1794

Disquisitio coniecturalis super formula integrali ∫∂ϕ cos(*i*ϕ)/(**α**+**β**cos(ϕ))^n

Leonhard Euler

prorsus uti supra invenimus pro casu $\int\frac{\partial\Phi\cos.3\Phi}{\Delta^3}$; talis autem consensus perpetuo deprehendi debebit.

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3) Disquisitio conjecturalis super formula integrali

$$\int\frac{\partial\Phi\cos.i\Phi}{(\alpha+\beta\cos.\Phi)^n}.$$

M. S. Academiae exhib. die 31. *Augusti* 1778.

§. 40. Incipiamus a casu simplicissimo quo $i=0$ et $n=1$, et formula integranda proponitur haec $\int\frac{\partial\Phi}{\alpha+\beta\cos.\Phi}$, ad quod praestandum commodissime in subsidium vocatur haec substitutio tang. $\frac{1}{2}\Phi=t$, unde statim fit $\partial\Phi=\frac{2\partial t}{1+tt}$: porro vero cum hinc sit

$$\sin.\tfrac{1}{2}\Phi=\frac{t}{\sqrt{(1+tt)}}\text{ et }\cos.\tfrac{1}{2}\Phi=\frac{1}{\sqrt{(1+tt)}},$$

erit $\cos.\Phi=\frac{1-tt}{1+tt}$, ideoque denominator nostrae formulae

$$\alpha+\beta\cos.\Phi=\frac{\alpha+\beta+(\alpha-\beta)tt}{1+tt},$$

sicque nostra formula integranda erit

$$\int\frac{2\partial t}{\alpha+\beta+(\alpha-\beta)tt}.$$

§. 41. Constat autem ex elementis esse

$$\int\frac{\partial t}{f+gtt}=\frac{1}{\sqrt{fg}}\text{ Arc. tang. }t\sqrt{\frac{g}{f}}.$$

Quare cum pro nostro casu sit $f=\alpha+\beta$ et $g=\alpha-\beta$, habebimus hanc integrationem

$$\int\frac{\partial\Phi}{\alpha+\beta\cos.\Phi}=\frac{2}{\sqrt{(\alpha\alpha-\beta\beta)}}\text{ Arc. tang. }t\sqrt{\frac{\alpha-\beta}{\alpha+\beta}},$$

existente $t=$ tang. $\frac{1}{2}\Phi$; quod ergo integrale evanescit casu $t=0$, ideoque casu $\Phi=0$. Quodsi ergo hoc integrale extendere velimus a termino $\phi = 0$ usque ad terminum $\phi = 180^\circ$, ubi fit $t = \infty$, istud integrale erit $\frac{2}{\sqrt{(\alpha\alpha - \beta\beta)}} \cdot \frac{\pi}{2}$, denotante π semiperipheriam circuli, cujus radius $= 1$.

§. 42. Quoniam igitur integrale nostrae formulae a termino $\phi = 0$ usque ad terminum $\phi = 180^\circ$ tam concinne et simpliciter exprimitur, etiam generatim in hac dissertatione in ea tantum integralia formulae generalis propositae

$$\int \frac{\partial \phi \cos. i\phi}{(\alpha + \beta \cos. \phi)^n},$$

sum inquisiturus, quae comprehenduntur inter terminos $\phi = 0$ et $\phi = 180^\circ$. Quia autem in casu tractato formula inest irrationalis $\sqrt{(\alpha\alpha - \beta\beta)}$, ad hoc incommodum tollendum, in sequentibus perpetuo assumemus $\alpha = 1 + aa$ et $\beta = -2a$, unde fit $\sqrt{(\alpha\alpha - \beta\beta)} = 1 - aa$, sicque nostrae disquisitiones versabuntur circa integrationem hujus formulae generalis

$$\int \frac{\partial \phi \cos. i\phi}{(1 + aa - 2a \cos. \phi)^n},$$

pro qua brevitatis gratia ubique statuamus

$$1 + aa - 2a \cos. \phi = \Delta,$$

ut nostra formula generalis jam sit

$$\int \frac{\partial \phi \cos. i\phi}{\Delta^n},$$

ubi ut jam notatum, eum tantum integralis valorem explorare nobis est propositum, qui intra terminos $\phi = 0$ et $\phi = 180^\circ$ contineatur, quem valorem ex casibus particularibus concludere conabimur. Praeterea vero hic in genere notetur, litteram i nobis perpetuo alios numeros non designare praeter integros, et quidem positivos, quandoquidem semper est

$$\cos. -i\phi = \cos. +i\phi.$$

I. De integratione formulae

$$\int\frac{\partial\Phi\cos.i\Phi}{\Delta}\left[\begin{array}{c}a\,\Phi=0\\ ad\,\Phi=180^\circ\end{array}\right].$$

§. 43. Hic ergo casus in generali continetur, ponendo exponentem $n=1$, quem casum ut simplicissimum spectamus, siquidem casus $n=0$ nulla prorsus laborat difficultate, cum sit

$$\int\partial\Phi\cos.i\Phi=\frac{1}{i}\sin.i\Phi,$$

quod integrale jam evanescit casu $i=0$, et quoniam i numeros tantum integros significat, sumto $\Phi=180^\circ$ hoc integrale iterum evanescit, solo casu excepto quo $i=0$, quippe quo casu integrale fiet $=\Phi$, ideoque sumto $\Phi=180^\circ$ erit pro terminis integrationis constitutis $\int\partial\Phi=\pi$.

§. 44. Iste postremus casus fundamentum continet, unde integralia formae hic propositae haurire conveniet; cum enim sit

$$\partial\Phi=\frac{(1+aa)\partial\Phi}{\Delta}-\frac{2a\partial\Phi\cos.\Phi}{\Delta},$$

erit integrando pro terminis praescriptis

$$\pi=(1+aa)\int\frac{\partial\Phi}{\Delta}-2a\int\frac{\partial\Phi\cos.\Phi}{\Delta};$$

supra autem invenimus esse $\int\frac{\partial\Phi}{\Delta}=\frac{\pi}{1-aa}$, quo valore substituto adipiscimur integrationem casus $i=1$, cum enim sit

$$\pi=\frac{(1+aa)\pi}{1-aa}-2a\int\frac{\partial\Phi\cos.\Phi}{\Delta},\text{ erit }\int\frac{\partial\Phi\cos.\Phi}{\Delta}=\frac{\pi a}{1-aa};$$

sicque jam duos casus sumus adepti, qui sunt

$$\int\frac{\partial\Phi}{\Delta}=\frac{\pi}{1-aa}\text{ et }\int\frac{\partial\Phi\cos.\Phi}{\Delta}=\frac{\pi a}{1-aa}.$$

§. 45. Ex his autem duobus casibus $i=0$ et $i=1$ sequentes omnes haud difficulter derivare licet ope hujus lemmatis; cum sit ut vidimus $\int\partial\Phi\cos.i\Phi=0$, erit

$$0=(1+aa)\int\frac{\partial\Phi\cos.i\Phi}{\Delta}-2a\int\frac{\partial\Phi\cos.\Phi\cos.i\Phi}{\Delta}.$$

Constat autem esse

$$2 \cos. \phi \cos. i\phi = \cos. (i-1)\phi + \cos. (i+1)\phi,$$

unde habebimus hanc aequationem

$$\frac{1+aa}{a}\int\frac{\partial\phi \cos. i\phi}{\Delta} = \int\frac{\partial\phi \cos. (i-1)\phi}{\Delta} + \int\frac{\partial\phi \cos. (i+1)\phi}{\Delta},$$

unde oritur istud lemma

$$\int\frac{\partial\phi \cos. (i+1)\phi}{\Delta} = \frac{1+aa}{a}\int\frac{\partial\phi \cos. i\phi}{\Delta} - \int\frac{\partial\phi \cos. (i-1)\phi}{\Delta}.$$

