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Methodus singularis resolvendi aequationes differentiales secundi gradus

Leonhard Euler

SUPPLEMENTUM IX.

AD SECT. I. TOM. II.

DE

RESOLUTIONE AEQUATIONUM DIFFERENTIALIUM SECUNDI GRADUS, DUAS TANTUM VARIABILES INVOLVENTIUM.

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1). Methodus singularis resolvendi aequationes differentiales secundi gradus. *M. S. Academiae exhib. die* 19 *Jan.* 1779.

§. 1. Si p et q fuerint functiones quaecunque ipsius x, atque proposita fuerit haec aequatio inter binas variabiles x et z

$$2p\partial z + z\partial p = \frac{\partial x}{q}\int\frac{z\partial x}{q},$$

evidens est ejus integrale facile inveniri posse, si ea multiplicetur per z, ut habeatur

$$2pz\partial z + zz\partial p = \frac{z\partial x}{q}\int\frac{z\partial x}{q}.$$

Prioris enim membri integrale est pzz, posterius verò membrum posito $\int\frac{z\partial x}{q} = v$, abit in $v\,\partial\,v$, cujus integrale est $\frac{1}{2}\,v\,v + \mathrm{C}$, ita ut hinc nanciscamur istam aequationem integralem $pzz = \frac{1}{2}vv + \mathrm{C}$, unde fit $vv = 2pzz - 2\,\mathrm{C}$, hincque

$$v = \int\frac{z\partial x}{q} = \sqrt{(2ppzz - \mathrm{C})},$$

quae differentiata dat

$$\frac{z\,\partial x}{q} = \frac{2pz\,\partial z + zz\,\partial p}{\sqrt{(2pzz - C)}},$$

facto ergo divisione per z, erit

$$\frac{\partial x}{q} = \frac{2p\,\partial z + z\,\partial p}{\sqrt{(2pzz - C)}},$$

quemadmodum autem hinc valor ipsius z per x, ejusque functiones p et q exprimi queat, non liquet. Ut autem istum scopum obtineamus, posito ut fecimus $\int \frac{z\,\partial x}{q} = v$ ut sit $vv = 2pzz - C$, retineamus quantitatem v in calculo, et cum sit

$$\frac{z\,\partial x}{q} = \partial v, \text{ erit } Z = \frac{q\,\partial v}{\partial x},$$

quo valore substituto habebimus

$$vv = \frac{2pqq\,\partial v^2}{\partial x^2} - C,$$

unde colligitur

$$\partial v = \frac{\partial x \sqrt{[vv + C]}}{q\sqrt{2p}},$$

quae sponte separationem admittit, cum sit

$$\frac{\partial v}{\sqrt{(vv + C)}} = \frac{\partial x}{q\sqrt{2p}}, \text{ ideoque}$$

$$\int \frac{\partial v}{\sqrt{(vv + C)}} = \int \frac{\partial x}{q\sqrt{2p}},$$

cujus valor, quoniam p et q sunt functiones ipsius x, tanquam cognitus spectari potest.

§. 2. Statuamus ergo hoc integrale

$$\int \frac{\partial x}{q\sqrt{2p}} = l\,X,$$

ut habeamus

$$\int \frac{\partial v}{\sqrt{(vv + C)}} = l\,X,$$

quare cum constet esse

$$\int \frac{\partial v}{\sqrt{(vv + C)}} = l\,[v + \sqrt{(vv + C)}], \text{ erit}$$

$$v + \sqrt{(vv + C)} = X,$$

unde colligitur $v = \frac{X^2 - C}{2X}$, ideoque per quantitatem X definitur.

