4\,\Phi = x^4 - 4xx + 2,$
$2 \cos. 5\,\Phi = x^5 - 5x^3 - 5x,$
$2 \cos. 6\,\Phi = x^6 - 6x^4 + 9xx - 2,$
$2 \cos. 7\,\Phi = x^7 - 7x^5 + 14x^3 - 7x,$
$2 \cos. 8\,\Phi = x^8 - 8x^6 + 20x^4 - 16xx + 2,$
$2 \cos. 9\,\Phi = x^9 - 9x^7 + 27x^5 - 30x^3 + 9x,$
etc. etc.

§. 2. In his formulis manifestum est primos terminos esse potestates ipsius x eiusdem exponentis, cuius multiplum proponitur; hinc vero exponentes continuo binario decrescere; tum vero signa terminorum alternari; praeterea vero coëfficiens secundi termini semper aequalis est ipsi multiplo; quod autem ad sequentes coëfficientes attinet, lex, qua progrediuntur, ita se habet, vti sequens forma ostendit:

$$2 \cos. n\Phi = x^n - nx^{n-2} + \frac{n(n-3)x^{n-4}}{1.\ 2.} - \frac{n(n-4)(n-5)x^{n-6}}{1.\ 2.\ 3.}$$
$$+ \frac{n(n-5)(n-6)(n-7)x^{n-8}}{1.\ 2.\ 3.\ 4.} - \frac{n(n-6)(n-7)(n-8)(n-9)x^{n-10}}{1.\ 2.\ 3.\ 4.\ 5.} \text{ etc.}$$

cuius formulae summus est vsus in cosinibus angulorum multiplorum quantumuis magnorum expedite assignandis. Veluti si proponatur duodecuplum anguli Φ, hinc statim obtinetur sequens expressio:

2 cos.

$$2 \cos. 12 \phi = x^{12} - 12 x^{10} + 54 x^8 - 112 x^6 + 105 x^4 - 36 x x + 2,$$

cuius veritas facile comprobatur, cum fit

$$2 \cos. 12 \phi = (2 \cos. 6 \phi)^2 - 2.$$

§. 3. Quanquam autem haec formula maximum praeftat vfum in multiplicatione angulorum, tamen fecundum rigorem geometricum neutiquam affirmari poteft, eam generaliter veritati effe confentaneam, quandoquidem pluribus cafibus maxime a veritate recedit. Quodfi enim fumamus $n = 0$, ifta formula praebet $2 \cos. 0 \phi = 1$, cum tamen eius valor fit 2. Multo magis formula aberrare deprehenditur, fi indici n valores negatiui tribuantur: pofito enim $n = -1$, inde prodiret

$$2 \cos. - \phi = \frac{1}{x} + \frac{1}{x^3} + \frac{2}{x^5} + \frac{5}{x^7} + \frac{14}{x^9} + \frac{42}{x^{11}} + \text{etc.}$$

in infinitum, quae expreffio manifefto eft falfiffima, cum fit

$$2 \cos. - \phi = 2 \cos. \phi = x,$$

illius vero feriei fumma $= \frac{x}{2} - \sqrt{(\frac{xx}{4} - 1)}$, quae igitur plurimum a veritate difcrepat, ficque aberratio formulae inuentae pro omnibus numeris negatiuis clariffime in oculos incurrit.

§. 4. Non folum autem haec formula pro numeris negatiuis fallit, fed etiam pro pofitiuis, fiquidem tota expreffio euoluatur, fumto enim $n = 1$, hinc nancifcemur iftam feriem:

$$2 \cos. \phi = x - \frac{1}{x} - \frac{1}{x^3} - \frac{2}{x^5} - \frac{5}{x^7} - \frac{14}{x^9} - \text{etc.}$$

quae ergo omnibus terminis poft primum fequentibus a veritate recedit: vera autem fumma huius expreffionis eft

$\frac{x}{2}$ +

Simili modo pro casu $n = 2$ ista formula praebet

$$2 \cos. 2\,\phi = xx - 2 - \frac{1}{xx} - \frac{2}{x^4} - \frac{5}{x^6} - \frac{14}{x^8} - \frac{42}{x^{10}} - \text{etc.}$$

vbi duo tantum primi termini veritati sunt conformes, sequentes omnes vero superflui atque adeo veritati contrarii. Hinc igitur intelligitur formulam illam pro $2 \cos. n\,\phi$ datam, siquidem in infinitum continuetur, semper in immanes errores praecipitare; sicque omnis eius vsus tantum restringitur ad numeros integros positiuos, vbi insuper caveri debet, ne series vltra terminos integros continuetur, quandoquidem nulli termini, vbi x in denominatores ingreditur, locum habere possunt.

§. 5. Quod autem ad numeros fractos attinet, nullo plane modo ista series cum veritate conciliari se patitur. Quodsi enim ponamus $n = \frac{1}{2}$, ista formula nobis praebebit

$$2 \cos. \tfrac{1}{2}\,\phi = \sqrt{x} - \frac{1}{2\,x\,\sqrt{x}} - \frac{5}{8\,x\,\sqrt{x}} - \frac{21}{16\,x^3\,\sqrt{x}} - \text{etc.}$$

Cum vero sit

$$\cos. \tfrac{1}{2}\,\phi = \sqrt{\frac{1 + \cos.\,\phi}{2}} = \sqrt{\frac{2 + x}{4}},$$

erit $2 \cos. \frac{1}{2}\,\phi = \sqrt{(2 + x)}$, quae autem expressio in seriem conuersa ab illa plurimum differt, cum sit

$$\sqrt{(x + 2)} = \sqrt{x} + \frac{1}{\sqrt{x}} - \frac{1}{2\,x\,\sqrt{x}} - \frac{1}{2\,x\,x\,\sqrt{x}} - \text{etc.}$$

Ex his iam manifestum est formulam inuentam, non obstante summo eius vsu, nullo modo tanquam veritati consentaneam admitti posse; vnde haec quaestio maximi momenti nascitur: cur ista formula a veritate ita aberret, vt certis tamen casibus ad veritatem perducat?

§. 6. Idem prorſus euenit in formula generali pro ſinibus angulorum multiplorum tradi ſolita. Si enim ponamus $2\,\text{ſin.}\,\phi = y$, quoniam etiamnunc eadem ſcala relationis $x, -1$ valet, ſinus angulorum multiplorum ſequenti modo progredientur:

$$\begin{aligned}
2\,\text{ſin.}\ 0\,\phi &= 0,\\
2\,\text{ſin.}\ 1\,\phi &= y,\\
2\,\text{ſin.}\ 2\,\phi &= y \cdot x,\\
2\,\text{ſin.}\ 3\,\phi &= y\,(xx - 1),\\
2\,\text{ſin.}\ 4\,\phi &= y\,(x^3 - 2x),\\
2\,\text{ſin.}\ 5\,\phi &= y\,(x^4 - 3xx + 1),\\
2\,\text{ſin.}\ 6\,\phi &= y\,(x^5 - 4x^3 + 3x),\\
2\,\text{ſin.}\ 7\,\phi &= y\,(x^6 - 5x^4 + 6xx - 1),\\
2\,\text{ſin.}\ 8\,\phi &= y\,(x^7 - 6x^5 + 10x^3 - 4x),\\
2\,\text{ſin.}\ 9\,\phi &= y\,(x^8 - 7x^6 + 15x^4 - 10xx + 1),\\
2\,\text{ſin.}\ 10\,\phi &= y\,(x^9 - 8x^7 + 21x^5 - 20x^3 + 5x).\\
&\text{etc.} \qquad\qquad \text{etc.}
\end{aligned}$$

Hic ſcilicet y ab x ita pendet, vt ſit $y = \sqrt{(4 - xx)}$.