Sumto nunc $i = 1$, istud lemma nobis suppeditat hunc casum

$$\int\frac{\partial\phi \cos. 2\phi}{\Delta} = \frac{1+aa}{a}\int\frac{\partial\phi \cos. \phi}{\Delta} - \int\frac{\partial\phi}{\Delta},$$

qui ergo per binos praecedentes expeditur; fiet enim

$$\int\frac{\partial\phi \cos. 2\phi}{\Delta} = \frac{\pi aa}{1-aa}.$$

Sumatur nunc $i = 2$, et lemma nobis dabit

$$\int\frac{\partial\phi \cos. 3\phi}{\Delta} = \frac{1+aa}{a}\int\frac{\partial\phi \cos. 2\phi}{\Delta} - \int\frac{\partial\phi \cos. \phi}{\Delta}, \text{ sive}$$

$$\int\frac{\partial\phi \cos. 3\phi}{\Delta} = \frac{\pi a^3}{1-aa}:$$

simili modo sumto $i = 3$, lemma dabit

$$\int\frac{\partial\phi \cos. 4\phi}{\Delta} = \frac{1+aa}{a}\int\frac{\partial\phi \cos. 3\phi}{\Delta} - \int\frac{\partial\phi \cos. 2\phi}{\Delta}, \text{ sive}$$

$$\int\frac{\partial\phi \cos. 4\phi}{\Delta} = \frac{\pi a^4}{1-aa}.$$

Porro casus $i = 4$ praebet

$$\int\frac{\partial\phi \cos. 5\phi}{\Delta} = \frac{1+aa}{a}\int\frac{\partial\phi \cos. 4\phi}{\Delta} - \int\frac{\partial\phi \cos. 3\phi}{\Delta}, \text{ sive}$$

$$\int\frac{\partial\phi \cos. 5\phi}{\Delta} = \frac{\pi a^5}{1-aa}, \text{ atque ita porro.}$$

§. 46. Hinc igitur patet, singulos istos casus ex binis praecedentibus determinari ope scalae relationis $\frac{1+aa}{a}$, -1, atque seriem recurrentem hinc oriundam abire in geometricam: quodsi enim bini termini postremi jam inventi fuerint

$$\frac{\pi a^\lambda}{1-aa} \text{ et } \frac{\pi a^{\lambda+1}}{1-aa},$$

sequens reperitur $= \frac{\pi a^{\lambda+2}}{1-aa}$, ex quo ergo sine ullo dubio sequitur, pro casu particulari hoc loco tractati in genere fore.

$$\int \frac{\partial \Phi \cos. i\Phi}{\Delta} = \frac{\pi a^i}{1-aa},$$

ubi autem probe est notandum, loco i non nisi numeros integros positivos assumi debere.

II. De integratione formulae.

$$\int \frac{\partial \Phi \cos. i\Phi}{\Delta^2} \begin{bmatrix} a\,\Phi = 0 \\ ad\,\Phi = 180^\circ \end{bmatrix}.$$

§. 47. Casus simplicissimus hic occurret $\int \frac{\partial \Phi}{\Delta^2}$, cujus ergo integrale ante omnia perscrutari oportet; hunc in finem consideremus hanc formulam finitam $\frac{\sin. \Phi}{\Delta} = V$, quae pro utroque termino $\Phi = 0$ et $\Phi = 180^\circ$ evanescit; hinc autem erit

$$\partial V = \frac{\partial \Phi \cos. \Phi}{\Delta} - \frac{2a\,\partial\Phi \sin. \Phi^2}{\Delta^2}, \text{ sive}$$

$$\partial V = \frac{(1+aa)\,\partial\Phi \cos. \Phi - 2a\,\partial\Phi}{\Delta^2};$$

unde integrando jam novimus esse

$$0 = (1+aa) \int \frac{\partial \Phi \cos. \Phi}{\Delta^2} - 2a \int \frac{\partial \Phi}{\Delta^2}.$$

Porro vero quoniam ante habuimus $\int \frac{\partial \Phi}{\Delta} = \frac{\pi}{1-aa}$, hanc formulam integralem supra et infra per Δ multiplicando, erit quoque

$$\frac{\pi}{1-aa} = (1+aa) \int \frac{\partial \Phi}{\Delta^2} - 2a \int \frac{\partial \Phi \cos \Phi}{\Delta^2}.$$

Ex praecedente autem colligitur

$$\int \frac{\partial \Phi \cos. \Phi}{\Delta^2} = \frac{2a}{1+aa} \cdot \int \frac{\partial \Phi}{\Delta^2},$$

quo valore substituto habebimus

$$\frac{\pi}{1-aa} = (1+aa) \int \frac{\partial \Phi}{\Delta^2} - \frac{4aa}{1+aa} \int \frac{\partial \Phi}{\Delta^2} = \frac{(1-aa)^2}{1+aa} \int \frac{\partial \Phi}{\Delta^2},$$

quamobrem hinc adipiscimur hanc integrationem principalem

$$\int \frac{\partial \Phi}{\Delta^2} = \frac{\pi (1+aa)}{(1-aa)^3},$$

ex quo immediate deducitur casus sequens

$$\int \frac{\partial \Phi \cos. \Phi}{\Delta^2} = \frac{2\pi a}{(1-aa)^3}.$$

§. 48. Pro sequentibus casibus consideremus integrationem in articulo praecedente inventam

$$\int \frac{\partial \Phi \cos. i \Phi}{\Delta} = \frac{\pi a^i}{1-aa},$$

quae formula integralis supra et infra per Δ multiplicando discerpitur in sequentes duas partes

$$\frac{\pi a^i}{1-aa} = (1+aa) \int \frac{\partial \Phi \cos. i \Phi}{\Delta^2} - 2a \int \frac{\partial \Phi \cos. \Phi \cos. i \Phi}{\Delta^2},$$

quae aequatio porro evolvitur in hanc formam

$$\frac{\pi a^i}{1-aa} = (1+aa) \int \frac{\partial \Phi \cos. i \Phi}{\Delta^2} - a \int \frac{\partial \Phi \cos. (i-1) \Phi}{\Delta^2}$$
$$- a \int \frac{\partial \Phi \cos. (i+1) \Phi}{\Delta^2};$$

unde deducitur hoc quasi lemma

$$\int \frac{\partial \Phi \cos. (i+1) \Phi}{\Delta^2} = \frac{1+aa}{a} \int \frac{\partial \Phi \cos. i \Phi}{\Delta^2}$$
$$- \int \frac{\partial \Phi \cos. (i-1) \Phi}{\Delta^2} - \frac{\pi a^{i-1}}{1-aa}.$$

§. 49. Sumamus nunc statim $i = 1$, atque istud lemma nobis praebet

$$\int \frac{\partial \Phi \cos. 2 \Phi}{\Delta^2} = \frac{1+aa}{a} \int \frac{\partial \Phi \cos. \Phi}{\Delta^2} - \int \frac{\partial \Phi}{\Delta^2} - \frac{\pi}{1-aa};$$

hic jam bini valores jam inventi substituantur, atque reperietur

$$\int \frac{\partial \Phi \cos. 2 \Phi}{\Delta^2} = \frac{\pi (1 + a a) - \pi (1 - a a)^2}{(1 - a a)^3},$$

hinc ergo sequitur fore

$$\int \frac{\partial \Phi \cos. 2 \Phi}{\Delta^2} = \frac{\pi (3 a a - a^4)}{(1 - a a)^5} = \frac{\pi a a (3 - a a)}{(1 - a a)^3}.$$

Sumatur nunc pro lemmate praemisso $i = 2$, eritque

$$\int \frac{\partial \Phi \cos. 3 \Phi}{\Delta^2} = \frac{1 + a a}{a} \int \frac{\partial \Phi \cos. 2 \Phi}{\Delta^2} - \int \frac{\partial \Phi \cos. \Phi}{\Delta^2} - \frac{\pi a}{1 - a a}, \text{ sive}$$

$$\int \frac{\partial \Phi \cos. 3 \Phi}{\Delta^2} = \frac{(1 + a a) \pi a (3 - a a) - 2 \pi a - \pi a (1 - a a)^2}{(1 - a a)^3},$$

quae expressio contrahitur in hanc

$$\int \frac{\partial \Phi \cos. 3 \Phi}{\Delta^2} = \frac{\pi a^3 (4 - 2 a a)}{(1 - a a)^3}.$$