§. 3. Cum igitur supra invenerimus $2pzz = vv + C$, erit

$$2pzz = \frac{(X^2 - C)^2}{4XX} + C = \frac{(XX + C)^2}{4XX},$$

consequenter erit

$$z\sqrt{2p} = \frac{X^2 + C}{2X},$$

sicque quantitas z ita per X exprimitur, ut sit

$$z = \frac{X^2 + C}{2X\sqrt{2p}},$$

ubi meminisse oportet esse

$$l\,X = \int \frac{\partial x}{q\sqrt{2p}}, \text{ sive } X = e^{\int \frac{\partial x}{q\sqrt{2p}}}.$$

§. 4. Manifestum autem est, aequationem nostram propositam, si a signo integrali liberetur, abire in aequationem differentialem secundi gradus, cujus ergo integrale completum modo elicuimus. Facta enim multiplicatione per q fiet

$$2pq\partial z + qz\partial p = \partial x \int \frac{z\partial x}{q},$$

et differentiatio sumto elemento ∂x constante praebebit sequentem aequationem differentialem secundi gradus

$$\left.\begin{array}{r} 2pq\partial\partial z + 2p\partial q\partial z + z\partial q\partial p \\ + 3q\partial p\partial z + qz\partial\partial p \end{array}\right\} = \frac{z\partial x^2}{q},$$

cujus ergo aequationis non parum abstrusae novimus esse integrale completum

$$z = \frac{X^2 + C}{2X\sqrt{2p}}, \text{ existente } X = e^{\int \frac{\partial x}{q\sqrt{2p}}},$$

ita ut ista quantitas X etiam constantem arbitrariam involvat.

§. 5. Cum autem haec aequatio non parum sit complicata, sequenti modo concinnius repraesentari potest; cum enim sit

$$\frac{q}{z}\partial . pzz = \partial x \int \frac{z \partial x}{q},$$

erit differentiationem tantum indicando

$$\partial . \frac{q \partial . pzz}{z} = \frac{z \partial x^2}{q},$$

quae manifesto integrabilis evadit, si multiplicetur per $\frac{2 q \partial . pzz}{z}$, quodsi enim brevitatis gratia statuatur $\frac{q \partial . pzz}{z} = s \partial x$, membrum sinistrum fit

$$2 s \partial x . \partial s \partial x = 2 \int \partial s \partial x^2,$$

ejusque ergo integrale $s s \partial x^2$: at vero ex parte dextra habebimus $2 \partial x^2 \partial . p z z$, cujus igitur integrale est

$$2 p z z \partial x^2 + C \partial x^2,$$

ita ut integratio nobis praebeat $ss = 2pzz + C$.

§. 6. Quo nunc hanc aequationem penitius evolvamus, statuamus ut ante $pzz = v$, ita ut sit $\frac{q \partial v}{z} = s \partial x$, eritque nostrum integrale inventum

$$ss = \frac{q q \partial v^2}{z z \partial x^2} = 2 v + C,$$

quae ob $zz = \frac{v}{p}$ abit in hanc

$$\frac{p q q \partial v^2}{v \partial x^2} = 2 v + C,$$

unde eruitur propémodum ut ante

$$\frac{\partial v}{\sqrt{v (2 v + C)}} = \frac{\partial x}{q \sqrt{p}},$$

quae a forma ante inventa non discrepat.

§. 7. Simili modo etiam aliae aequationes differentiales magis complicatae resolvi poterunt, veluti si proponatur ista aequatio

$$3\,p\,\partial z + z\,\partial p = \frac{\partial x}{q}\int\frac{z\,z\,\partial x}{q},$$

ubi iterum p et q denotant functiones quascunque ipsius x. Cum enim sit

$$3\,p\,\partial z + z\,\partial p = \frac{\partial \cdot p\,z^3}{z\,z};$$

erit per $z\,z$ multiplicando

$$\partial \cdot p\,z^3 = \frac{z\,z\,\partial x}{q}\int\frac{z\,z\,\partial x}{q},$$

quae posito $\int\frac{z\,z\,\partial x}{q} = v$ abit in $\partial \cdot p\,z^3 = v\,\partial v$, ideoque integrando $2\,p\,z^3 = v\,v + \mathrm{C}$.

