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Specimen aequationum differentialium indefiniti gradus earumque integrationis

Leonhard Euler

exprimantur. Facile autem patet hoc modo ad aequationes differentiales adhuc altiorum graduum progredi licere. Hac igitur ratione calculo integrali haud contemnendum incrementum allatum est censendum. Cum igitur hic praecipuum negotium versetur in integratione completa hujusmodi aequationis

$$y + \frac{f\partial y}{\partial x} + \frac{g\partial\partial y}{\partial x^2} + \frac{h\partial^3 y}{\partial x^3} + \text{etc.} = z,$$

ubi z est functio quaecunque ipsius x, ejus resolutionem jam passim exhibitam huc accommodemus et breviter ostendamus. Formetur haec aequatio

$$1 + fu + gu^2 + hu^3 + iu^4 + \text{etc.} = 0,$$

cujus radices u designentur litteris α, β, γ, δ, etc. quibus inventis erit uti jam olim ostendi

$$y = \frac{e^{\alpha x}\int e^{-\alpha x} z\,\partial x}{f + 2g\alpha + 3h\alpha^2 + 4i\alpha^3 + \text{etc.}} + \frac{e^{\beta x}\int e^{-\beta x} z\,\partial x}{f + 2g\beta + 3h\beta^2 + 4i\beta^3 + \text{etc.}} + \text{etc.}$$

Hae scilicet formulae ex singulis radicibus α, β, γ, δ, etc. formatae et junctim sumtae dabunt valorem ipsius y atque adeo integrale completum, quia singulae formulae integrales constantem arbitrariam involvunt.

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2) Specimen aequationum differentialium indefiniti gradus earumque integrationis. *M. S. Academiae exhib. die* 13 *Decembris*, 1781.

§. 19. Quando aequationes differentiales secundum gradus differentialium distinguuntur, ipsa rei natura gradus intermedios excludere videtur: cum enim totidem integrationibus opus sit, harum numerus certe non integer esse non potest. Incidi tamen nuper in aequationem differentialem indefiniti gradus, cujus exponens etiam numerus fractus esse potest, atque adeo mihi licuit ejus integrale assignare; quod cum omni attentione dignum videatur, totam analysin, qua sum usus, hic dilucide exponam.

§. 20. Cum miras proprietates unciarum potestatum binomii, quas hoc charactere indicare soleo $\left(\frac{p}{q}\right)$, cujus valor est hoc productum

$$\frac{p}{1} \cdot \frac{p-1}{2} \cdot \frac{p-2}{3} \cdot \ldots \ldots \frac{p-q+1}{q},$$

considerassem, in mentem mihi venit valorem hujusmodi formulae $\left(\frac{p}{q}\right)$ ad formulam integralem revocare, unde etiam casus, quibus p et q non sunt numeri integri, assignari queant. Directe quidem talem reductionem non succedere observavi, unde ejus valorem reciprocum $\frac{1}{\left(\frac{p}{q}\right)}$ sum contemplatus, cujus valor est

$$\frac{1}{p} \cdot \frac{2}{p-1} \cdot \frac{3}{p-2} \cdot \ldots \ldots \frac{q}{p-q+1}.$$

Hunc in finem statuo

$$\frac{1 . 2 . 3 \ldots \ldots q \times x^p}{p(p-1)(p-2) \ldots (p-q+1)} = s,$$

ita ut posito $x = 1$ desideratus valor ipsius $1 : \left(\frac{p}{q}\right)$ obtineatur.

§. 21. Sit nunc brevitatis gratia $1 . 2 . 3 \ldots q = \mathrm{N}$, ut habeatur $s = \frac{\mathrm{N} x^p}{p \ldots (p-q+1)}$, in cujus denominatore tenendum est factores continuo unitate decrescere. Quod si jam ista formula differentietur, prodibit

$$\frac{\partial s}{\partial x} = \frac{N x^{p-1}}{(p-1)\ldots\ldots(p-q+1)},$$

sicque primus factor denominatoris est sublatus, ac differentiatione denuo instituta prodibit

$$\frac{\partial \partial s}{\partial x^2} = \frac{N x^{p-1}}{(p-2)\ldots\ldots(p-q+1)}.$$