§. 7. Contemplatio harum formularum ſimili modo vt ante pro angulo indefinito $n.\phi$ ſequentem ſuppeditabit formulam generalem:

$$\begin{aligned}
2\,\text{ſin.}\,n\,\phi = y\Big(x^{n-1} &- \frac{(n-2)}{1}x^{n-3} + \frac{(n-3)(n-4)}{1.\;2}x^{n-5}\\
&- \frac{(n-4)(n-5)(n-6)}{1.\;2.\;3}x^{n-7} + \frac{(n-5)(n-6)(n-7)(n-8)}{1.\;2.\;3.\;4}x^{n-9}\\
&- \frac{(n-6)(n-7)(n-8)(n-9)(n-10)}{1.\;2.\;3.\;4.\;5}x^{n-11} + \text{etc.}\Big)
\end{aligned}$$

Haec autem formula cum veritate conſiſtere nequit, niſi ita reſtringatur, vt primo tantum ad numeros integros poſitiuos pro n aſſumendos applicetur; deinde vt termini non vlte-

vlterius continuentur, quam quoad ad exponentes negatiuos ipſius x peruẽniatur. Ita ſumendo $n = 12$ hinc repe-rietur:

$$2 \sin. 12 \Phi = y (x^{11} - 10 x^9 + 36 x^7 - 56 x^5 + 35 x^3 - 6 x).$$

§. 8. Cum igitur ambae iſtae formulae generales tam pro ſinibus, quam coſinibus angulorum multiplorum dari ſolitae tam enormiter a veritate diſſentiant, hinc quaeſtio naſcitur maximi momenti: quomodo hi errores euitari atque eiusmodi ſeries erui debeant, quae cum veritate perpetuo, atque omnibus plane caſibus perfecte conſentiant? Tales igitur ſeries ex ipſis Analyſeos principiis hic inuoſtigare conſtitui. Vocabo igitur $\cos. \Phi = z$ et $\cos. n \Phi = s$, et in ſeriem per poteſtates ipſius z procedentem inquiram, quae verum valorem ipſius s exhibeat, quicunque numeri, ſiue poſitiui, ſiue negatiui, ſiue integri, ſiue fracti pro n ſubſtituantur.

§. 9. Cum igitur ſit $z = \cos. \Phi$, erit $\partial \Phi = \frac{-\partial z}{\sqrt{(1 - zz)}}$; ſimilique modo, cum ſit $s = \cos. n \Phi$, erit $n \partial \Phi = \frac{-\partial s}{\sqrt{(1 - ss)}}$, vnde ſequitur fore $\frac{\partial s}{\sqrt{(1 - ss)}} = \frac{n \partial z}{\sqrt{(1 - zz)}}$. Haec vero eadem aequatio prodiiſſet, ſi litterae z et s deſignaſſent ſinus angulorum Φ et $n \Phi$; ſi enim altera ſinum, altera coſinum ſignificaſſet, prodiiſſet $\frac{\partial s}{\sqrt{(1 - ss)}} = \frac{-n \partial z}{\sqrt{(1 - zz)}}$. Hanc obrem ſi quadrata ſumamus, haec aequatio: $\frac{\partial s^2}{1 - ss} = \frac{nn \partial z^2}{1 - zz}$, omnes illas varietates in ſe complectetur. Conſideremus igitur hanc aequationem differentialem:

$$\partial s^2 (1 - zz) = nn \partial z^2 (1 - ss),$$

ex qua quo commodius ſeries pro s erui queat, eam denuo differentiemus, ſumto elemento ∂z conſtante, ſicque

 nan-

nancifcemur hanc aequationem differentialem fecundi gradus:

$$\partial\partial s(1 - zz) - z\partial z\partial s + nns\partial z^2 = 0,$$

quae latiffimo fenfu omnia in fe complectitur, quae tam circa finus quam cofinus angulorum multiplorum defiderari poffunt. Hanc igitur aequationem omni cura pertractemus, ac primo quidem fine vllo refpectu ad doctrinam angulorum.

Problema 1.

Propofita aequatione differentiali fecundi gradus

$$\partial\partial s(1 - zz) - z\partial z\partial s + nns\partial z^2 = 0,$$

eius integrale completum per duplicem integrationem inueftigare.

Solutio.

§. 10. Hic ftatim patet, hanc aequationem integrabilem fieri, fi ducatur in ∂s; multiplicetur igitur in $2\partial s$ et integrale erit

$$\partial s^2(1 - zz) + nnss\partial z^2 = C\partial z^2.$$

Ex hac aequatione deducimus $\partial s^2 = \frac{\partial z^2(C - nnss)}{1 - zz}$, quae formula ita repraefentetur: $\partial s^2 = \frac{\partial z^2(nnss - C)}{zz - 1}$, ex qua eruitur: $\frac{\partial s}{\sqrt{(nnss - C)}} = \frac{\partial z}{\sqrt{(zz - 1)}}$.
Cum igitur fit

$$\int\frac{\partial z}{\sqrt{(zz - 1)}} = l[z + \sqrt{(zz - 1)}] \text{ et}$$

$$\int\frac{\partial s}{\sqrt{(nnss - C)}} = \frac{1}{n}l[ns + \sqrt{(nnss - C)}],$$

noftra aequatio erit

$$\frac{1}{n}l[ns + \sqrt{(nnss - C)}] = l[z + \sqrt{(zz - 1)}] + lD.$$

Multiplicemus per n et ad numeros afcendendo reperiemus

$$ns + \sqrt{(nnss - C)} = D[z + \sqrt{(zz - 1)}]^n.$$

§. 11.

§. 11. Quo nunc hinc valorem ipfius s eruamus, ponamus breuitatis gratia: $ns + \sqrt{(nnss - C)} = Q - ns$, vt fit

$$Q = D[z + \sqrt{(zz-1)}]^n, \text{ eritque}$$

$$\sqrt{(nnss - C)} = Q - ns, \text{ vnde elicitur:}$$

$$s = \frac{QQ + C}{2nQ} = \frac{Q}{2n} + \frac{C}{2nQ}.$$

Quodfi iam forma conftantium arbitrariarum immutetur, valor integralis completus quantitatis s per variabilem z ita concinne exprimi poterit:

$$s = f[z + \sqrt{(zz-1)}]^n + g[z + \sqrt{(zz-1)}]^n,$$

quae forma etiam hoc modo exhiberi poteft:

$$s = f[z + \sqrt{(zz-1)}]^n + g[z - \sqrt{(zz-1)}]^n.$$

Alia folutio fuccinctior.

§. 12. Quanquam hic integrale completum quaerimus, tamen fufficiet bina integralia particularia inueftigasse. Quoniam enim in aequatione propofita variabilis s vbique vnicam tantum habet dimenfionem, fi ei fatisfaciant valores $s = p$ et $s = q$, etiam fatisfaciet valor $s = p + q$, atque adeo in genere $s = fp + gq$, denotantibus litteris f et g conftantes quascunque. Hoc obferuato negligatur conftans in prima integratione adiecta, eritque $\partial s^2 = \frac{nnss\partial z^2}{zz-1}$, ideoque $\frac{\partial s}{s} = \frac{n\partial z}{\sqrt{(zz-1)}}$, hincque porro fit

$$s = [z + \sqrt{(zz-1)}]^n.$$

§. 13. Quia in aequationem differentialem tantum quadratum n ingreditur, cuius radix aeque eft $-n$ ac $+n$, integrale quoque particulare erit

$$s = [z + \sqrt{(zz-1)}]^n = [z - \sqrt{(zz-1)}]^n,$$

 ficſicque duo habemus integralia particularia, per litteras p et q deſignata, ex iis igitur conflatur iſtud integrale completum:

$$s = f[z + \sqrt{(zz - 1)}]^n + g[z - \sqrt{(zz - 1)}]^n.$$

Problema 2.

Eiusdem aequationis differentialis ſecundi gradus:

$$\partial\partial s(1 - zz) - z\partial z\partial s + nns\partial z^2 = 0,$$

integrale completum per ſeriem infinitam exprimere, cuius termini per poteſtates ipſius z deſcendendo progrediantur.

Solutio.