Sit nunc in lemmate praemisso $i = 3$, eritque

$$\int \frac{\partial \Phi \cos. 4 \Phi}{\Delta^2} = \frac{1 + a a}{a} \int \frac{\partial \Phi \cos. 3 \Phi}{\Delta^2} - \int \frac{\partial \Phi \cos. 2 \Phi}{\Delta^2} - \frac{\pi a a}{1 - a a}, \text{ sive}$$

$$\int \frac{\partial \Phi \cos. 4 \Phi}{\Delta^2} = \frac{(1 + a a) \pi a a (4 - 2 a a) - \pi a a (3 - a a) - \pi a a (1 - a a)^2}{(1 - a a)^3},$$

quae expressio contrahitur in hanc

$$\int \frac{\partial \Phi \cos. 4 \Phi}{\Delta^2} = \frac{\pi a^4 (5 - 3 a a)}{(1 - a a)^3}.$$

Sit nunc in lemmate nostro $i = 4$, eritque

$$\int \frac{\partial \Phi \cos. 5 \Phi}{\Delta^2} = \frac{1 + a a}{a} \int \frac{\partial \Phi \cos. 4 \Phi}{\Delta^2} - \int \frac{\partial \Phi \cos. 3 \Phi}{\Delta^2} = \frac{\pi a^3}{1 - a a}, \text{ sive}$$

$$\int \frac{\partial \Phi \cos. 5 \Phi}{\Delta^2} = \frac{(1 + a a) \pi a^3 (5 - 3 a a) - \pi a^3 (4 - 2 a a) - \pi a^3 (1 - a a)^2}{(1 - a a)^3},$$

quae expressio contrahitur in hanc

$$\int \frac{\partial \Phi \cos. 5 \Phi}{\Delta^2} = \frac{\pi a^5 (6 - 4 a a)}{(1 - a a)^3}.$$

Sit nunc in lemmate nostro $i = 5$, eritque

$$\int \frac{\partial \Phi \cos. 3 \Phi}{\Delta^2} = \frac{1 + a a}{a} \int \frac{\partial \Phi \cos. 5 \Phi}{\Delta^2} - \int \frac{\partial \Phi \cos. 4 \Phi}{\Delta^2} - \frac{\pi a^4}{1 - a a}, \text{ sive}$$

$$\int \frac{\partial \Phi \cos. 6 \Phi}{\Delta^2} = \frac{(1 + a a) \pi a^4 (6 - 4 a a) - \pi a^4 (5 - 3 a a) - \pi a^4 (1 - a a)^2}{(1 - a a)^3},$$

quae expressio contrahitur in hanc

$$\int \frac{\partial \Phi \cos. 6 \Phi}{\Delta^2} = \frac{\pi a^6 (7 - 5 a a)}{(1 - a a)^3}.$$

§. 50. Qui has formulas earumque generationem attentius perpendet, nullo certe modo dubitabit, inde hanc conclusionem deducere, quin in genere pro casu hic proposito futurum sit

$$\int \frac{\partial \Phi \cos. i \Phi}{\Delta^2} = \frac{\pi a^i [i + 1 - (i - 1) a a]}{(1 - a a)^3}$$

cujus lex cum non sit tam manifesta, quam in casu praecedente, omnes formulas inventas junctim ante oculos ponamus

$$\int \frac{\partial \Phi}{\Delta^2} = \frac{\pi(1 + aa)}{(1 - aa)^3}$$

$$\int \frac{\partial \Phi \cos. \Phi}{\Delta^2} = \frac{\pi a (2 - 0 aa)}{(1 - aa)^3}$$

$$\int \frac{\partial \Phi \cos. 2\Phi}{\Delta^2} = \frac{\pi aa (3 - aa)}{(1 - aa)^3}$$

$$\int \frac{\partial \Phi \cos. 3\Phi}{\Delta^2} = \frac{\pi a^3 (4 - 2aa)}{(1 - aa)^3}$$

$$\int \frac{\partial \Phi \cos. 4\Phi}{\Delta^2} = \frac{\pi a^4 (5 - 3aa)}{(1 - aa)^3}$$

$$\int \frac{\partial \Phi \cos. 5\Phi}{\Delta^2} = \frac{\pi a^5 (6 - 4aa)}{(1 - aa)^3}$$

$$\int \frac{\partial \Phi \cos. 6\Phi}{\Delta^2} = \frac{\pi a^6 (7 - 5aa)}{(1 - aa)^3}$$

III. De integratione formulae.

$$\int \frac{\partial \Phi \cos. i \Phi}{\Delta^3} \begin{bmatrix} a \Phi = 0 \\ ad \Phi = 180 \end{bmatrix}.$$

§. 51. Pro casu simplicissimo $\int \frac{\partial \Phi}{\Delta^3}$ eruendo, utamur hac formula

$$V = \frac{\sin. \Phi}{\Delta^2}, \text{ eritque } \partial V = \frac{\partial \Phi \cos. \Phi}{\Delta^2} - \frac{2 \partial \Phi \sin. \Phi^2}{\Delta^3}, \text{ sive}$$

$$\partial V = \frac{(1 + aa) \partial \Phi \cos. \Phi - 2a \partial \Phi \cos. \Phi^2 - 4a \partial \Phi \sin. \Phi^2}{\Delta^3}.$$

Hic loco $\sin. \Phi^2$ scribatur $1 - \cos. \Phi^2$, atque integrando, ob $V = 0$ habebimus hanc aequationem

$$0 = (1 + a a) \int \frac{\partial \Phi \cos. \Phi}{\Delta^3} - 4 a \int \frac{\partial \Phi}{\Delta^3} + 2 a \int \frac{\partial \Phi \cos. \Phi^2}{\Delta^3}.$$

§. 52. Huc addamus hanc formam indefinitam

$$s = A\int\frac{\partial\Phi}{\Delta} + B\int\frac{\partial\Phi}{\Delta^2}$$

cujus differentiale ad denominationem Δ^3 perducatur, litterae vero A et B ita definiantur, ut membra $\partial\Phi \cos.\Phi$ et $\partial\Phi \cos.\Phi^2$ evanescant, eritque formulis differentialibus additis

$$\frac{\Delta^3(\partial V + \partial s)}{\partial\Phi} = -4a \qquad + (1+aa)\cos.\Phi \qquad + 2a\cos.\Phi^2$$
$$+ A(1+aa)^2 - 4Aa(1+aa)\cos.\Phi + 4Aaa\cos.\Phi^2$$
$$+ B(1+aa) - 2Ba\cos.\Phi.$$

Nunc igitur ut termini cos. Φ^2 abigantur, statuatur

$2a + 4Aaa = 0$, ideoque $A = \frac{-1}{2a}$.

Nunc etiam termini cos. Φ e medio tollantur, eritque

$1 + aa - 4Aa(1+aa) - 2Ba = 0$, unde fit

$B = \frac{3(1+aa)}{2a}$.