Hoc igitur modo continuo differentiando, omnes factores denominatoris tollentur, ac pervenietur tandem ad hanc aequationem

$$\frac{\partial^q s}{\partial x^q} = N x^{p-q}.$$

§. 22. Pervenimus igitur, loco N valorem suum substituendo, ad hanc aequationem differentialem

$$\frac{\partial^q s}{1\ldots\ldots q\,\partial x^q} = x^{p-q},$$

quam ergo tot vicibus integrari oporteret, quot q continet unitates, atque singulae integrationes ita sunt instituendae ut, posito $x = 0$ integralia evanescant, et postquam omnes integrationes fuerint absolutae, loco x scribi debebit unitas, hocque modo valor ipsius s resultans dabit valorem formulae $1 : \left(\frac{p}{q}\right)$. Quo autem istas integrationes generalius expediamus, loco x^{p-q} scribamus X, ut habeamus hanc aequationem resolvendam

$$\frac{\partial^q s}{1.2\ldots\ldots q\,\partial x^q} = X.$$

§. 23. Hanc aequationem primo multiplicemus per ∂x, ejusque integrale dabit

$$\frac{\partial^{q-1} s}{1.2.3\ldots\ldots q\,\partial x^{q-1}} = \int X\,\partial x.$$

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Istam aequationem ducamus in $1 . \partial x$, eritque integrando

$$\frac{\partial^{q-2} s}{2 . 3 \ldots . q . \partial x^{q-2}} = \int \partial x \int X \partial x = x \int X \partial x - \int X x \partial x.$$

Per notas enim reductiones ejusmodi integralia repetita ad simplicia reduci possunt. Haec aequatio jam per $2 \partial x$ multiplicata eodemque modo integrata praebebit

$$\frac{\partial^{q-3} s}{3 . 4 \ldots . q . \partial x^{q-3}} = x^2 \int X \partial x - 2 x \int X x \partial x + \int X x^2 \partial x.$$

Nunc per $3 \partial x$ multiplicando et integrando proveniet

$$\frac{\partial^{q-4} s}{4 . 5 \ldots . q . \partial x^{q-4}} = x^3 \int X \partial x - 3 x^2 \int X x \partial x + 3 x \int X x^2 \partial x - \int X x^3 \partial x.$$

Eodem modo reperietur

$$\frac{\partial^{q-5} s}{5 . 6 \ldots . q . \partial x^{q-5}} = x^4 \int X \partial x - 4 x^3 \int X x \partial x + 6 x^2 \int X x^2 \partial x$$
$$- 4 x \int X x^3 \partial x + \int X x^4 \partial x,$$

sicque in genere nostros characteres in usum vocando erit

$$\frac{\partial^{q-n} s}{n (n+1) \ldots . . q . \partial x^{q+n}} = x^{n-1} \int X \partial x - \left(\frac{n-1}{1}\right) x^{n-2} \int X x \partial x$$
$$+ \left(\frac{n-1}{2}\right) x^{n-3} \int X x^2 \partial x - \left(\frac{n-1}{3}\right) x^{n-4} \int X x^3 \partial x + \text{etc.}$$

§. 24. Statuamus nunc $n = q$, et cum sit $\partial^0 s = s$, orietur haec aequatio finita

$$\frac{s}{q} = x^{q-1} \int X \partial x - \left(\frac{q-1}{1}\right) x^{q-2} \int X x \partial x$$
$$+ \left(\frac{q-1}{2}\right) x^{q-3} \int X x^2 \partial x - \text{etc.}$$

cujus singula membra ita integrari debent, ut posito $x = 0$ evanescant, quod quidem semper eveniet, si modo sit $q - 1 > 0$, quamobrem ipsae formulae integrales $\int X \partial x$, $\int X x \partial x$, etc. tantum sive adjectione constantis integrari debent. Etsi enim hoc modo x forte in denominatorem ingrediatur, per potestatem ipsius x, qua multiplicari debent, iterum tolletur.