§. 14. Quaerimus igitur pro valore litterae s ſeriem, cuius ſinguli termini ſint poteſtates ipſius z, quarum exponentes continuo decreſcant. Ex ipſa autem aequationis propoſitae forma facile concludere licet, iſtos exponentes continuo binario diminui debere, propterea quod in hac aequatione variabilis z cum ſuo differentiali ∂z in ſingulis terminis vel nullam, vel duas habet dimenſiones; vnde ſi primus terminus contineat poteſtatem z^λ, ſequentes termini poteſtates $z^{\lambda-2}$; $z^{\lambda-4}$; $z^{\lambda-6}$; etc. continebunt, ita vt ſeries, quam quaerimus, talem ſit habitura formam:

$$s = Az^\lambda - Bz^{\lambda-2} + Cz^{\lambda-4} - Dz^{\lambda-6} + Ez^{\lambda-8} - Fz^{\lambda-10} + \text{etc.}$$

vbi ergo totum negotium huc redit, vt valores ſingulorum coëfficientium rite determinemus.

§. 15. Ante omnia autem hic inueſtigari oportet primum exponentem λ, a quo haec ſeries ſit incipienda, qui ita comparatus eſſe debet, vt facta ſubſtitutione coëfficientes primi termini ſponte ſe deſtruant. Ad iſtum exponentem inueniendum ſufficiet tantum duos terminos priores

res consideraffe, scilicet $s = Az^{\lambda} - Bz^{\lambda-2}$, quae substitutio quo facilius fieri queat, ipsam aequationem propositam, per ∂z^2 diuidendo, ita referamus:

$$\frac{\partial\partial s}{\partial z^2} - \frac{zz\partial\partial s}{\partial z^2} - \frac{z\partial s}{\partial z} + nns = 0,$$

et cum iam fit

$$\frac{\partial s}{\partial z} = \lambda Az^{\lambda-1} - (\lambda-2)Bz^{\lambda-3} \text{ et}$$

$$\frac{\partial\partial s}{\partial z^2} = \lambda(\lambda-1)Az^{\lambda-2} - (\lambda-2)(\lambda-3)Bz^{\lambda-4},$$

facta substitutione fieri debebit:

$$\left.\begin{array}{ll} -\lambda(\lambda-1)Az^{\lambda} & +\lambda(\lambda-1)Az^{\lambda-2} \\ -\lambda Az^{\lambda} & +(\lambda-2)(\lambda-3)Bz^{\lambda-2} \\ +nnAz^{\lambda} & +(\lambda-2)Bz^{\lambda-2} \\ & -nnBz^{\lambda-2} \end{array}\right\} = 0.$$

Hic ergo ante omnia coëfficiens primi termini Az^{λ} nihilo aequalis statui debet, sicque fiet $-\lambda(\lambda-1) - \lambda + nn = 0$, ideoque $\lambda\lambda = nn$, vnde duo valores pro λ obtinentur, scilicet vel $\lambda = +n$ vel $\lambda = -n$.

§. 16. Quoniam igitur pro λ geminum nacti sumus valorem, hinc pro quantitate s duas eruemus series infinitas, quae iunctim sumtae valorem integralem completum expriment, namque verus iste valor ita exhibebitur:

$$s = \begin{cases} Az^{n} - Bz^{n-2} + Cz^{n-4} - Dz^{n-6} + Ez^{n-8} - \text{etc.} \\ +\mathfrak{A}z^{-n} - \mathfrak{B}z^{-n-2} + \mathfrak{C}z^{-n-4} - \mathfrak{D}z^{-n-6} + \mathfrak{E}z^{-n-8} - \text{etc.} \end{cases}$$

et quoniam facta substitutione pro vtraque serie coëfficientes primorum terminorum Az^{n} et $\mathfrak{A}z^{-n}$ sponte se tollunt, binae litterae A et $\mathfrak{A}$ non determinabuntur, sed penitus arbitrio nostro relinquentur, ideoque vicem gerent binarum constantium, quae per duplicem integrationem ingredi sunt censendae.

dae. Euidens porro eſt vtramque hanc ſeriem ſeorſim inueſtigari poſſe; vnde ſolutio noſtra duabus conſtabit partibus:

I. Inueſtigatio ſeriei prioris

$$s = Az^n - Bz^{n-2} + Cz^{n-4} - Dz^{n-6} + Ez^{n-8} - \text{etc.}$$

§. 17. Quo commodius huius ſeriei coëfficientes determinemus, aequationem propoſitam mutatis ſignis ita repraeſentemus:

$$\frac{zz\partial\partial s}{\partial z^2} - \frac{\partial\partial s}{\partial z^2} + \frac{z\partial s}{\partial z} - nns = 0;$$

et cum ſit

$$\frac{\partial s}{\partial z} = nAz^{n-1} - (n-2)Bz^{n-3} + (n-4)Cz^{n-5} - (n-6)Dz^{n-7} \text{ etc.}$$

et

$$\frac{\partial\partial s}{\partial z^2} = n(n-1)Az^{n-2} - (n-2)(n-3)Bz^{n-4} + (n-4)(n-5)Cz^{n-6} \text{ etc.}$$

ordinetur ſubſtitutio ſequenti modo:

$$\begin{array}{rl}
\frac{zz\partial\partial s}{\partial z^2} = & n(n-1)Az^n - (n-2)(n-3)Bz^{n-2} + (n-4)(n-5)Cz^{n-4} \text{ etc.} \\
-\frac{\partial\partial s}{\partial z^2} = & \qquad - (n)(n-1)Az^{n-2} + (n-2)(n-3)Bz^{n-4} \text{ etc.} \\
+\frac{z\partial s}{\partial z} = & nAz^n - (n-2)Bz^{n-2} + (n-4)Cz^{n-4} \text{ etc.} \\
-nns = & -nnAz^n + nnBz^{n-2} - nnCz^{n-4} \text{ etc.} \\
\hline
\end{array}$$

$$0 = \left\{ \begin{array}{l}
-(n-2)^2Bz^{n-2} + (n-4)^2Cz^{n-4} - (n-6)^2Dz^{n-6} \\
+ \quad nnB \qquad - \quad nnC \qquad + \quad nnD \\
-n(n-1)A + (n-2)(n-3)B - (n-4)(n-5)C
\end{array} \right\} \text{etc.}$$

§. 18. Nunc igitur quemlibet coëfficientem per ſuum antecedentem ſatis concinne determinare licebit; erit enim

$$B = \frac{n(n-1)A}{2 \cdot 2(n-1)} = \frac{nA}{4};$$

$$C = \frac{(n-2)(n-3)B}{nn-(n-4)^2} = \frac{(n-2)(n-3)B}{4 \cdot 2(n-2)} = \frac{(n-3)B}{8};$$

D =

$$D = \frac{(n-4)(n-5)C}{nn-(n-6)^2} = \frac{(n-4)(n-5)C}{12(n-3)};$$
$$E = \frac{(n-6)(n-7)D}{nn-(n-8)^2} = \frac{(n-6)(n-7)D}{16(n-4)};$$
$$F = \frac{(n-8)(n-9)E}{20(n-5)} \text{ etc.}$$
etc.