Ex quibus valoribus nanciscimur

$$\frac{\Delta^3(\partial V + \partial s)}{\partial\Phi} = \frac{(1-aa)^2}{a};$$

hinc ergo vicissim integrando habebimus

$$V + s = \frac{(1-aa)^2}{a}\int\frac{\partial\Phi}{\Delta^3}.$$

§. 53. Cum igitur, ut jam notavimus, sit $V = 0$, atque ex casibus jam tractatis

$$s = \frac{-1}{2a}\cdot\frac{\pi}{1-aa} + \frac{3(1+aa)}{2a}\cdot\frac{\pi(1+aa)}{(1-aa)^3},$$

habebimus hanc aequationem

$$\frac{(1-aa)^2}{a}\int\frac{\partial\Phi}{\Delta^3} = \frac{3\pi(1+aa)^2 - \pi(1-aa)^2}{2a(1-aa)^3},$$

unde colligitur

$$\int\frac{\partial\Phi}{\Delta^3} = \frac{\pi(1+4aa+a^4)}{(1-aa)^5}.$$

§. 54. Cum sit $\int \frac{\partial \phi}{\Delta^2} = \frac{\pi(1+aa)}{(1-aa)^3}$, erit per reductionem hactenus usitatam

$$\frac{\pi(1+aa)}{(1-aa)^3} = (1+aa)\int \frac{\partial \phi}{\Delta^3} - 2a\int \frac{\partial \phi \cos. \phi}{\Delta^3},$$

unde concludimus

$$\int \frac{\partial \phi \cos. \phi}{\Delta^3} = \frac{1+aa}{2a}\int \frac{\partial \phi}{\Delta^3} - \frac{\pi(1+aa)}{2a(1-aa)^3}, \text{ ideoque}$$

$$\int \frac{\partial \phi \cos. \phi}{\Delta^3} = \frac{1+aa}{2a} \cdot \frac{\pi(1+4aa+a^4)}{(1-aa)^5} - \frac{\pi(1+aa)}{2a(1-aa)^3}$$

$$= \frac{3\pi a(1+aa)}{(1-aa)^5} = \frac{\pi a(3+3aa)}{(1-aa)^5}.$$

§. 55. Cum igitur in articulo praecedente invenimus

$$\int \frac{\partial \phi \cos. i\phi}{\Delta^2} = \frac{\pi a^i[i+1-(i-1)aa]}{(1-aa)^3},$$

hanc formulam integralem supra et infra per Δ multiplicando habebimus

$$\frac{\pi a^i[i+1-(i-1)aa]}{(1-aa)^3} = (1+aa)\int \frac{\partial \phi \cos. i\phi}{\Delta^3}$$

$$-2a\int \frac{\partial \phi \cos. i\phi \cos. \phi}{\Delta^3}, \text{ sive}$$

$$\frac{\pi a^i[i+1-(i-1)aa]}{(1-aa)^3} = (1+aa)\int \frac{\partial \phi \cos. i\phi}{\Delta^3}$$

$$-a\int \frac{\partial \phi \cos. (i-1)\phi}{\Delta^3} - a\int \frac{\partial \phi \cos. (i+1)\phi}{\Delta^3};$$

unde deducitur hoc quasi lemma

$$\int \frac{\partial \phi \cos. (i+1)\phi}{\Delta^3} = \frac{1+aa}{a}\int \frac{\partial \phi \cos. i\phi}{\Delta^3}$$

$$-\int \frac{\partial \phi \cos. (i-1)\phi}{\Delta^3} - \frac{\pi a^{i-1}[i+1-(i-1)aa]}{(1-aa)^3}$$

§. 56. Sumamus nunc statim $i = 1$, atque istud lemma nobis praebet

$$\int\frac{\partial\Phi\cos.2\Phi}{\Delta^3} = \frac{1+aa}{2a}\int\frac{\partial\Phi\cos.\Phi}{\Delta^3} - \int\frac{\partial\Phi}{\Delta^3} - \frac{2\pi}{2(1-aa)^3};$$

hic jam bini valores jam inventi substituantur, reperieturque

$$\int\frac{\partial\Phi\cos.2\Phi}{\Delta^3} = \frac{1+aa}{a}\cdot\frac{\pi a(3+3aa)}{(1-aa)^5} - \frac{\pi(1+4aa+a^4)}{(1-aa)^5}$$
$$- \frac{\pi(1-aa)^2}{(1-aa)^5} = \frac{\pi aa(6)}{(1-aa)^5};$$

sumto $i = 2$, erit

$$\int\frac{\partial\Phi\cos.3\Phi}{\Delta^3} = \frac{\pi a^3(10-5aa+a^4)}{(1-aa)^5};$$

sumto $i = 3$, nanciscimur

$$\int\frac{\partial\Phi\cos.4\Phi}{\Delta^3} = \frac{\pi a^4(15-12aa+3a^4)}{(1-aa)^5};$$

sumto $i = 4$, prodit

$$\int\frac{\partial\Phi\cos.5\Phi}{\Delta^3} = \frac{\pi a^5(21-21aa+6a^4)}{(1-aa)^5};$$

posito $i = 5$, erit

$$\int\frac{\partial\Phi\cos.6\Phi}{\Delta^3} = \frac{\pi a^6(28-32aa+10a^4)}{(1-aa)^5};$$

et in genere

$$\int\frac{\partial\Phi\cos.i\Phi}{\Delta^3} = \pi a^i\left[\frac{i(i+3)+2}{2} - 2(ii-4)aa + \left[\frac{i(i-3)+2}{2}\right]a^4\right],$$

quae forma facile transformatur in hanc

$$\int\frac{\partial\Phi\cos.i\Phi}{\Delta^3} = \frac{\pi a^i}{(1-aa)^5}\left[\frac{(i+1)(i+2)}{2} - (i+2)(i-2)aa + \frac{(i-1)(i-2)}{2}a^4\right].$$

§. 57. Hoc modo procedere liceret ad sequentes formulas, in quibus denominator est Δ^4, Δ^5, Δ^6, etc. verum integralium formae ita continuo magis fierent complicatae, ut vix ullus ordo in iis observari posset, quamobrem aliam viam inire conveniet, qua numerum i pro dato assumimus, et continuo a minoribus ad majores numeros n procedemus. Primo igitur sumamus $i = 0$, et investigemus valorem integralem formulae $\int \frac{\partial \Phi}{\Delta^{n+1}}$.

Integratio formulae.

$$\int \frac{\partial \Phi}{\Delta^{n+1}} \begin{bmatrix} a\,\Phi = 0 \\ ad\,\Phi = 180 \end{bmatrix}$$

existente $\Delta = 1 + aa - 2\,a\cos.\Phi$.

§. 58. Ex praecedentibus colligere licet, quemlibet casum exponentis $n+1$ a duobus praecedentibus pendere, ita ut sit sub terminis integrationis praescriptis

$$\int \frac{\partial \Phi}{\Delta^{n+1}} = \alpha \int \frac{\partial \Phi}{\Delta^{n}} + \beta \int \frac{\partial \Phi}{\Delta^{n-1}};$$

ubi totum negotium eo redit, ut coefficientes α et β rite determinentur: hunc in finem statuamus in genere esse

$$\int \frac{\partial \Phi}{\Delta^{n+1}} = \alpha \int \frac{\partial \Phi}{\Delta^{n}} + \beta \int \frac{\partial \Phi}{\Delta^{n-1}} + \gamma\, \frac{\sin.\Phi}{\Delta^{n}},$$

quippe qui postremus terminus pro utroque integrationis termino evanescit.