§. 25. His circa singula integralia observatis extra signa summatoria jam ponere licebit $x = 1$, quippe qui est casus quaestionis propositae; sicque reperietur

$$1 : q\left(\frac{p}{q}\right) = \int X \partial x \left[1 - \left(\frac{q-1}{1}\right)x + \left(\frac{q-1}{2}\right)x^2 - \left(\frac{q-1}{3}\right)x^3 + \text{etc.}\right],$$

cujus seriei valor manifesto est $(1-x)^{q-1}$, ita ut habeamus hanc expressionem determinatam

$$\frac{1}{q\left(\frac{p}{q}\right)} = \int X \partial x (1-x)^{q-1},$$

cujus ergo valor etiam casibus quibus q non est numerus integer per quadraturas exhiberi potest, sicque aequationis differentialis indefiniti gradus $\partial^q s = N X \partial x^q$ integrale feliciter elicuimus, et quia $X = x^{p-q}$, omnes unciae hoc modo ad formas integrales redigentur

$$\left(\frac{p}{q}\right) = \frac{1}{q \int x^{p-q} \partial x (1-x)^{q-1}},$$

et quia exponentes ipsius x et ipsius $1-x$ permutari possunt, erit etiam

$$\left(\frac{p}{q}\right) = \frac{1}{q \int x^{q-1} \partial x (1-x)^{p-q}},$$

hancque formulam ex principio diversissimo non ita pridem sum adeptus.

Theorema 1.

§. 26. Valor hujus characteris $\left(\frac{p}{q}\right)$ reduci potest ad formulam integralem, cum sit

$$\left(\frac{p}{q}\right) = \frac{1}{q \int x^{q-1} \partial x (1-x)^{p-q}},$$

siquidem hoc integrale ab $x = 0$ ad $x = 1$ extendatur.

Corollarium 1.

§. 27. Sumto ergo $p = 0$ erit

$$\left(\frac{0}{q}\right) = \frac{1}{q \int x^{q-1} \partial x (1-x)^{-q}}.$$

Ostendi autem olim esse

$$\int x^{q-1} \partial x (1-x)^{-q} = \frac{\pi}{\sin. \pi q},$$

unde ergo fiet

$$\left(\frac{0}{q}\right) = \frac{\sin. \pi q}{\pi q}.$$

Corollarium 2.

§. 28. Deinde per notam integralium reductionem reperitur

$$\int x^{q-1} \partial x (1-x)^{p-q} = \frac{p}{\sin. \pi q} : \left(\frac{p-q}{p}\right),$$

cujus ergo valor, quoties p est numerus integer, absolute assignari potest, quamobrem in genere erit

$$\left(\frac{p}{q}\right) = \frac{\sin. \pi q}{\pi q} : \left(\frac{p-q}{p}\right).$$

Corollarium 3.

§. 29. Cum igitur vicissim sit

$$\int x^{q-1} \partial x (1-x)^{p-q} = \frac{1}{q \left(\frac{p}{q}\right)},$$

si hic loco $q - 1$ scribamus f, et g loco $p - q$, habebimus

$$\int x^{f} \partial x (1-x)^{g} = \frac{1}{(1+f)\left(\frac{f+g+1}{f+1}\right)}.$$

Scholion.

§. 30. Quoniam igitur hanc formulam integralem nacti sumus ex aequatione integrali indefiniti gradus, eandem investigationem latius extendamus in sequente problemate.

Problema 12.

§. 31. *Proposita serie sive finita sive infinita*

$$S = \frac{A}{\left(\frac{p}{q}\right)} + \frac{B}{\left(\frac{p+1}{q}\right)} + \frac{C}{\left(\frac{p+2}{q}\right)} + \frac{D}{\left(\frac{p+3}{q}\right)} + etc.$$

ejus valorem per formulam integralem exprimere.

Solutio.