Hinc igitur omnes coëfficientes per primum A sequenti modo determinantur:

$$B = \frac{n}{4}A;$$
$$C = \frac{n(n-3)}{4.\ 8}A;$$
$$D = \frac{n(n-4)(n-5)}{4.\ 8.\ 12}A;$$
$$E = \frac{n(n-5)(n-6)(n-7)}{4.\ 8.\ 12.\ 16}A;$$
$$F = \frac{n(n-6)(n-7)(n-8)(n-9)}{4.\ 8.\ 12.\ 16.\ 20}A; \text{ etc.}$$

sicque prior series valorem ipsius s exhibens erit:

$$s = A\left(z^n - \frac{n}{4}z^{n-2} + \frac{n(n-3)}{4.\ 8}z^{n-4} - \frac{n(n-4)(n-5)}{4.\ 8.\ 12}z^{n-6} + \frac{n(n-5)(n-6)(n-7)}{4.\ 8.\ 12.\ 16}z^{n-8} - \text{etc.}\right)$$

II. Inuestigatio seriei posterioris.

$$\mathfrak{A}z^{-n} - \mathfrak{B}z^{-n-2} + \mathfrak{C}z^{-n-4} - \mathfrak{D}z^{-n-6} + \mathfrak{E}z^{-n-8} - \mathfrak{F}z^{-n-10} + \text{etc.}$$

§. 19. Determinatio coëfficientium huius seriei simili prorsus modo institui potest, quo praecedentem elicuimus, neque tamen opus est omnes operationes hic repetere; nam quia totum discrimen harum duarum serierum in solo signo numeri n consistit, quo ipsa aequatio proposita non afficitur, singularum litterarum Germanicarum valores immediate ex Latinis deriuare licebit, si modo in iis loco n scribatur $-n$, sicque obtinebimus:

$$\mathfrak{B} = -\frac{n}{4}\mathfrak{A};$$
$$\mathfrak{C} = +\frac{n(n+3)}{4.\ 8}\mathfrak{A};$$
$$\mathfrak{D} = -\frac{n(n+4)(n+5)}{4.\ 8.\ 12}\mathfrak{A};$$

$$\mathfrak{E} = + \frac{n(n+5)(n+6)(n+7)}{4.\,8.\,12.\,16}\mathfrak{A};$$
$$\mathfrak{F} = - \frac{n(n+6)(n+7)(n+8)(n+9)}{4.\,8.\,12.\,16.\,20}\mathfrak{A};\ \text{etc.}$$

§. 20. Ex his ergo valor integralis completus quantitatis s ex duabus sequentibus seriebus infinitis componetur:

$$s = \begin{cases} \mathrm{A}\left(z^n - \frac{n}{4}z^{n-2} + \frac{n(n-3)}{4.\,8}z^{n-4} - \frac{n(n-4)(n-5)}{4.\,8.\,12}z^{n-6} + \text{etc.}\right) \\ + \mathfrak{A}\left(z^{-n} + \frac{n}{4}z^{-n-2} + \frac{n(n+3)}{4.\,8}z^{-n-4} + \frac{n(n+4)(n+5)}{4.\,8.\,12}z^{-n-6} + \text{etc.}\right) \end{cases}$$

Problema 3.

Quoniam valorem completum ipsius s duplici modo expressum inuenimus, alterum scilicet per duplicem integrationem in problemate 1. alterum per duas series infinitas in problemate 2. constantes arbitrarias ita determinare, vt duae illae expressiones inter se consentiant.

Solutio.

§. 21. In problemate primo binas constantes arbitrarias per litteras f et g designauimus, vbi inuenimus hunc valorem:

$$s = f(z + \sqrt{[zz-1]})^n + g(z + \sqrt{[zz-1]})^{-n},$$

in praecedente vero problemate per binas series infinitas inuenimus esse:

$$s = \begin{cases} \mathrm{A}\left(z^n - \frac{n}{4}z^{n-2} + \frac{n(n-3)}{4.\,8}z^{n-4} - \frac{n(n-4)(n-5)}{4.\,8.\,12}z^{n-6} + \text{etc.}\right) \\ + \mathfrak{A}\left(z^{-n} + \frac{n}{4}z^{-n-2} + \frac{n(n+3)}{4.\,8}z^{-n-4} + \frac{n(n+4)(n+5)}{4.\,8.\,12}z^{-n-6} + \text{etc.}\right) \end{cases}$$

vbi binae constantes arbitrariae in litteris A et $\mathfrak{A}$ continentur. Quaeritur ergo, quomodo has litteras A et $\mathfrak{A}$ per f et g definiri oporteat, vt hae duae expressiones inter se consentientes reddantur?

§. 22. Perpetuo autem, quoties de constantibus arbitrariis definiendis quaestio mouetur, ad casum quempiam specialem est respiciendum, quo valor per seriem expressus euadat cognitus, quandoquidem, si constantes vnico casui spe-

ſpeciali ſatisfaciunt, eorum valor rite erit determinatus. Hanc ob rem diſpiciamus, an non quispiam caſus ſpecialis detur, quo bini valores innoteſcant. Conſideremus igitur primam formam finitam, ac manifeſtum quidem eſt, ibi caſum $z = 0$ in hunc finem adhiberi non poſſe, propterea quod iſte valor ad imaginaria rediret. Deinde vero etiam binae ſeries, poſito $z = 0$, partim terminis euaneſcentibus, partim infinitis conſtabunt; ſicque ex hoc caſu nihil plane concludi poteſt.

§. 23. Caſus autem $z = 1$ aliquid polliceri videtur; tum enim prior forma finita praebebit $s = f + g$; at vero poſito $z = 1$ ambae ſeries nihilo minus in infinitum excurrunt, ita vt earum valores nobis neutiquam euadant cogniti; quamobrem eiusmodi caſu nobis erit opus, quo binae ſeries inuentae deſinant in infinitum excurrere, et omnes termini prae primis quaſi euaneſcant, ita vt terminos tantum primos conſideraſſe ſufficiat. Manifeſtum autem eſt in vtraque ſerie prae primo termino ſequentes omnes eſſe euaniturος, ſi ipſi z valor infinite magnus tribuatur. Statuamus igitur $z = \infty$, atque ex binis ſeriebus valor ipſius s pro hoc caſu erit $s = A\,\infty^n + \frac{\mathfrak{A}}{\infty^n}$.

§. 24. Faciamus igitur etiam in priori forma finita $z = \infty$, et cum hoc caſu ſit $\sqrt{(zz - 1)} = \infty$, valor ipſius s hinc orietur:

$$s = f(2\,\infty)^n + g(2\,\infty)^{-n} = 2^n f\,\infty^n + \frac{g}{2^n\,\infty^n},$$

quam ergo expreſſionem cum ante inuenta, quae erat $A\,\infty^n + \frac{\mathfrak{A}}{\infty^n}$, congruere oportet, atque manifeſto ſequitur, hoc fieri, ſi ſtatuamus $A = 2^n f$ et $\mathfrak{A} = \frac{g}{2^n}$.

§. 25.

§. 25. Quodſi igitur valor completus finitus fuerit

$$s = f(z + \sqrt{[zz - 1]})^n + g(z + \sqrt{[zz - 1]})^{-n},$$

idem valor per duas ſequentes ſeries iunctim ſumtas exprimetur:

$$s = \begin{cases} f[(2z)^n - \frac{n}{1}(2z)^{n-2} + \frac{n(n-3)}{1.2}(2z)^{n-4} - \frac{n(n-4)(n-5)}{1.2.3}(2z)^{n-6} \text{etc.}] \\ + g\left[\frac{1}{(2z)^n} + \frac{n}{1} \cdot \frac{1}{(2z)^{n+2}} + \frac{n(n+3)}{1.2} \cdot \frac{1}{(2z)^{n+4}} + \text{etc.}\right] \end{cases}$$

Problema 4.

Si z denotet coſinum cuiuspiam anguli ϕ, vt ſit $z = \cos.\phi$, inueſtigare ſeries pro coſinu anguli ncupli $n\phi$.

Solutio.

§. 26. Supra vidimus, ſi fuerit $z = \cos.\phi$, ac vocetur $\cos.n\phi = s$, tum relationem inter s et z per eam ipſam aequationem differentio-differentialem exprimi, quam hactenus tractauimus (vid. §. 9.). Neceſſe igitur eſt vt valor quaeſitus $\cos.n\phi$ in ſuperioribus expreſſionibus, tam in finita, quam in infinita contineatur, totumque negotium huc redit, vt binae conſtantes f et g rite pro hoc caſu definiantur. At vero pro expreſſione finita, ob $z = \cos.\phi$ et $\sqrt{(zz - 1)} = \sqrt{-1}\sin.\phi$ habebimus

$$s = f(\cos.\phi + \sqrt{-1}\sin.\phi)^n + g(\cos.\phi + \sqrt{-1}\sin.\phi)^{-n}.$$

Conſtat autem eſſe

$$(\cos.\phi + \sqrt{-1}\sin.\phi)^n = \cos.n\phi + \sqrt{-1}\sin.\phi \text{ et}$$
$$(\cos.\phi + \sqrt{-1}\sin.\phi)^{-n} = \cos.n\phi - \sqrt{-1}\sin.n\phi,$$

ſicque habebimus

$$s = (f + g)\cos.n\phi + (f - g)\sqrt{-1}\sin.n\phi;$$

quamobrem, vt prodeat $s = \cos.n\phi$, ſtatui oportet $f = g = \frac{1}{2}$.