§. 8. Quoniam autem posuimus $\int\frac{z\,z\,\partial x}{q} = v$, erit $z\,z = \frac{q\,\partial v}{\partial x}$, hincque $z^3 = \frac{q\,\partial v}{\partial x}\sqrt{\frac{q\,\partial v}{\partial x}}$,

unde fit

$$\frac{2\,p\,q\,\partial v}{\partial x}\sqrt{\frac{q\,\partial v}{\partial x}} = v\,v + \mathrm{C}.$$

Sumtis ergo quadratis erit

$$\frac{4\,p\,p\,q^3\,\partial v^3}{\partial x^3} = (v\,v + \mathrm{C})^2, \text{ ideoque}$$

$$\frac{\partial v^3}{(v\,v + \mathrm{C})^2} = \frac{\partial x^3}{4\,p\,p\,q^3},$$

cujus radix cubica praebet

$$\frac{\partial v}{\sqrt[3]{(v\,v + \mathrm{C})^2}} = \frac{\partial x}{q\sqrt[3]{4\,p\,p}}.$$

Hinc igitur quantitas v per x definitur, ita ut jam v spectare queamus tanquam veram functionem ipsius x, qua inventa erit

$$z^3 = \frac{v\,v + \mathrm{C}}{2\,p}, \text{ hincque } z = \sqrt[3]{\frac{v\,v + \mathrm{C}}{2\,p}}.$$

§. 9. Eadem ista aequatio adhuc alio modo resolvi poterit, quandoquidem per q multiplicata ita repraesentatur

$$\frac{q\partial \cdot p z^3}{zz} = \partial x \int \frac{zz\partial x}{q}, \text{ sive}$$

$$\partial \cdot \frac{q\partial \cdot p z^3}{zz} = \frac{zz\partial x^2}{q},$$

quae manifesto integrabilis redditur, multiplicando per $\frac{2q\partial \cdot p z^3}{zz}$, prodit enim

$$\left(\frac{q\partial \cdot p z^3}{zz}\right)^2 = 2 p z^3 \partial x^2 + C \partial x^2.$$

§. 10. Jam ponatur $p z^3 = v$, ita ut sit

$$z^3 = \frac{v}{p}, \text{ et } z^4 = \frac{v}{p}\sqrt[3]{\frac{v}{p}},$$

quo valore substituto habebimus

$$\frac{p q q \partial v^2 \sqrt[3]{p}}{v \sqrt[3]{v}} = 2 v \partial x^2 + C \partial x^2,$$

unde concluditur

$$\frac{\partial v^2}{v(2v+C)\sqrt[3]{v}} = \frac{\partial x^2}{p q q \sqrt[3]{p}}, \text{ sive}$$

$$\frac{\partial v}{\sqrt{v(2v+C)\sqrt[3]{v}}} = \frac{\partial v}{v^{\frac{2}{3}}\sqrt{(2v+C)}} = \frac{\partial x}{q\sqrt[3]{pp}},$$

haec aequatio simplicior evadit, ponendo $v = u^3$, scilicet

$$\frac{3\partial u}{\sqrt{(2u^3+C)}} = \frac{\partial x}{q\sqrt[3]{pp}}.$$

Hinc intelligitur, innumerabilia exempla per has formulas expediri posse.

§. 11. Quin etiam hujusmodi aequationes multo generaliores tractari poterunt; namque aequatio generalior ita potest repraesentari

$$\frac{\partial \cdot p z^m}{z^n} = \frac{\partial x}{q} \int \frac{z^n \partial x}{q},$$

quae evoluta dat

$$m p z^{m-n-1} \partial z + z^{m-n} \partial p = \frac{\partial x}{q} \int \frac{z^n \partial x}{q}.$$

Facta autem multiplicatione per z^n, prodit aequatio sponte integrabilis

$$\partial . p z^m = \frac{z^n \partial x}{q} \int \frac{z^n \partial x}{q},$$

si quidem prodit

$$2 p z^m = \left(\int \frac{z^n \partial x}{q} \right)^2 + C.$$

§. 12. Ad hanc aequationem ulterius evolvendam statuamus

$$\int \frac{z^n \partial x}{q} = v, \text{ eritque } z^n = \frac{q \partial v}{\partial x},$$

unde primo $2 p z^m = v v + C$, et hinc porro

$$(2 p)^{\frac{n}{m}} . z^n = (2 p)^{\frac{n}{m}} . \frac{q \partial v}{\partial x} = (v v + C)^{\frac{n}{m}},$$

quae cum sponte sit separabilis, dabit

$$\frac{\partial v}{(v v + C)^{\frac{n}{m}}} = \frac{\partial x}{q (2 p)^{\frac{n}{m}}},$$

unde ergo quantitas v per x determinabitur, qua inventa ipsa quantitas quaesita z ita exprimetur, ut sit $z^m = \frac{v v + C}{2 p}$.