Tribuamus singulis terminis potestates ipsius x, ac statuamus

$$S = \frac{A\,x^p}{\left(\frac{p}{q}\right)} + \frac{B\,x^{p+1}}{\left(\frac{p+1}{q}\right)} + \frac{C\,x^{p+2}}{\left(\frac{p+2}{q}\right)} + \text{etc.},$$

quae series ergo, posito $x = 1$, praebebit ipsam seriem propositam. Ubi observandum, in omnibus terminis litteram q eundem retinere valorem, alteram vero p continuo unitate augeri, unde productum indefinitum $1.\ 2.\ 3 \ldots\ldots\ q = N$ in omnibus terminis eundem retinebit valorem. Quare cum supra ex aequatione $s = \frac{x^p}{\left(\frac{p}{q}\right)}$ deduxerimus hanc aequationem differentialem indefiniti gradus

$$\frac{\partial^q\, s}{\partial\, x^q} = N\, x^{p-q},$$

ex singulis terminis nostrae seriei idem resultabit differentiale, si modo exponentem p unitate augeamus, unde ergo reperiemus

$$\frac{\partial^q\, s}{\partial\, x^q} = N\,A\,x^{p-q} + N\,B\,x^{p-q+1} + \text{etc.}$$

§. 32. Ponamus nunc

$$A + Bx + Cx^2 + Dx^3 + \text{etc.} = V,$$

eritque

$$\frac{\partial^q s}{N \partial x^q} = x^{p-q} V,$$

quamobrem si statuamus $x^{p-q} V = X$, habebimus ipsam aequationem jam ante tractatam

$$\frac{\partial^q s}{1 . 2 \ldots q \partial x^q} = X,$$

cujus integratio q vicibus repetita nos perduxit ad hanc expressionem $s = q \int X \partial x (1-x)^{q-1}$, unde ergo pro X et V valores substituendo nanciscemur summam quaesitam S, scilicet

$$S = q \int x^{p-q} \partial x (A + Bx + Cx^2 + Dx^3 + \text{etc.}) (1-x)^{q-1},$$

si modo hoc integrale ab $x = 0$ ad $x = 1$ extendatur, vel ut ante innuimus, si modo in integratione nulla constans adjiciatur, deinde vero sumatur $x = 1$.

Exemplum.

§. 33. Sit $V = (1-x)^n$, ita ut sit

$$A = 1, \; B = -\left(\tfrac{n}{1}\right), \; C = +\left(\tfrac{n}{2}\right), \; D = -\left(\tfrac{n}{3}\right), \; \text{etc.},$$

et series proposita erit

$$S = \frac{1}{\left(\frac{p}{q}\right)} - \frac{\left(\frac{n}{1}\right)}{\left(\frac{p+1}{q}\right)} + \frac{\left(\frac{n}{2}\right)}{\left(\frac{p+2}{q}\right)} - \frac{\left(\frac{n}{3}\right)}{\left(\frac{p+3}{q}\right)} + \text{etc.}$$

tum igitur summa hujus seriei erit

$$S = q \int x^{p-q} \partial x (1-x)^{q+n-1},$$

sive permutatis exponentibus ipsius x et $1-x$, erit quoque

$$S = q \int x^{q+n-1} \partial x (1-x)^{p-q}.$$

Nunc autem evidens est hanc ipsam formulam integralem iterum ad characterem hic usitatum reduci posse ope §. 29. erit enim $f = q + n - 1$ et $g = p - q$, atque hinc prodibit

$$S = \frac{q}{(q+n)\left(\frac{p+n}{q+n}\right)}.$$

Hinc ergo sive formulis integralibus habebimus hanc summationem seriei infinitae maxime notabilem

$$\frac{1}{\left(\frac{p}{q}\right)} - \frac{\left(\frac{n}{1}\right)}{\left(\frac{p+1}{q}\right)} + \frac{\left(\frac{n}{2}\right)}{\left(\frac{p+2}{q}\right)} - \frac{\left(\frac{n}{3}\right)}{\left(\frac{p+3}{q}\right)} + \frac{\left(\frac{n}{4}\right)}{\left(\frac{p+4}{q}\right)} - \text{etc.}$$

$$= \frac{q}{(q+n)\left(\frac{p+n}{q+n}\right)}.$$

Corollarium 1.