§. 27.

§. 27. Nunc igitur cognitis binis litteris f et g, quibus euadit $s = \text{cos.}\, n\,\Phi$, etiam ambas illas series infinitas exhibere poterimus, quibus coniunctis idem valor $\text{cos.}\, n\,\Phi$ exprimatur. Scilicet cum sit $f = g = \frac{1}{2}$, multiplicemus vtrinque per 2, atque impetrabimus:

$$2\,\text{cos.}\, n\,\Phi = \left\{ \begin{array}{l} (2z)^n - \frac{n}{1}.(2z)^{n-2} + \frac{n(n-3)}{1.2}.(2z)^{n-4} - \text{etc.} \\ + \frac{1}{(2z)^n} + \frac{n}{1}.\frac{1}{(2z)^{n+2}} + \frac{n(n+3)}{1.2}.\frac{1}{(2z)^{n+4}} + \text{etc.} \end{array} \right\}$$

vbi est $z = \text{cos.}\, \Phi$. Hinc si, vti initio fecimus, vocemus $x = 2\,\text{cos.}\, \Phi$, vt sit $2z = x$, vera expressio pro $\text{cos.}\, n\,\Phi$ erit:

$$2\,\text{cos.}\, n\,\Phi = \left\{ \begin{array}{l} x^n - \frac{n}{1}.x^{n-2} + \frac{n(n-3)}{1.2}.x^{n-4} - \text{etc.} \\ + \frac{1}{x^n} + \frac{n}{1}.\frac{1}{x^{n+2}} + \frac{n(n+3)}{1.2}.\frac{1}{x^{n+4}} + \text{etc.} \end{array} \right\}.$$

§. 28. Omnes igitur defectus, quos supra circa formulam pro $\text{cos.}\, n\,\Phi$ tradi solitam recensuimus, inde originem traxere, quod posterior series, quae potestates ipsius x in denominatoribus exhibet, negligi solet: ea autem adiecta perpetuo pulcherrimus consensus cum veritate deprehendetur, quoscunque etiam numeros pro n, siue positiuos, siue negatiuos, siue integros siue fractos accipiamus. Ita si ponamus $n = 0$, hinc statim consequimur $2\,\text{cos.}\, 0\,\Phi = 2$. Deinde cum sit $\text{cos.} - n\,\Phi = \text{cos.}\, n\,\Phi$, haec conuenientia statim ex praesente forma elucet, quippe quae eadem manet, etiamsi loco n scribatur $-n$.

§. 29. Examinemus nunc etiam aliquot casus simpliciores, ac primo quidem sumamus $n = 1$ reperieturque:

2 cos.

$$2\,\text{cof.}\,\phi = \begin{Bmatrix} x - \frac{1}{x} - \frac{1}{x^3} - \frac{2}{x^5} - \frac{5}{x^7} - \frac{14}{x^9} - \frac{42}{x^{11}} - \text{etc.} \\ \frac{1}{x} + \frac{1}{x^3} + \frac{2}{x^5} + \frac{5}{x^7} + \frac{14}{x^9} + \frac{42}{x^{11}} + \text{etc.} \end{Bmatrix} = x,$$

vbi feries pofterior manifefto terminos fuperfluos prioris tollit, ita vt prodeat $2\,\text{cof.}\,\phi = x$. Ponamus etiam $n = 2$ ac reperiemus:

$$2\,\text{cof.}\,2\phi = \begin{Bmatrix} xx - 2 - \frac{1}{xx} - \frac{2}{x^4} - \frac{5}{x^6} - \frac{14}{x^8} - \frac{42}{x^{10}} - \text{etc.} \\ \frac{1}{xx} + \frac{2}{x^4} + \frac{5}{x^6} + \frac{14}{x^8} + \frac{42}{x^{10}} + \text{etc.} \end{Bmatrix} = xx - 2.$$

Sit etiam $n = 3$ ac reperiemus

$$2\,\text{cof.}\,3\phi = \begin{Bmatrix} x^3 - 3x + \frac{0}{x} - \frac{1}{x^3} - \frac{3}{x^5} - \frac{9}{x^7} - \frac{28}{x^9} - \text{etc.} \\ + \frac{1}{x^3} + \frac{3}{x^5} + \frac{9}{x^7} + \frac{28}{x^9} + \text{etc.} \end{Bmatrix} = x^3 - 3x.$$

§. 30. Ex his exemplis fatis manifeftum eft quoties numerus n fuerit integer pofitiuus, tum omnes terminos, quos fupra tanquam inutiles reiicere iuffimus, hic fponte per feriem pofteriorem auferri. Praeterea vero hic nullum dubium fupereffe poteft, quin etiam pro omnibus numeris fractis loco n affumtis veritas fit proditura. Sit enim $n = \frac{1}{2}$ ac fiet:

$$2\,\text{cof.}\,\tfrac{1}{2}\phi = \begin{Bmatrix} \sqrt{x} - \frac{1}{2x\sqrt{x}} - \frac{5}{8x^3\sqrt{x}} - \frac{21}{16x^5\sqrt{x}} - \text{etc.} \\ \frac{1}{\sqrt{x}} + \frac{1}{2xx\sqrt{x}} + \frac{7}{8x^4\sqrt{x}} + \text{etc.} \end{Bmatrix},$$

hae duae autem feries permixtae eam ipfam feriem producunt, quam fupra §. 5. indicauimus.

Theorema.

Quoties n eft numerus integer pofitiuus, tum omnes termini fracti prioris feriei a ferie pofteriore deftruuntur, ita vt tantum remaneant termini integri prioris feriei, quibus adeo valor ipfius $2\,\text{cof.}\,n\phi$ exprimetur.

De-

Demonstratio.

§. 31. Contemplemur accuratius priorem seriem a termino x^n incipientem, in qua cum signa $+$ et $-$ alternentur, ne hinc sequens ratiocinium turbetur, hanc seriem hoc modo repraesentemus:

$$x^n - \frac{n}{1}x^{n-2} - \frac{n(3-n)}{1.\,2}x^{n-4} - \frac{n(4-n)(5-n)}{1.\,2.\,3}x^{n-6} - \text{etc.}$$

Nunc autem breuitatis gratia singulos hos coëfficientes sequentibus characteribus designemus:

$$x^n - [n-2]x^{n-2} - [n-4]x^{n-4} - [n-6]x^{n-6} - [n-8]x^{n-8} - \text{etc.}$$

sicque erit

$$[n-2] = \frac{n}{1};$$

$$[n-4] = \frac{n(3-n)}{1.\,2};$$

$$[n-6] = \frac{n(4-n)(5-n)}{1.\,2.\,3};$$

$$[n-8] = \frac{n(5-n)(6-n)(7-n)}{1.\,2.\,3.\,4};$$

$$[n-10] = \frac{n(6-n)(7-n)(8-n)(9-n)}{1.\,2.\,3.\,4.\,5};\ \text{etc.}$$

§. 32. Hinc ergo in genere, si potestatis x^{n-2i} coëfficientem per $-[n-2i]$ designemus, erit

$$[n-2i] = \frac{n}{1}\cdot\frac{(i+1-n)}{2}\cdot\frac{(i+2-n)}{3}\cdot\frac{(i+3-n)}{4}\ \ldots\ \frac{(2i-1-n)}{i}.$$