§. 59. Differentietur nunc ista aequatio, et facta divisione per $\partial \Phi$, orietur sequens aequatio

$$\frac{1}{\Delta^{n+1}} = \frac{\alpha}{\Delta^{n}} + \frac{\beta}{\Delta^{n-1}} + \frac{\gamma\cos.\Phi(1+aa-2a\cos.\Phi) - 2\gamma\, an\sin.\Phi^2}{\Delta^{n+1}},$$

haecque aequatio multiplicata per Δ^{n+1} abibit in hanc formam

$$1 = \alpha(1+aa-2a\cos.\Phi) + \beta(1+aa)^2 - 2\beta\, a\cos.\Phi(1+aa)$$
$$+ 4\beta\, aa\cos.\Phi^2 + \gamma\cos.\Phi(1+aa-2a\cos.\Phi) - 2\gamma\, an\sin.\Phi^2.$$

Cum nunc sit

$$2\cos.\Phi^2 = 1 + \cos.2\Phi \text{ et } 2\sin.\Phi^2 = 1 - \cos.2\Phi,$$

hac reductione adhibita pervenietur ad sequentem aequationem

$$\begin{array}{lll} 1=\alpha(1+aa) & -2\alpha a\cos.\Phi & +2\beta aa\cos.2\Phi \\ +\beta(1+aa)^2 & -4\beta a(1+aa)\cos.\Phi & -\gamma a\cos.2\Phi \\ +2\beta aa & +\gamma(1+aa)\cos.\Phi & +\gamma na\cos.2\Phi \\ -\gamma a & & \\ -\gamma na. & & \end{array}$$

§. 60. Ut nunc hanc aequationem resolvamus, necesse est, ut tam termini involventes cos. Φ, quam cos. 2Φ, seorsim ad nihilum redigantur; unde ex postremo termino deducimus

$$2\beta aa-\gamma a+\gamma na=0;$$

ideoque

$$\beta=\frac{\gamma(1-n)}{2a}=-\frac{\gamma(n-1)}{2a},$$

qui valor in terminis cos. Φ affectis substitutus perducit ad hanc aequationem

$$-2\alpha a+2\gamma(n-1)(1+aa)+\gamma(1+aa)=0,$$

unde fit

$$2\alpha a=2\gamma n(1+aa)-\gamma(1+aa);$$

ideoque erit

$$\alpha=\frac{\gamma(1+aa)(2n-1)}{2a}.$$

Jam hic valores loco α et β inventi substituantur in prima parte, atque deducemur ad hanc aequationem

$$1=\frac{\gamma n(1+aa)^2}{a}-\frac{\gamma(n-1)(1+aa)^2}{2a}-\gamma a(n-1)-\gamma a-\gamma na, \text{ sive}$$

$$2a=2\gamma n(1+aa)^2-\gamma(n-1)(1+aa)^2-2\gamma aa(n-1)-2\gamma aa-2\gamma naa,$$

$$\text{vel } 2a=\gamma(n+1)(1+aa)^2-4\gamma naa,$$

unde fit

$$\gamma=\frac{2a}{n(1-aa)^2}.$$

§. 61. Invento jam isto valore γ, hinc eliciemus

$$\alpha = \frac{(2n-1)(1+aa)}{n(1-aa)^2} \text{ et } \beta = \frac{-(n-1)}{n(1-aa)^2},$$

hincque per $n(1-aa)^2$ multiplicando, adipiscimur

$$n(1-aa)^2 \int \frac{a\,\Phi}{\varDelta^{n+1}} = (2n-1)(1+aa)\int \frac{\partial\,\Phi}{\varDelta^n} - (n-1)\int \frac{\partial\,\Phi}{\varDelta^{n-1}},$$

cujus beneficio ex cognitis jam duobus casibus assignari poterit casus sequens.

§. 62. Jam ante autem invenimus esse $\int \frac{\partial\,\Phi}{\varDelta} = \frac{\pi}{1-aa}$. Pro sequentibus vero ponamus

$$\int \frac{\partial\,\Phi}{\varDelta^2} = \frac{\pi\,A}{(1-aa)^3};\ \int \frac{\partial\,\Phi}{\varDelta^3} = \frac{\pi\,B}{(1-aa)^5};\ \int \frac{\partial\,\Phi}{\varDelta^4} = \frac{\pi\,C}{(1-aa)^7};$$

$$\int \frac{\partial\,\Phi}{\varDelta^5} = \frac{\pi\,D}{(1-aa)^9};\ \int \frac{\partial\,\Phi}{\varDelta^6} = \frac{\pi\,E}{(1-aa)^{11}};\ \text{etc.}$$

Ubi jam ante invenimus $A = 1 + aa$ et $B = 1 + 4aa + a^4$, unde sequentes valores omnes C, D, E, etc. ope reductionis inventae definiri poterunt.

§. 63. Introducamus ergo istos valores, atque sequentes nanciscemur aequationes

I. $A = 1 + aa$,

II. $2B = 3(1+aa)A - (1-aa)^2$,

III. $3C = 5(1+aa)B - 2(1-aa)^2 A$,

IV. $4D = 7(1+aa)C - 3(1-aa)^2 B$,

V. $5E = 9(1+aa)D - 4(1-aa)^2 C$,

VI. $6F = 11(1+aa)E - 5(1-aa)^2 D$,

VII. $7G = 13(1+aa)F - 6(1-aa)^2 E$,

VIII. $8H = 15(1+aa)G - 7(1-aa)^2 F$,

etc.

§. 64 Harum aequationum prima statim dat valorem ante inventum $A = 1 + aa$; secunda vero praebet

$$2\,B = \left\{ \begin{array}{l} 3 + 6\,aa + 3\,a^4 \\ -1 + 2\,aa + a^4 \end{array} \right.$$

unde fit

$$B = 1 + 4\,aa + a^4.$$

Deinde vero tertia aequatio praebet

$$3\,C = \left\{ \begin{array}{l} 5 + 25\,aa + 25\,a^4 + 5\,a^6 \\ -2 + 2\,aa + 2\,a^4 - 2\,a^6 \end{array} \right.$$

unde elicitur

$$C = 1 + 9\,aa + 9\,a^4 + a^6.$$

Porro quarta aequatio

$$4\,D = \left\{ \begin{array}{l} 7 + 70\,aa + 126\,a^4 + 70\,a^6 + 7\,a^8 \\ -3 - 6\,aa + 18\,a^4 - 6\,a^6 - 3\,a^8 \end{array} \right.$$

unde colligitur

$$D = 1 + 16\,aa + 36\,a^4 + 16\,a^6 + a^8.$$

Simili modo ex aequatione quinta colligimus

$$5\,E = \left\{ \begin{array}{l} 9 + 153\,aa + 468\,a^4 + 468\,a^6 + 153\,a^8 + 9\,a^{10} \\ -4 - 28\,aa + 32\,a^4 + 32\,a^6 - 28\,a^8 - 4\,a^{10} \end{array} \right.$$

unde colligitur

$$E = 1 + 25\,aa + 100\,a^4 + 100\,a^6 + 25\,a^4 + a^{10}.$$

Evolvamus etiam sextam aequationem quae praebet

$$6\,F = \left\{ \begin{array}{l} 11 + 286\,aa + 1375\,a^4 + 2200\,a^6 + 1375\,a^8 + 286\,a^{10} + 11\,a^{12} \\ -5 - 70\,aa - 25\,a^4 + 200\,a^6 - 25\,a^8 - 70\,a^{10} - 5\,a^{12}, \end{array} \right.$$

hincque concluditur

$$F = 1 + 36\,aa + 225\,a^4 + 400\,a^6 + 225\,a^8 + 36\,a^{10} + a^{12}$$

§. 65. Hic non sine admiratione deprehendimus, omnes coefficientes harum formarum esse numeros quadratos, quorum radices occurrunt in potestatibus respondentibus binomii $1 + aa$, sicque pro littera sequente habebimus

$$G = 1 + 7^2 aa + 21^2 a^4 + 35^2 a^6 + 35^2 a^8 + 21^2 a^{10} + 7^2 a^{12} + a^{14},$$

quae littera respondet formulae integrali $\int \frac{\partial \Phi}{\Delta^{7+1}}$, ita ut hic sit $n = 7$. Quodsi ergo formae generalis $\int \frac{\partial \Phi}{\Delta^{n+1}}$ integrale statuamus $= \frac{\pi V}{(1 - aa)^{in+1}}$, erit valor litterae

$$V = 1 + (\tfrac{n}{1})^2 aa + (\tfrac{n}{2})^2 a^4 + (\tfrac{n}{3})^2 a^6 + (\tfrac{n}{4})^2 a^8 + (\tfrac{n}{5})^2 a^{10} + \text{etc.}$$

adhibitis scilicet characteribus, quibus coefficientes potestatum binomii designare consuevimus, dum scilicet est

$$(\tfrac{n}{1}) = n;\ (\tfrac{n}{2}) = \frac{n}{1} \cdot \frac{n-1}{2};\ (\tfrac{n}{3}) = \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}\ \text{etc.}$$

§. 66. Haec quidem conclusio tantum per inductionem quasi conjectura est deducta; vix enim quisquam reperietur, cui ista conjectura suspecta videatur, quamquam rigorosa demonstratione nondum sit corroborata; casu enim fortuito neutiquam evenire certe potest, ut omnes istos coefficientes prodierint numeri quadrati, atque adeo ipsorum coefficientium qui in evolutione potestatis $(1 + aa)^n$ occurrunt, interim tamen deinceps vidi pro hac veritate solidam demonstrationem adornari posse.