§. 34. Si ergo fuerit $n = 0$, oritur aequatio manifeste identica scilicet $\frac{1}{\left(\frac{p}{q}\right)} = \frac{1}{\left(\frac{p}{q}\right)}$. At si $n = 1$ prodit

$$\frac{q}{(q+1)\left(\frac{p+1}{q+1}\right)} = \frac{1}{\left(\frac{p}{q}\right)} - \frac{1}{\left(\frac{p+1}{q+1}\right)}.$$

Si $n = 2$ fiet

$$\frac{q}{(q+2)\left(\frac{p+2}{q+2}\right)} = \frac{1}{\left(\frac{p}{q}\right)} - \frac{2}{\left(\frac{p+1}{q}\right)} + \frac{1}{\left(\frac{p+2}{q}\right)}.$$

Corollarium 2.

§. 35. Quo consensus cum veritate clarius appareat evolvamus casum determinatum, quo $p = 3$, $q = 2$, $n = 4$, eritque

$$\frac{q}{q+n} = \frac{1}{3}, \text{ et } \left(\frac{p+n}{q+n}\right) = \left(\frac{7}{6}\right) = \left(\frac{7}{1}\right) = 7.$$

Deinde fit

$\left(\frac{p}{q}\right) = \left(\frac{3}{2}\right) = 3$; $\left(\frac{p+1}{q}\right) = \left(\frac{4}{2}\right) = 6$; $\left(\frac{p+2}{q}\right) = \left(\frac{5}{2}\right) = 10$; $\left(\frac{p+3}{q}\right) = 15$;

quae est progressio numerorum trigonalium; tum vero erit

$$\left(\frac{n}{1}\right) = 4;\ \left(\frac{n}{2}\right) = 6;\ \left(\frac{n}{3}\right) = 4;\ \left(\frac{n}{4}\right) = 1.$$

His igitur valoribus substitutis erit

$$\frac{1}{3 \cdot 7} = \frac{1}{3} - \frac{4}{6} + \frac{6}{10} - \frac{4}{15} + \frac{1}{21},$$

quod egregie convenit.

Exemplum. 2.

§. 36. Statuamus $V = (1 + x)^{q-1}$, ut fiat

$$S = q \int x^{p-q} \partial x (1 - x x)^{q-1};$$

tum vero erit

$$A = 1;\ B = \left(\frac{q-1}{1}\right);\ C = \left(\frac{q-1}{2}\right);\ D = \left(\frac{q-1}{3}\right);\ \text{etc.}$$

sicque series proposita erit

$$S = \frac{1}{\left(\frac{p}{q}\right)} + \frac{\left(\frac{q-1}{1}\right)}{\left(\frac{p+1}{q}\right)} + \frac{\left(\frac{q-1}{2}\right)}{\left(\frac{p+2}{q}\right)} + \frac{\left(\frac{q-1}{3}\right)}{\left(\frac{p+3}{q}\right)} + \text{etc.}$$

Evidens autem est, hanc formulam integralem etiam ad nostros characteres reduci posse. Ponamus enim $x x = y$, erit

$$S = \frac{q}{2} \int y^{\frac{p-q-1}{2}} \partial y (1 - y)^{q-1},$$

sive permutatis exponentibus

$$S = \frac{q}{2} \int y^{q-1} \partial y (1 - y)^{\frac{p-q-1}{2}},$$

quae comparata cum §. 29. dat $f = q - 1$, $g = \frac{p-q-1}{2}$, quibus valoribus substitutis colligitur

$$S = \frac{q}{2q\left(\frac{\frac{p+q-1}{2}}{q}\right)} = \frac{1}{2\left(\frac{\frac{p+q-1}{2}}{q}\right)} = \frac{1}{\left(\frac{p}{q}\right)} + \frac{\left(\frac{q-1}{1}\right)}{\left(\frac{p+1}{q}\right)} + \frac{\left(\frac{q-2}{2}\right)}{\left(\frac{p+2}{q}\right)} + \text{etc.}$$

vel si ponatur $\frac{p+q-1}{2} = r$, erit

$$S = \frac{1}{2\left(\frac{r}{q}\right)} = \frac{1}{\left(\frac{p}{q}\right)} + \frac{\left(\frac{q-1}{1}\right)}{\left(\frac{p+1}{q}\right)} + \frac{\left(\frac{q-1}{2}\right)}{\left(\frac{p+2}{q}\right)} + \text{etc.}$$

Corollarium 1.