Scilicet ista forma composita erit ex i factoribus, quorum primus semper est $\frac{n}{1}$; secundus $\frac{i+1-n}{2}$; tertius $= \frac{i+2-n}{3}$; quartus $= \frac{i+3-n}{4}$, donec perueniatur ad vltimum, qui est $\frac{2i-1-n}{i}$; vnde patet factorem quemlibet intermedium indici λ respondentem fore $\frac{i+\lambda-1-n}{\lambda}$, si modo fuerit $\lambda < i$; tum vero sumto $\lambda = i$ prodibit factor vltimus $\frac{2i-1-n}{i}$. Hoc ergo modo pro qualibet potestate x^{n-2i}, eius coëfficientem, quem per $[n-2i]$ designamus, facile exhibere licebit. Ita potestatis x^{n-20} coëfficiens erit

$$[n-20] = \frac{n}{1}\cdot\frac{(11-n)}{2}\cdot\frac{(12-n)}{3}\cdot\frac{(13-n)}{4}\cdot\frac{(14-n)}{5}\cdot\frac{(15-n)}{6}\cdot\frac{(16-n)}{7}\cdot\frac{(17-n)}{8}\cdot\frac{(18-n)}{9}\cdot\frac{(19-n)}{10},$$

vnde

vnde patet, quoties n fuerit numerus integer, fiue 11, fiue 12, fiue 13, vsque ad 19, iftum coëfficientem femper fore $= 0$, atque hoc adeo in genere eueniet, quando n fuerit numerus integer, vel $i + 1$, vel $i + 2$, vel $i + 3$, vsque ad $2i - 1$.

§. 33. Sumamus nunc $i = n$, quae pofitio ergo locum habere nequit, nifi n fit numerus integer, quandoquidem manifeftum eft loco i alios numeros praeter integros accipi non poffe. Hoc igitur modo obtinebitur coëfficiens poteftatis x^{-n}, quem defignamus per $-[-n]$, eritque:

$$[-n] = \frac{n. \ 1. \ 2. \ 3. \ 4, \ 5 \ . \ . \ . \ . \ . \ . \ . \ . \ n-1}{1. \ 2. \ 3. \ 4. \ 5 \ . \ . \ . \ . \ . \ . \ . \ . \ . \ n},$$

cuius expreffionis valor manifefto eft $= 1$, ita vt terminus hinc natus fit $-1 . x^{-n} = -\frac{1}{x^n}$, qui terminus primum in fractis occurrit. Si enim capiatur $i < n$, vt tamen fit $n - 2i$ numerus negatiuus, coëfficientes, vt fupra vidimus, erunt $= 0$, namque fumto $i = n - 1$, iam factor fecundus euanefcit; fumto autem $i = n - 2$, tertius euanefcit; quartus deinde fi $i = n - 3$ et ita porro, ficque omnes termini hunc praecedentes erunt integri.

§. 34. Quaeramus nunc etiam terminos hunc: $-\frac{1}{x^n}$ fequentes, ac pro fecundo erit $i = n + 1$, vnde poteftatis x^{-n-2} coëfficiens erit

$$-\frac{n. \ 2. \ 3. \ 4, \ 5 \ . \ . \ . \ . \ . \ . \ . \ . \ n+1}{1. \ 2. \ 3. \ 4. \ . \ . \ . \ . \ . \ . \ . \ . \ n+1} = n,$$

ita vt poteftatum negatiuarum fecunda fit $-\frac{n}{x^{n+2}}$. Simili modo pro tertio termino fit $i = n + 2$, et poteftatis x^{-n-4} coëfficiens erit

$$-\frac{n. \ 3. \ 4. \ 5. \ 6 \ . \ . \ . \ . \ . \ . \ . \ n+3}{1, \ 2. \ 3. \ 4. \ 5 \ . \ . \ . \ . \ . \ . \ . \ n+2} = -\frac{n(n+3)}{1. \ 2.}.$$

Sit

Sit nunc $i = n + 3$, ac poteſtatis x^{-n-6} coëfficiens erit

$$-\frac{n.\ 4.\ 5.\ 6.\ 7\ \dots\ (n+5)}{1.\ 2.\ 3.\ 4\ \dots\ (n+3)} = -\frac{n(n+4)(n+5)}{1.\ 2.\ 3}.$$

Eodem modo euidens eſt poteſtatis x^{-n-8} coëfficientem fore

$$-\frac{n.\ 5.\ 6.\ 7.\ 8\ \dots\ (n+7)}{1.\ 2.\ 3.\ 4\ \dots\ (n+4)} = -\frac{n(n+5)(n+6)(n+7)}{1.\ 2.\ 3.\ 4}.$$

§. 35. Ex his igitur manifeſtum eſt terminos fractos, ad quos prima ſeries perducit, fore

$$-\frac{1}{x^n} - \frac{n}{1} \cdot \frac{1}{x^{n+2}} - \frac{n(n+3)}{1.\ 2} \cdot \frac{1}{x^{n+4}} - \frac{n(n+4)(n+5)}{1.\ 2.\ 3} \cdot \frac{1}{x^{n+6}} - \text{etc.}$$

Cum igitur altera ſeries pro $2 \cos. n\,\Phi$ adiicienda ſit

$$\frac{1}{x^n} + \frac{n}{1} \cdot \frac{1}{x^{n+2}} + \frac{n(n+3)}{1.\ 2} \cdot \frac{1}{x^{n+4}} + \frac{n(n+4)(n+5)}{1.\ 2.\ 3} \cdot \frac{1}{x^{n+6}} + \text{etc.}$$

nunc firmiter a nobis eſt euictum, omnes terminos fractos prioris ſeriei per ſeriem poſteriorem penitus tolli, ita vt ex ſerie priore termini tantum integri relinquantur, quibus valor ipſius $2 \cos. n\,\Phi$ exprimatur. Ex ipſa autem demonſtratione apparet, hanc deſtructionem locum habere non poſſe, niſi exponens n fuerit numerus integer. Pro exponentibus igitur fractis ambas illas ſeries in infinitum continuari oportet, quippe quae iunctim ſumtae demum valorem pro $2 \cos. n\,\Phi$ exhibebunt; ſicque omnia ſunt perſpicua, quae initio circa neceſſarias reſtrictiones formulae pro coſinibus datae ſunt tradita.

§. 36. Hic autem imprimis notaſſe iuuabit, valorem vtriusque ſeriei ſeorſim ſumtae reuera eſſe imaginarium. Vidimus enim priorem ſeriem natam eſſe ex euolutione formulae $[z + \sqrt{(zz - 1)}]^n$, poſteriorem vero ex euolutione huius: $[z - \sqrt{(zz - 1)}]^n$, poſito $z = \cos. \Phi$; tum autem prior formula transformatur in hanc: $\cos. n\,\Phi + \sqrt{-1} \sin. n\,\Phi$, poſterior vero in hanc: $\cos. n\Phi - \sqrt{-1} \sin. n\Phi$, quarum vtraque manifeſto eſt imaginaria, earum tamen ſumma praebet $2\cos. n\Phi$; ſin autem poſteriorem a priore auferamus, relinquetur $2\sqrt{-1} \sin. n\Phi$, cui formulae ergo aequaretur differentia noſtrarum ſerierum; vnde patet ſinum anguli multipli $n\Phi$ hoc modo per ſeries realiter exprimi non poſſe, cui incommodo autem remedium afferemus in ſequente problemate.

Problema 5.

Propoſita aequatione differentio-differentiali

$$\partial\partial s(zz-1) + z\partial z\partial s - nns\partial z^2 = 0,$$

ſi ponatur $s = v\sqrt{(zz-1)}$, *valorem huius quantitatis* v *per ſeriem exprimere.*

Solutio.