§. 13. Illustremus haec unico exemplo a primo casu petito, sumendo scilicet $p = 1 + xx$ et $q = \sqrt{2}$, ita ut aequatio proposita sit

$$2\,\partial z(1 + xx) + 2\,z\,x\,\partial x = \frac{\partial x}{2}\int z\,\partial x,$$

quae in hanc aequationem secundi gradus evolvitur

$$4\,\partial\partial z(1 + xx) + 12\,x\,\partial x\,\partial z + 3\,z\,\partial x^2 = 0,$$

cujus ergo integrale quaeritur.

§. 14. Faciamus ergo applicationem solutionis supra §. 3. inventae, ubi cum hic sit $p = 1 + xx$ et $q = \sqrt{2}$, erit

$$l\,X = \tfrac{1}{2}\int\frac{\partial x}{\sqrt{(1+xx)}} = \tfrac{1}{2}\,l\,[x + \sqrt{(1 + xx)}] - \tfrac{1}{2}\,l\,a,$$

unde fit

$$X = \frac{\sqrt{[x + \sqrt{(1 + xx)}]}}{\sqrt{a}},$$

hoc igitur valore substituto habebimus

$$z = \frac{a\,C + x + \sqrt{(1 + xx)}}{2\sqrt{2\,a\,(1 + xx)\,[x + \sqrt{(1 + xx)}]}},$$

quae hoc modo simplicius exprimitur

$$z = \frac{[a\,C + x + \sqrt{(1 + xx)}]\,\sqrt{[-x + \sqrt{(1 + xx)}]}}{2\sqrt{2\,a\,(1 + xx)}}.$$

Ubi ergo duae quantitates constantes arbitrariae sunt involutae, atque adeo hoc integrale completum algebraice determinetur. Posito ergo $C = 0$, integrale particulare erit ex prima forma petitum

$$z = \frac{\sqrt{[x + \sqrt{(1 + xx)}]}}{2\sqrt{2\,a\,(1 + xx)}}.$$

§. 15. Aliud integrale particulare hinc exhiberi potest, constantes ita sumendo ut sit $a\,C$ infinitum, at vero $C\sqrt{a}$ finitum $= b$, tum enim erit

$$z = \frac{a\,C}{2\sqrt{2\,a\,(1 + xx)\,[x + \sqrt{(1 + xx)}]}} = \frac{b}{2\sqrt{2\,(1 + xx)\,[x + \sqrt{(1 + xx)}]}},$$

quae forma redigitur ad hanc

$$z = \frac{\alpha \sqrt{[-x + \sqrt{(1 + xx)}]}}{\sqrt{(1 + xx)}}.$$

2.) Methodus nova investigandi omnes casus, quibus hanc aequationem differentio-differentialem

$$\partial \partial y (1 - a x x) - b x \partial x \partial y - c y \partial x^2 = 0$$

resolvere licet. *M. S. Academiae exhib. die* 13 *Januarii*, 1780.

§. 16. Hic quidem in usum vocari posset methodus a me et ab aliis jam passim exposita, qua valor ipsius y per seriem infinitam exprimitur. Tunc enim omnibus casibus, quibus haec series alicubi abrumpitur, habebitur integrale particulare aequationis propositae; unde quidem haud difficulter integrale completum erui poterit. Verum etsi hoc modo infiniti casus integrabiles reperiuntur, tamen non omnes innotescunt, sed dantur praeterea infiniti alii casus, qui resolutionem admittunt. Quamobrem hic methodum prorsus singularem proponam, cujus ope omnes plane casus integrabiles elici poterunt. Haec autem methodus ita est comparata, ut cognito casu quocunque resolutionem admittente, ex eo innumerabiles alii deduci queant.

§. 17. Statim autem se offerunt duo casus simplicissimi, quibus resolutio succedit, quorum alter est, si $c = 0$, alter vero si $b = a$, quos ergo binos casus principales ante omnia evolvi oportet.