§. 37. Hic casu $q = 1$ summa inventa ipsi termino primo aequatur. Sumamus autem $q = 2$, erit

$$\frac{1}{2\binom{\frac{p+1}{2}}{q}} = \frac{1}{\binom{p}{2}} + \frac{1}{\binom{p-1}{2}},$$

hoc est

$$\frac{4}{pp-1} = \frac{2}{p(p-1)} + \frac{2}{p(p+1)},$$

unde patet istam summationem esse veritati consentaneam; de quo quidem nullum superesse potest dubium, quoties q est numerus integer positivus; quamobrem quosdam casus consideremus ubi non est talis.

Corollarium 2.

§. 38. Quo autem evolutio facilior evadat, contemplemur casum quo $r = q$, ut fiat $\binom{r}{q} = 1$, tum autem erit $p = 1 + q$ hincque

$$\binom{p}{q} = 1 + q;\ \binom{p+1}{q} = \frac{q+1}{1}\cdot\frac{q+2}{2};\ \binom{p+2}{q} = \frac{q+1}{1}\cdot\frac{q+2}{2}\cdot\frac{q+3}{3},$$

quibus substitutis orietur haec series

$$\tfrac{1}{2} = \frac{1}{q+1} + \frac{2(q-1)}{(q+1)(q+2)} + \frac{3(q-1)(q-2)}{(q+1)(q+2)(q+3)} + \frac{4(q-1)(q-2)(q-3)}{(q+1)(q+2)(q+3)(q+4)} + \text{etc.}$$

quae series notatu maxime est digna, quia ejus summa semper est $\frac{1}{2}$, quicunque valores litterae q tribuantur. Si enim sit $q = 0$, habebitur

$$\tfrac{1}{2} = 1 - 1 + 1 - 1 + 1 - \text{etc.}$$

quae est series notissima. Sit nunc $q = -1$, et ob $q + 1 = 0$ multiplicemus omnes terminos per $q + 1$, prodibitque haec series

$$0 = 1 - 4 + 9 - 16 + 25 - \text{etc.}$$

uti differentias sumendo facile patet. Ponamus $q = \frac{1}{2}$, et haec series prodibit

$$\tfrac{1}{2}=\tfrac{2}{3}-\frac{2.2}{3.5}+\frac{2.3}{5.7}-\frac{2.4}{7.9}+\frac{2.5}{9.11}-\text{etc.}$$

Cum igitur sit

$$\tfrac{2}{3}=1-\tfrac{1}{3};\quad \frac{4}{3.5}=\tfrac{2}{3}-\tfrac{2}{5};\ \frac{6}{5.7}=\tfrac{3}{5}-\tfrac{3}{7};\ \frac{8}{7.9}=\tfrac{4}{7}-\tfrac{4}{9};$$

et ita porro, his substitutis prodibit haec series

$$\tfrac{1}{2}=1-1+1-1+1-1+1-\text{etc.}$$

At si sumamus $q=-\frac{1}{2}$ erit

$$\tfrac{1}{2}=2-4+6-8+10-12+\text{etc.},$$

quod per differentias fit manifestum.

Corollarium 3.

§. 39. Sumamus nunc $r=0$, ut fiat $p=1-q$. Demonstravi autem esse $\left(\frac{0}{q}\right)=\frac{\sin. q\pi}{q\pi}$, unde orietur

$$\frac{\pi q}{2\sin.\pi q}=\frac{1}{\left(\frac{1-q}{q}\right)}+\frac{\left(\frac{q-1}{1}\right)}{\left(\frac{2-q}{q}\right)}+\frac{\left(\frac{q-1}{2}\right)}{\left(\frac{3-q}{q}\right)}+\text{etc.}$$

cujus casum $q=\frac{1}{2}$ evolvisse pretium erit, membrum enim sinistrum fit $\frac{\pi}{4}$. Pro parte dextra autem habebimus