§. 37. Quod ad expreſſionem finitam huius quantitatis v attinet, ea ſponte patet ex valore finito pro ipſa quantitate s inuento, cum ſit

$$v\sqrt{(zz-1)} = f[z+\sqrt{(zz-1)}]^n + g[z-\sqrt{(zz-1)}]^n.$$

Nunc autem nobis propoſitum eſt valorem ipſius v per eiusmodi ſeriem inueſtigare, cuius termini etiam per poteſtates ipſius z pariter deſcendentes progrediantur, quam ergo ſeriem ex ipſa aequatione propoſita deduci oportet, poſtquam ſcilicet loco s valor $v\sqrt{(zz-1)}$ fuerit introductus.

§. 38. Quo autem hanc ſubſtitutionem facilius facere queamus, ſumamus logarithmos $ls = lv + \frac{1}{2}l(zz-1)$, vnde differentiando nanciſcimur $\frac{\partial s}{s} = \frac{\partial v}{v} + \frac{z\partial z}{zz-1}$, quae aequatio denuo differentiata praebet:

$$\frac{s\partial\partial s - \partial s^2}{ss} = \frac{\partial\partial v}{v} - \frac{\partial v^2}{vv} + \frac{\partial z^2}{zz-1} - \frac{2zz\partial z^2}{(zz-1)^2}.$$

Iam

Iam addatur vtrinque

$$\frac{\partial s^2}{ss} = \frac{\partial v^2}{vv} + \frac{2z\partial z\partial v}{v(zz-1)} + \frac{zz\partial z^2}{(zz-1)^2},$$

atque obtinebimus

$$\frac{\partial\partial s}{s} = \frac{\partial\partial v}{v} + \frac{\partial z^2}{zz-1} - \frac{zz\partial z^2}{(zz-1)^2} + \frac{2z\partial z\partial v}{v(zz-1)}, \text{ fiue}$$

$$\frac{\partial\partial s}{s} = \frac{\partial\partial v}{v} + \frac{2z\partial z\partial v}{v(zz-1)} - \frac{\partial z^2}{(zz-1)^2}.$$

§. 39. Aequatio autem propofita per s diuifa fit

$$\frac{\partial\partial s}{s}(zz-1) + \frac{z\partial z\partial s}{s} - nn\partial z^2 = 0,$$

eaque facta fubftitutione induet fequentem formam:

$$\frac{\partial\partial v}{v}(zz-1) + \frac{3z\partial z\partial v}{v} - (nn-1)\partial z^2 = 0,$$

quae ducta in $\frac{v}{\partial z^2}$ dabit

$$\frac{\partial\partial v}{\partial z^2}(zz-1) + \frac{3z\partial v}{\partial z} - v(nn-1) = 0,$$

atque ex hac aequatione feriem defideratam pro v elici oportebit.

§. 40. Hic igitur iterum ante omnia primum terminum inueftigare debemus, quem in finem ftatuamus

$$v = Az^\lambda + Bz^{\lambda-2} + Cz^{\lambda-4} \text{ etc.}$$

et cum fit

$$\frac{\partial v}{\partial z} = \lambda Az^{\lambda-1} + (\lambda-2)Bz^{\lambda-3} \text{ etc.}$$

$$\frac{\partial\partial v}{\partial z^2} = \lambda(\lambda-1)Az^{\lambda-2} + (\lambda-2)(\lambda-3)Bz^{\lambda-4} + \text{etc.}$$

facta fubftitutione orietur fequens aequalitas:

$$\begin{aligned} 0 = \lambda(\lambda-1)Az^\lambda &+ (\lambda-2)(\lambda-3)Bz^{\lambda-2} + \text{etc.}\\ &- \lambda(\lambda-1)Az^{\lambda-2} - \text{etc.}\\ 3\lambda Az^\lambda &+ 3(\lambda-2)Bz^{\lambda-2} + \text{etc.}\\ -(nn-1)Az^\lambda &- (nn-1)Bz^{\lambda-2} - \text{etc.} \end{aligned}$$

Vt nunc prima poteftas z^λ fponte tollatur, neceffe eft vt fit ſit $\lambda(\lambda-1)+3\lambda-nn+1=0$, ſiue $(\lambda+1)^2-nn=0$, hincque duo valores pro λ reperiuntur: $\lambda=-n-1$ et $\lambda=n-1$; vnde ſequitur pro valore completo ipſius v exprimendo requiri duas ſeries infinitas, quas ſequenti modo referamus:

$$v=\begin{cases} Az^{n-1}-Bz^{n-3}+Cz^{n-5}-Dz^{n-7}+Ez^{n-9}-\text{etc.} \\ +\mathfrak{A}z^{-n-1}-\mathfrak{B}z^{-n-3}+\mathfrak{C}z^{-n-5}-\mathfrak{D}z^{-n-7}+\mathfrak{E}z^{-n-9}-\text{etc.} \end{cases}$$

quarum autem ſufficiet priorem determinaſſe, quoniam poſterior inde naſcitur, ſcribendo $-n$ loco n; quamobrem habebimus vt ſequitur:

$$v=Az^{n-1}-Bz^{n-3}+Cz^{n-5}-Dz^{n-7}+Ez^{n-9}-\text{etc.}$$

$$\frac{\partial v}{\partial z}=(n-1)Az^{n-2}-(n-3)Bz^{n-4}+(n-5)Cz^{n-6}-\text{etc.}$$

$$\frac{\partial\partial v}{\partial z^2}=(n-1)(n-2)Az^{n-3}-(n-3)(n-4)Bz^{n-5}+\text{etc.}$$

§. 41. Fiat nunc ſubſtitutio in ſingulis terminis noſtrae aequationis ſequenti modo:

$$\begin{array}{rcccc}
 & z^{n-1} & z^{n-3} & z^{n-5} & \\
\frac{zz\partial\partial v}{\partial z^2}= & (n-1)(n-2)A- & (n-3)(n-4)B+ & (n-5)(n-6)C & -\text{etc.} \\
-\frac{\partial\partial v}{\partial z^2}= & & -(n-1)(n-2)A+ & (n-3)(n-4)B & -\text{etc.} \\
\frac{3z\partial v}{\partial z}=+ & 3(n-1)A- & 3(n-3)B+ & 3(n-5)C & -\text{etc.} \\
nnv=- & nnA+ & nnB- & nnC & +\text{etc.} \\
+v=+ & A- & B+ & C & -\text{etc.} \\
\hline
0= & 0A+ & 4(n-1)B- & 8(n-2)C & +\text{etc.} \\
 & & -(n-1)(n-2)A+ & (n-3)(n-4)B & -\text{etc.}
\end{array}$$

§. 42. Cum nunc ſingularum poteſtatum coëfficientes ſe deſtruere debeant, per primum A, qui arbitrio noſtro relinquitur, ſequentes omnes hoc modo determinabuntur:

B =

$B = \frac{(n-2)}{4} A;$

$C = \frac{(n-3)(n-4)}{8(n-2)} B = \frac{(n-3)(n-4)}{4.8} A;$

$D = \frac{(n-5)(n-6)}{12(n-3)} C = \frac{(n-4)(n-5)(n-6)}{4.8.12} A;$

$E = \frac{(n-7)(n-8)}{16(n-4)} D = \frac{(n-5)(n-6)(n-7)(n-8)}{4.8.12.16} A;$

etc.