§. 67. Hac igitur lege stabilita, valores litterarum A, B, C, D etc., quas in expressiones integralium induximus, sequenti modo se habebunt

$$A = 1^2 + 1^2 aa,$$
$$B = 1^2 + 2^2 aa + 1^2 a^4,$$
$$C = 1^2 + 3^2 aa + 3^2 a^4 + 1^2 a^6,$$

$$D = 1^2 + 4^2 aa + 6^2 a^4 + 4^2 a^6 + 1^2 a^8,$$
$$E = 1^2 + 5^2 aa + 10^2 a^4 + 10^2 a^6 + 5^2 a^8 + 1^2 a^8,$$
$$F = 1^2 + 6^2 aa + 15^2 a^4 + 20^2 a^6 + 15^2 a^8 + 6^2 a^{10} + 1^2 a^{12},$$
$$G = 1^2 + 7^2 aa + 21^2 a^4 + 35^2 a^6 + 35^2 a^8 + 21^2 a^{10} + 7^2 a^{12} + 1^2 a 14,$$

etc. etc.

Integratio formulae generalis

$$\int \frac{\partial \phi \cos. i \phi}{\Delta^{n+1}} \left[\begin{array}{l} \text{a } \phi = 0 \\ \text{ad } \phi = 180 \end{array} \right]$$

existente $\Delta = 1 + aa - 2a \cos. \phi$.

§. 68. Haec formula generalis perinde tractari potest ac praecedens, dum valor integralis cujusque casus etiam a duobus casibus praecedentibus pendet, ita ut ponere queamus

$$\int \frac{\partial \phi \cos. i \phi}{\Delta^{n+1}} = \alpha \int \frac{\partial \phi \cos. i \phi}{\Delta^{n}} + \beta \int \frac{\partial \phi \cos. i \phi}{\Delta^{n-1}},$$

quatenus scilicet integralia ad binos terminos integrationis stabilitos referuntur; quia autem necesse est, ut aequationem generalem ob ista conditione liberam constituamus, aliquot membra adjungi oportet, quae pro utroque termino evanescant, neque enim hic sufficit, ut ante unicum terminum adjunxisse, verum adeo ternos hujusmodi terminos adjungi debebunt, cujus ratio mox ex ipso calculo elucebit; hanc ob rem constituamus sequentem aequationem

$$\int \frac{\partial \phi \cos. i \phi}{\Delta^{n+1}} = \alpha \int \frac{\partial \phi \cos. i \phi}{\Delta^{n}} + \beta \int \frac{\partial \phi \cos. i \phi}{\Delta^{n-1}}$$
$$+ \gamma \frac{\sin i \phi}{\Delta^n} + \delta \frac{\sin. (i-1) \phi}{\Delta^n} + \varepsilon \frac{\sin. (i+1) \phi}{\Delta^n},$$

quae postrema membra, quoniam i est numerus integer, pro utroque termino integrationis evanescunt.

§. 69. Differentietur igitur nunc ista aequatio, ac posito brevitatis gratia $1 + aa = b$, ut sit $\Delta = b - 2a\cos.\Phi$, negligantur denominatores, qui erunt Δ^{n+1} una cum elemento $\partial\Phi$. Primo notetur esse

$$\Delta\cos.i\Phi = b\cos.i\Phi - a\cos.(i-1)\Phi - a\cos.(i+1)\Phi,$$

tum vero ob

$$\Delta^2 = bb - 4ab\cos.\Phi + 4aa\cos.\Phi^2 = 2aa + bb$$
$$- 4ab\cos.\Phi + 2aa\cos.2\Phi, \text{ erit}$$
$$\Delta^2\cos.i\Phi = (bb + 2aa)\cos.i\Phi - 2ab\cos.(i-1)\Phi$$
$$- 2ab\cos.(i+1)\Phi + aa\cos.(i-2)\Phi$$
$$+ aa\cos.(i+2)\Phi.$$

Deinde vero habebitur

$$\partial.\frac{\sin.i\Phi}{\Delta^n} = i\Delta\cos.i\Phi - 2na\sin.i\Phi\sin.\Phi = ib\cos.i\Phi$$
$$+ ia\cos.(i-1)\Phi - ia\cos.(i+1)\Phi$$
$$- na\cos.(i-1)\Phi + na\cos.(i+1)\Phi.$$

Simili modo erit

$$\partial.\frac{\sin.(i-1)\Phi}{\Delta^n} = (i-1)b\cos.(i-1)\Phi - (i-1)a\cos.(i-2)\Phi$$
$$- (i-1)a\cos.i\Phi - na\cos.(i-2)\Phi + na\cos.i\Phi),$$

ac denique

$$\partial.\frac{\sin.(i+1)\Phi}{\Delta^n} = (i+1)b\cos.(i+1)\Phi - (i+1)a\cos.i\Phi$$
$$- (i+1)a\cos.(i+2)\Phi - na\cos.i\Phi + na\cos.(i+2)\Phi.$$

§. 70. Hic igitur occurrunt quinque anguli scilicet

$$i\Phi,\ (i-1)\Phi,\ (i+1)\Phi,\ (i-2)\Phi \text{ et } (i+2)\Phi,$$

unde patet ratio, cur terni termini absoluti sint supra adjuncti; diffeerentiale ergo facta evolutione singulorum terminorum, per quinque columnas sequenti modo repraesentetur, ita ut membrum sinistrum, quod est $\cos. i\,\Phi$, aequetur sequenti expressioni

$\cos. i\,\Phi$	$\cos.(i-1)\Phi$	$\cos.(i+1)\Phi$	$\cos.(i-2)\Phi$	$\cos.(i+2)\Phi$
$+ab$	$-aa$	$-aa$		
$+\beta(bb+2aa)$	$-2\beta ab$	$-2\beta ab$	$+\beta aa$	$+\beta aa$
$+\gamma ib$	$-\gamma ia$	$-\gamma ia$		
	$-\gamma na$	$+\gamma na$		
$-\delta(i-1)a$	$+\delta(i-1)b$	$+\varepsilon(i+1)b$	$-\delta(i-1)a$	$-\varepsilon(i+1)a$
$+\delta na$			$-\delta na$	$+\varepsilon na$
$-\varepsilon(i+1)a$				
$-\varepsilon na$				

§. 71. Hic igitur omnes quatuor posteriores columnae ad nihilum redigi debent, propterea quod sola prima columna membro sinistro aequari potest; incipiamus igitur a binis columnis ultimis, unde deducimus

$$\delta = \frac{\beta a}{i+n-1} \text{ et } \varepsilon = \frac{\beta a}{i-n+1}.$$

His valoribus introductis, pro secunda columna erit

$$-2\beta ab+\delta(i-1)b = \frac{\beta ab(1-i-2n)}{i+n-1} = -\frac{\beta ab(i+2n-1)}{i+n-1}.$$

Pro tertia vero columna erit

$$-2\beta ab+\varepsilon(i+1)b = -\frac{\beta ab(i-2n+1)}{i-n+1};$$

unde haec binae columnae nobis praebent has duas aequationes

$$-aa-\gamma(i+n)a-\frac{\beta ab(i+2n-1)}{i+n-1} = 0,$$

$$-aa-\gamma(i-n)a-\frac{\beta ab(i-2n+1)}{i-n+1} = 0$$

§. 72. Harum duarum aequationum subtrahatur posterior a priore, ac prodibit

$$-2\gamma na-\frac{2\beta inab}{ii-(n-1)^2} = 0,$$

unde colligimus

$$\gamma = -\frac{\beta i b}{ii-(n-1)^2}.$$

Atque hinc porro ex secunda deduci potest valor ipsius α, cum sit

$$\alpha a = -\gamma(i+n)a - \frac{\beta a b(i+2n-1)}{i+n-1},$$

erit

$$\alpha = \frac{\beta i(i+n)b}{ii-(n-1)^2} - \frac{\beta(i+2n-1)b}{i+n-1} = \frac{\beta(2nn-3n+1)b}{ii-(n-1)^2}$$
$$= \frac{\beta(n-1)(2n-1)b}{ii-(n-1)^2}.$$