$$\left(\frac{q-1}{1}\right)=-\tfrac{1}{2};\ \left(\frac{q-1}{2}\right)=\frac{1.3}{2.4};\ \left(\frac{q-1}{3}\right)=-\frac{1.3.5}{2.4.6};\ \text{etc.}$$

tum vero pro denominatore

$$\left(\frac{1-q}{q}\right)=1;\ \left(\frac{2-q}{q}\right)=\tfrac{3}{2};\ \left(\frac{3-q}{q}\right)=\frac{3.5}{2.4};\ \left(\frac{4-q}{q}\right)=\frac{3.5.7}{2.4.};\ \text{etc.}$$

quibus valoribus substitutis orietur haec series

$$\tfrac{\pi}{4}=1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}-\text{etc.}$$

quae est series notissima. Ponamus autem adhuc $q=-\frac{1}{2}$, et membrum sinistrum erit ut ante $\frac{\pi}{4}$; pro parte dextra autem erit

$$\left(\frac{q-1}{1}\right)=-\tfrac{3}{2};\ \left(\frac{q-1}{2}\right)=\frac{3.5}{2.4};\ \left(\frac{q-2}{3}\right)=-\frac{3.5.7}{2.4.6};\ \text{etc. tum}$$

$$\left(\frac{1-q}{q}\right)=\frac{1.3}{2.4};\ \left(\frac{2-q}{q}\right)=\frac{1.3.5}{2.4.6};\ \left(\frac{3-q}{q}\right)=\frac{1.3.5.7}{2.4.6.8};\ \text{etc. hinc}$$

$$\frac{\pi}{4} = \frac{2.\,4}{1.\,3} - \frac{4.\,6}{1.\,5} + \frac{6.\,8}{1.\,7} - \frac{8.\,10}{1.\,9} + \text{etc.},$$

ujus veritas ita ostenditur. Cum sit

$$\frac{2.\,4}{1.\,3} = 3 - \tfrac{1}{3};\quad \frac{4.\,6}{1.\,5} = 5 - \tfrac{1}{5};\quad \frac{6.\,8}{1.\,7} = 7 - \tfrac{1}{7};\quad \frac{8.\,10}{1.\,9} = 9 - \tfrac{1}{9};\ \text{etc.}$$

erit illa series aequalis huic

$$\tfrac{\pi}{4} = 3 - \tfrac{1}{3} - 5 + \tfrac{1}{5} + 7 - \tfrac{1}{7} - 9 + \tfrac{1}{9} - \text{etc.}$$

quae series in has duas discerpatur

$$\tfrac{\pi}{4} = \begin{cases} 3 - 5 + 7 - 9 + 11 - 13 + \text{etc.} \\ -\tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \tfrac{1}{13} - \text{etc.} \end{cases}$$

De superiore notetur, ejus summam per differentias erutam esse

$$3 - 5 + 7 - 9 + 11 - 13 + \text{etc.} = 1;$$

inferioris summa ex serie supra inventa, qua erat

$$\tfrac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \text{etc. erit}$$

$$-\tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \text{etc.} = \tfrac{\pi}{4} - 1,$$

unde jam manifestum est fore

$$3 - \tfrac{1}{3} - 5 + \tfrac{1}{5} + 7 - \tfrac{1}{7} - 9 + \tfrac{1}{9} - \text{etc.} = 1 + \tfrac{\pi}{4} - 1 = \tfrac{\pi}{4}.$$

Hinc igitur patet, pro q etiam numeros negativos atque adeo fractos accipi posse.

Theorema generale.

§. 40. Si X denotet functionem quamcunque ipsius x, et proposita fuerit haec aequatio differentialis cujuscunque gradus,

$$\partial^q y = 1.\,2.\,3 \ldots\ldots q\, X\, \partial x^q,$$

ubi exponens q denotet numeros quoscunque sive integros sive fractos sive positivos sive negativos, cujus ergo aequationis resolutio totidem integrationes requirit, quae si singulae ab $x = 0$ inchoentur omnibusque peractis statuatur $x = 1$, tum semper erit $y = q \int X \partial x (1 - x)^{q-1}$, hoc scilicet integrali ab $x = 0$ ad $x = 1$ extenso.