ſicque prior ſeries pro quantitate v erit

$$A z^{n-1} - \frac{(n-2)}{4} A z^{n-3} + \frac{(n-3)(n-4)}{4.8} A z^{n-5} - \text{etc.}$$

§. 43. Ex hac autem ſerie altera ſponte eruitur, ſi modo loco A ſcribamus 𝔄 et $-n$ loco n, vnde valor completus ipſius v binis ſequentibus ſeriebus exprimetur:

$$v = \begin{cases} A\left(z^{n-1} - \frac{(n-2)}{4} z^{n-3} + \frac{(n-3)(n-4)}{4.8} z^{n-5} - \text{etc.}\right) \\ + \mathfrak{A}\left(z^{-n-1} + \frac{(n+2)}{4} z^{-n-3} + \frac{(n+3)(n+4)}{4.8} z^{-n-5} + \text{etc.}\right) \end{cases}.$$

§. 44. Nunc autem ſupereſt, vt iſtam formam per ſeries inuentam cum forma finita

$$v\sqrt{(zz-1)} = f[z + \sqrt{(zz-1)}]^n + g[z + \sqrt{(zz-1)}]^{-n},$$

concordem reddamus, quem in finem tribuamus ipſi z valorem infinite magnum, quandoquidem hoc modo in binis ſeriebus prae terminis primis omnes ſequentes euaneſcunt, ita vt hinc fit $v = A z^{n-1} + \mathfrak{A} z^{-n-1}$; at vero ex forma finita, ob $\sqrt{(zz-1)} = z$, erit $vz = f(2z)^n + g(2z)^{-n}$, ideoque $v = 2^n f z^{n-1} + \frac{g}{2^n z^{n+1}}$, qua forma cum illa comparata euidens eſt ſumi debere $A = 2^n f$ et $\mathfrak{A} = \frac{g}{2^n}$.

Problema 6.

Propofito angulo quocunque ϕ inuenire formam generalem pro finibus angulorum quorumuis multiplorum.

Solutio.

§. 45. Statuatur vt ante $z = \text{cof.}\,\phi$, eritque $\sqrt{(zz-1)} = +\sqrt{-1}\,\text{fin.}\,\phi$, hincque porro

$$[z + \sqrt{(zz-1)}]^n = \text{cof.}\,n\phi + \sqrt{-1}\,\text{fin.}\,n\phi \text{ et}$$
$$[z - \sqrt{(zz-1)}]^n = \text{cof.}\,n\phi - \sqrt{-1}\,\text{fin.}\,n\phi.$$

Nunc igitur ex forma finita habebimus

$$s = v\sqrt{-1}\,\text{fin.}\,\phi = f(\text{cof.}\,n\phi + \sqrt{-1}\,\text{fin.}\,n\phi)$$
$$+ g(\text{cof.}\,n\phi - \sqrt{-1}\,\text{fin.}\,n\phi).$$

Faciamus iam $f = 1$ et $g = -1$, ac prodibit

$$s = v\sqrt{-1}\,\text{fin.}\,\phi = 2\sqrt{-1}\,\text{fin.}\,n\phi,$$

ficque erit $2\,\text{fin.}\,n\phi = v\,\text{fin.}\,\phi$.

§. 46. Cum nunc fit $f = 1$ et $g = -1$, erit $A = 2^n$ et $\mathfrak{A} = -\frac{1}{2^n}$, vnde pro v binas fequentes feries nancifcimur:

$$v = \left\{ \begin{array}{l} 2(2z)^{n-1} - \frac{2(n-2)}{1}(2z)^{n-3} + \frac{2(n-3)(n-4)}{1.\;2}(2z)^{n-5} - \text{etc.} \\ -2(2z)^{-n-1} + \frac{2(n+2)}{1}(2z)^{-n-3} - \frac{2(n+3)(n+4)}{1.\;2}(2z)^{-n-5} + \text{etc.} \end{array} \right\}.$$

Ponamus nunc, vt fupra fecimus, $x = 2\,\text{cof.}\,\phi$, vt $2z = x$, tum vero infuper $y = 2\,\text{fin.}\,\phi$, et quoniam inuenimus $2\,\text{fin.}\,n\phi = v\,\text{fin.}\,\phi = \frac{1}{2}vy$, tantum opus eft feries fuperiores pro v inventas per $\frac{1}{2}y$ multiplicare, hocque modo fequentem expreffionem generalem pro finibus angulorum multiplorum obtinebimus:

2 fin.

$$2\,\text{fin.}\,n\Phi = \left\{ \begin{array}{l} y[x^{n-1} - \frac{(n-2)}{1}x^{n-3} + \frac{(n-3)(n-4)}{1.\,2}x^{n-5} - \text{etc.}] \\ -y[x^{-n-1} + \frac{(n+2)}{1}x^{-n-3} + \frac{(n+3)(n+4)}{1.\,2}x^{-n-5} + \text{etc.}] \end{array} \right\},$$

haecque expressio veritati erit consentanea, quicunque valores litterae n tribuantur, siue positiui siue negatiui, siue integri siue fracti; vnde patet formulam initio datam hoc defectu laborare, quod ibi series posterior est omissa. Primo autem statim patet, si $n = 0$, vtique prodire $2\,\text{fin.}\,n\Phi = 0$, propterea quod singuli termini binarum serierum se mutuo tollunt; deinde etiam clarum est, pro numeris negatiuis n sinus etiam negatiuos prodire.

§. 47. Euoluamus nunc etiam aliquot casus pro numeris integris, ac primo quidem sit $n = 1$, eritque

$$2\,\text{fin.}\,\Phi = \left\{ \begin{array}{l} y\,(1 + \frac{1}{xx} + \frac{3}{x^4} + \frac{10}{x^6} + \frac{35}{x^8} + \text{etc.}) \\ -y\qquad(\frac{1}{xx} + \frac{3}{x^4} + \frac{10}{x^6} + \frac{35}{x^8} + \text{etc.}) \end{array} \right\},$$

sicque binis seriebus iunctis fiet $2\,\text{fin.}\,\Phi = y$. Sit nunc $n = 2$, eritque

$$2\,\text{fin.}\,2\Phi = \left\{ \begin{array}{l} y(x - 0 + \frac{1}{x^3} + \frac{4}{x^5} + \frac{15}{x^7} + \frac{35}{x^9} + \text{etc.}) \\ -y\qquad(\frac{1}{x^3} + \frac{4}{x^5} + \frac{15}{x^7} + \frac{35}{x^9} + \text{etc.}) \end{array} \right\}.$$

Sit nunc etiam $n = 3$, eritque

$$2\,\text{fin.}\,3\Phi = \left\{ \begin{array}{l} y(xx - 1 + 0 + \frac{1}{x^4} + \frac{5}{x^6} + \frac{21}{x^8} + \frac{84}{x^{10}} + \text{etc.}) \\ -y\qquad(\frac{1}{x^4} + \frac{5}{x^6} + \frac{21}{x^8} + \frac{84}{x^{10}} + \text{etc.}) \end{array} \right\},$$

ideoque $2\,\text{fin.}\,3\Phi = y\,(xx - 1)$. Ponatur etiam $n = 4$, eritque

$$2\,\text{fin.}\,4\Phi = \left\{ \begin{array}{l} y(x^3 - 2x + 0 + 0 + \frac{1}{x^5} + \frac{6}{x^7} + \frac{28}{x^9} + \text{etc.}) \\ -y\qquad(\frac{1}{x^5} + \frac{6}{x^7} + \frac{28}{x^9} + \text{etc.}) \end{array} \right\},$$

ideoque $2\,\text{fin.}\,4\Phi = y\,(x^3 - 2x)$.

§. 48.

§. 48. Ex his exemplis iam ſatis elucet, poſteriorem ſeriem in priori omnes terminos fractos deſtruere, ita vt ſufficiat ex priore ſerie ſolos terminos integros ſumſiſſe, quoties ſcilicet numerus n fuerit integer; hocque adeo facile in genere ſimili modo demonſtrari poſſet, quo vſi ſumus pro coſinibus, ſuperfluumque foret ſimilem demonſtrationem hic adornare. Caeterum in hac euolutione ſingulare phaenomenon ſe exerit, quod non parum ſuſpectum videri queat, in eo conſiſtens, quod ad conſenſum ſerierum infinitarum cum valore integrali finito ſtabiliendum, vſi ſumus caſu quo $z = \infty$, quae poſitio inſtituto noſtro, quo loco z coſinum anguli aſſumimus, maxime aduerſatur. Verum ſi perpendamus iſtum conſenſum in genere eſſe conſtitutum, ſine vllo reſpectu habito ad applicationem angulorum, omnia dubia ſponte euaneſcere debent, imprimis cum iam pleniſſimus conſenſus cum veritate luculanter eluceſcat.

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