§. 73. Hi jam valores substituantur in prima columna, atque orietur sequens aequatio

$$\left.\begin{matrix} \frac{\beta(n-1)(2n-1)bb}{ii-(n-1)^2} & +2\beta aa \\ +\beta bb & -\frac{\beta(i-n-1)aa}{i+n-1} \\ -\frac{\beta iibb}{ii-(n-1)^2} & -\frac{\beta(i+n+1)aa}{-i-n+1} \end{matrix}\right\} = 1.$$

Multiplicando igitur per $ii-(n-1)^2$, prodibit haec aequatio

$$ii-(n-1)^2 = 2\beta aa[ii-(n-1)^2] + \beta bb(n-1)(2n-1)$$
$$-\beta aa(i-n-1)(i-n+1) + \beta bb[ii-(n-1)^2]$$
$$-\beta aa(i+n+1)(i+n-1) - \beta iibb.$$

Facta autem reductione, terminus βaa multiplicabitur per

$$2[ii-(n-1)^2] - (i-n)^2 + 1 - (i+n)^2 + 1,$$

sive per $-4n(n-1)$; at vero βbb multiplicabitur per

$$(n-1)(2n-1) + ii - (n-1)^2 - ii,$$

sive per $n(n-1)$, sicque erit

$$ii(n-1)^2 = -4\beta n(n-1)aa + \beta n(n-1)bb$$
$$= \beta n(n-1)(bb-4aa).$$

Cum igitur posuerimus $b = 1 + aa$, erit

$$bb - 4aa = (1 - aa)^2,$$

consequenter hinc elicimus

$$\beta = \frac{ii - (n-1)^2}{n(n-1)(1-aa)^2}.$$

§. 74. Invento jam valore litterae β, ex eo deducimus valorem $\alpha = \frac{(2n-1)b}{n(1-aa)^2}$: valores autem litterarum γ, δ, et ε non amplius in censum veniunt, et reductio quam quaerimus erit

$$\int \frac{\partial \Phi \cos. i\Phi}{\Delta^{n+1}} = \alpha \int \frac{\partial \Phi \cos. \Phi}{\Delta^n} + \beta \int \frac{\partial \Phi \cos. i\Phi}{\Delta^{n-1}},$$

sive sublatis fractionibus habebitur ista aequatio

$$n(n-1)(1-aa)^2 \int \frac{\partial \Phi \cos. i\Phi}{\Delta^{n+1}} = (n-1)(2n-1)(1+aa) \int \frac{\partial \Phi \cos. i\Phi}{\Delta^n}$$
$$+ [ii - (n-1)^2] \int \frac{\partial \Phi \cos. i\Phi}{\Delta^{n-1}},$$

quae aequatio casu $i = 0$ redit ad reductionem praecedentis sectionis.

§. 75. Inventa hac reductione generali, pro ejus applicatione cum sit

$$\int \frac{\partial \Phi \cos. i\Phi}{\Delta} = \frac{\pi a^i}{1 - aa}, \text{ ubi } n = 0,$$

ponamus pro sequentibus

$$\int \frac{\partial \Phi \cos. i\Phi}{\Delta^2} = \frac{\pi a^i}{(1-aa)^3} \text{ A, ubi } n = 1$$

$$\int \frac{\partial \Phi \cos. i\Phi}{\Delta^3} = \frac{\pi a^i}{(1-aa)^5} \text{ B, ubi } n = 2$$

$$\int \frac{\partial \Phi \cos. i\Phi}{\Delta^4} = \frac{\pi a^i}{(1-aa)^7} \text{ C, ubi } n = 3$$

$$\int \frac{\partial \phi \cos. i \phi}{\Delta^5} = \frac{\pi a^i}{(1-aa)^9} \text{ D, ubi } n = 4$$

$$\int \frac{\partial \phi \cos. i \phi}{\Delta^6} = \frac{\pi a^i}{(1-aa)^{11}} \text{ E, ubi } n = 5,$$

atque adeo in genere sit

$$\int \frac{\partial \phi \cos. i \phi}{\Delta^{n+1}} = \frac{\pi a^i}{(1-aa)^{2n+1}} \text{ V}:$$

supra autem jam i invenimus esse

$$\text{A} = i + 1 - (i-1)\, a a,$$

sive terminos positive repraesentando

$$\text{A} = 1 + i + (1-i)\, a a.$$

§. 76. Quodsi in reductione nostra inventa poneremus $n = 1$, ea daret $i\, i \int \partial \phi \cos. i \phi = 0$, quod primo verum est casu $i = 0$, tum vero ob $\int \partial \phi \cos. i \phi = \frac{1}{i} \sin. i \phi = 0$, quod quidem per se patet. Incipiamus igitur a casu $n = 2$, et procedendo per sequentes valores $n = 3$, $n = 4$, $n = 5$, etc. nanciscemur sequentes aequationes

I. Si $n = 2$, erit
$$2 \cdot 1\, \text{B} = 1 \cdot 3\, (1 + aa)\, \text{A} + (ii - 1)(1 - aa)^2.$$

II. Si $n = 3$, erit
$$3 \cdot 2\, \text{C} = 2 \cdot 5\, (1 + aa)\, \text{B} + (ii - 4)(1 - aa)^2\, \text{A}.$$

III. Si $n = 4$, erit
$$4 \cdot 3\, \text{D} = 3 \cdot 7\, (1 + aa)\, \text{C} + (ii - 9)(1 - aa)^2\, \text{B}.$$

IV. Si $n = 5$, erit
$$5 \cdot 4\, \text{E} = 4 \cdot 9\, (1 + aa)\, \text{D} + (ii - 16)(1 - aa)^2\, \text{C}.$$

V. Si $n = 6$, erit
$$6 \cdot 5\, \text{F} = 5 \cdot 11\, (1 + aa)\, \text{E} + (ii - 25)(1 - aa)^2\, \text{D}.$$

etc. etc.

§. 77. Cum igitur sit

$$A = 1 + i + (1 - i)\,a\,a,$$

pro prima aequatione erit

$$(1 + a\,a)\,A = 1 + i + 2\,a\,a + (1 - i)\,a^4,$$

hujus triplo addi oportet

$$(i\,i - 1)\,(1 - a\,a)^2 = i\,i - 1 - 2\,(i\,i - 1)\,a\,a + (i\,i - 1)\,a^4,$$

unde oritur primo terminus absolutus $= (2 + i)\,(1 + i)$, deinde coefficiens ipsius $a\,a$ erit $8 - 2\,i\,i$, coefficiens vero ipsius a^4 erit $(2 - i)\,(1 - i)$, unde concludimus litteram

$$B = \frac{(2+i)}{1.}\,\frac{(1+i)}{2} + (2+i)\,(2-i)\,a\,a + \frac{(2-i)}{1.}\,\frac{(1-i)}{2}\,a^4.$$

§. 78. Ista forma nos manuducit ad coefficientes potestatum binomii, quos ut jam monimus per characteres peculiares repraesentamus, sicque per tales characteres erit

$$A = \left(\frac{1+i}{1}\right) + \left(\frac{1-i}{1}\right)a\,a, \text{ tum vero}$$

$$B = \left(\frac{2+i}{2}\right) + \left(\frac{2+i}{1}\right)\left(\frac{2-i}{1}\right)a\,a + \left(\frac{2-i}{2}\right)a^4$$

Videamus autem, quomodo haec lex in sequentibus valoribus se sit habitura.

§. 79. Evolvamus igitur aequationem secundam, pro qua sequentes duas multiplicationes institui oportet

$$10\left[\frac{2+3\,i+i\,i}{2} + (4 - i\,i)\,a\,a + \frac{2-3\,i+i\,i}{2}\,a^4\right] \text{ per } 1 + a\,a,$$

ultimum autem membrum postulat hanc multiplicationem

$$(i\,i - 4)\,(1 - 2\,a\,a + a^4) \text{ per } 1 + i + (1 - i)\,a\,a;$$

unde primo oritur iste terminus absolutus

$$10 + 15\,i + 5\,i\,i + (i\,i - 4)\,(1 + i),$$

quae reducitur ad hanc formam $(2 + i)\,(1 + i)\,(3 + i)$. Pro termino autem $a\,a$ erit

$$40 - 10ii + 5(2+i)(1+i) + (ii-4)[-2(1+i) + 1 - i]$$
$$= (4 - ii)(11 + 3i) + 5(2+i)(1+i),$$

quae expressio reducitur ad

$$(2+i)(27 - 3ii) = 3(2+i)(3+i)(3-i).$$

Porro coefficiens ipsius a^4 erit

$$(2-i)(27 - 3ii) = 3(2-i)(3+i)(3-i).$$

Denique coefficiens ipsius a^6 erit $(2-i)(1-i)(3-i)$.

§. 80. Calculo ergo hoc peracto habebimus

$$3.2\,C = (3+i)(2+i)(1+i) + 3(3+i)(2+i)(3-i)aa$$
$$+ 3(3+i)(2-i)(3-i)a^4 + (3-i)(2-i)(1-i)a^6,$$

quae forma commode redigitur ad istam per characteres coefficientium binomii

$$C = (\tfrac{3+i}{3}) + (\tfrac{3+i}{2})(\tfrac{3-i}{1})aa + (\tfrac{3+i}{1})(\tfrac{3-i}{2})a^4 + (\tfrac{3-i}{3})a^6.$$

Hic ordo maxime confirmat conjecturam ex casibus praecedentibus deductam, neque dubium ullum esse potest, quin sequentes litterae istos sortiantur valores

$$D = (\tfrac{4+i}{4}) + (\tfrac{4+i}{3})(\tfrac{4-i}{1})aa + (\tfrac{4+i}{2})(\tfrac{4-i}{2})a^4$$
$$+ (\tfrac{4+i}{1})(\tfrac{4-i}{3})a^6 + (\tfrac{4-i}{4})a^8.$$

$$E = (\tfrac{5+i}{5}) + (\tfrac{5+i}{4})(\tfrac{5-i}{1})aa + (\tfrac{5+i}{3})(\tfrac{5-i}{2})a^4$$
$$+ (\tfrac{5+i}{2})(\tfrac{5-i}{3})a^6 + (\tfrac{5+i}{1})(\tfrac{5-i}{4})a^8 + (\tfrac{5-i}{5})a^{10}.$$

etc. etc.

Interim tamen fatendum est, hunc ordinem egregium tantum per conjecturam se nobis obtulisse; cujus ergo demonstratio rigorosa adhuc desideratur.

§. 81. Cum igitur supra ingenere posuerimus

$$\int \frac{\partial \phi \cos. i \phi}{\Delta^{n+1}} \left[\begin{array}{l} \text{a } \phi = 0 \\ \text{ad } \phi = 180 \end{array}\right] = \frac{\pi a^i}{(1 - a a)^{2n+1}} V,$$

erit nunc

$$V = \left(\frac{n+i}{n}\right) + \left(\frac{n+i}{n-1}\right)\left(\frac{n-i}{1}\right) a a + \left(\frac{n+i}{n-2}\right)\left(\frac{n-i}{2}\right) a^4$$
$$+ \left(\frac{n+i}{n-3}\right)\left(\frac{n-i}{3}\right) a^6 + \left(\frac{n+i}{n-4}\right)\left(\frac{n-i}{4}\right) a^8 + \text{etc.}$$

unde sponte deducitur forma in articulo praecedenti conclusa, ubi erat $i = 0$. Pro hoc enim casu erit

$$V = \left(\frac{n}{n}\right) + \left(\frac{n}{n-1}\right)\left(\frac{n}{1}\right) a a + \left(\frac{n}{n-2}\right)\left(\frac{n}{2}\right) a^4 + \left(\frac{n}{n-3}\right)\left(\frac{n}{3}\right) a^6 + \text{etc.}$$

Cum autem in hujusmodi characteribus perpetuo sit $\left(\frac{n}{p}\right) = \left(\frac{n}{n-p}\right)$, erit prorsus uti supra conjectavimus

$$V = \left(\frac{n}{0}\right) + \left(\frac{n}{1}\right)^2 a a + \left(\frac{n}{2}\right)^2 a^4 + \left(\frac{n}{3}\right)^2 a^6 + \left(\frac{n}{4}\right)^2 a^8 + \text{etc.}$$

Hinc igitur operae pretium erit sequens theorema constituere.

Theorema generale.

§. 82. Si formula integralis

$$\int \frac{\partial \phi \cos. i \phi}{(1 + a a - 2 a \cos. \phi)^{n+1}},$$

a termino $\phi = 0$ usque ad terminum $\phi = 180^{\circ}$ extendatur, valor integralis semper habebit talem formam

$$\frac{\pi a^i}{(1 - a a)^{2n+1}} V, \text{ existente}$$

$$V = \left(\frac{n+i}{i}\right) + \left(\frac{n+i}{i+1}\right)\left(\frac{n-i}{1}\right) a a + \left(\frac{n+i}{i+2}\right)\left(\frac{n-i}{2}\right) a^4$$
$$+ \left(\frac{n+i}{i+3}\right)\left(\frac{n-i}{3}\right) a^6 + \left(\frac{n+i}{i+4}\right)\left(\frac{n-i}{4}\right) a^8 + \text{etc.}$$

dummodo fuerit i numerus integer, atque adeo tam positivus quam negativus; quandoquidem etiam posteriori casu ista forma veritati consentanea deprehenditur, ita ut ista expressio latius pateat, quam omnes casus speciales junctim sumti, unde eam per conjecturam conclusimus; namque in omnibus casibus specialibus littera i necessario denotabat numeros integros tantum positivos.

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4) Demonstratio Theorematis insignis per conjecturam eruti, circa integrationem formulae

$$\int \frac{\partial \phi \cos. i\phi}{(1 + aa - 2a\cos.\phi)^{n+1}}$$

M. S. Academiae exhib. die 10 *Septembris* 1778.

§. 83. Cum nuper hanc formulam integralem tractassem, ac potissimum in ejus valorem inquisivissem, quem accipit, si integrale a termino $\phi = 0$ ad terminum $\phi = 180^\circ$ usque extendatur; ex pluribus casibus, quos evolvere licuit, conclusi ejus integrale in genere ita expressum iri

$$\frac{\pi a^i}{(1 - aa)^{2n+1}} V,$$

ubi V denotat summam hujus seriei

$$V = \left(\frac{n-i}{0}\right)\left(\frac{n+i}{i}\right) + \left(\frac{n-i}{1}\right)\left(\frac{n+i}{i+1}\right) a^2 + \left(\frac{n-i}{2}\right)\left(\frac{n+i}{i+2}\right) a^4 + \text{etc.}$$

Hic scilicet isti characteres clausulis inclusi designant coefficientes potestatis binomialis, dum statuimus

$$(1 + x)^m = 1 + \left(\frac{m}{1}\right) x + \left(\frac{m}{2}\right) x^2 + \left(\frac{m}{3}\right) x^3 + \left(\frac{m}{4}\right) x^4 + \text{etc.}$$

§. 84. Circa hanc autem formulam integralem ante omnia tenendum est, litteram i perpetuo significare numeros integros, quandoquidem in analysi constanter assumitur, casu $\phi = 180^\circ$ sem-