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De formulis differentialibus angularibus maxime irrationalibus, quas tamen per logarithmos et arcus circulares integrare licet

Leonhard Euler

SUPPLEMENTUM IV.

AD TOM. I. CAP. V.

DE

INTEGRATIONE FORMULARUM ANGULOS SINUSVE ANGULORUM IMPLICANTIUM.

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1) De formulis differentialibus angularibus maxime irrationalibus, quas tamen per logarithmos et arcus circulares integrare licet. *M. S. Academiae exhibit. die* 5. *Maii* 1777.

§. 1. Quae jam saepius sum commentatus de formulis differentialibus irrationalibus, quae nulla substitutione ad rationalitatem revocari possunt, nihilo vero minus integrationem per logarithmos et arcus circulares admittunt: etiam transferri possunt ad ejusmodi formulas angulares, quae sinus et cosinus cujuspiam anguli involvunt. Forma autem generalis hujusmodi differentialium, quae hoc modo tractari possunt, sequenti modo repraesentari potest: denotante ϕ angulum quemcunque, designet Φ functionem quamcunque rationalem ipsius tang. $n\phi$, atque inveni istam formulam

$$\frac{\Phi \partial \phi (f \sin. \lambda \phi + g \cos. \lambda \phi)}{\sqrt[n]{(a \sin. n\phi + b \cos. n\phi)^{\lambda}}}$$

semper per logarithmos et arcus circulares integrari posse, id quod a casibus simplicioribus inchoando in sequentibus problematibus ostendere constitui.

Problema 1.

§. 2. *Proposita formula differentiali* $\frac{\partial \Phi \cos. \Phi}{\sqrt[n]{\cos. n \Phi}}$, *ejus integrale per logarithmos et arcus circulares investigare.*

Solutio.

Quoniam mihi quidem alia adhuc via non patet istud praestandi, nisi per imaginaria procedendo, formulam $\sqrt{-1}$ littera i in posterum designabo, ita ut sit $ii = -1$, ideoque $\frac{1}{i} = -i$. Jam ante omnia in numeratore nostrae formulae loco $\cos. \Phi$ has duas partes substituamus

$$\tfrac{1}{2}(\cos. \Phi + i \sin. \Phi) + \tfrac{1}{2}(\cos. \Phi - i \sin. \Phi),$$

atque ipsam formulam propositam per duas hujusmodi partes repraesentemus, quae sint

$$\partial p = \frac{\partial \Phi (\cos. \Phi + i \sin. \Phi)}{\sqrt[n]{\cos. n \Phi}} \text{ et } \partial q = \frac{\partial \Phi (\cos. \Phi - i \sin. \Phi)}{\sqrt[n]{\cos. n \Phi}},$$

ita ut ipsa formula nostra proposita sit $\frac{1}{2}\partial p + \frac{1}{2}\partial q$, ideoque ejus integrale $\frac{p+q}{2}$.

§. 3. Nunc ambas istas partes seorsim sequenti modo tractemus. Pro formula scilicet priore

$$\partial p = \frac{\partial \Phi (\cos. \Phi + i \sin. \Phi)}{\sqrt[n]{\cos. n \Phi}} \text{ statuamus } \frac{\cos. \Phi + i \sin. \Phi}{\sqrt{\cos. n \Phi}} = x,$$

ut sit $\partial p = x \partial \Phi$, ac sumtis potestatibus exponentis n habebimus

$$x^n = \frac{(\cos.\,\Phi + i \sin.\,\Phi)^n}{\cos.\,n\,\Phi}.$$

Constat autem esse

$$(\cos.\,\Phi + i \sin.\,\Phi)^n = \cos.\,n\,\Phi + i \sin.\,n\,\Phi,$$

sicque erit $x^n = 1 + i\,\text{tang.}\,n\,\Phi$, unde colligitur

$$\text{tang.}\,n\,\Phi = \frac{x^n - 1}{i} = i\,(1 - x^n):$$

hinc cum posito in genere tang. $\omega = Z$, sit $\partial\,\omega = \frac{\partial Z}{1 + ZZ}$, erit pro nostro casu

$$n\,\partial\,\Phi = \frac{-\,n\,i\,x^{n-1}\,\partial\,x}{1 + i\,i - 2\,i\,i\,x^n + i\,i\,x^{2n}},$$

quae formula ob $i\,i = -1$ transmutatur in hanc

$$\partial\,\Phi = \frac{-\,i\,x^{n-1}\,\partial\,x}{2\,x^n - x^{2n}},$$

hincque ipsa formula

$$\partial\,p = x\,\partial\,\Phi = \frac{-\,i\,\partial\,x}{2 - x^n},$$

quae cum sit rationalis, ejus integratio nulli difficultati est subjecta.

§. 4. Quodsi jam simili modo pro altera formula

$$\partial\,q = \frac{\partial\,\Phi\,(\cos.\,\Phi - i \sin.\,\Phi)}{\sqrt[n]{\cos.\,n\,\Phi}}, \text{ statuatur } \frac{\cos.\,\Phi - i \sin.\,\Phi}{\sqrt[n]{\cos.\,n\,\Phi}} = y,$$

ut sit $\partial\,q = y\,\partial\,\Phi$, per similes operationes, quae a praecedentibus in hoc solo discrepabunt, quod littera i negative sit accipienda, resultabit ista transformatio $\partial\,q = \frac{i\,\partial\,y}{2 - y^n}$, quae cum priori prorsus sit similis, eadem integratione totum negotium conficietur, et pro ipso integrali quaesito habebimus

$$p + q = -i\int\frac{\partial x}{2 - x^n} + i\int\frac{\partial y}{2 - y^n}.$$

§. 5. Constat autem integralia talium formularum ex duplicis generis partibus, scilicet logarithmicis et arcubus circularibus constare, ita ut illarum forma generalis sit $f\,l\,(\alpha + \beta x + \gamma xx)$, harum vero g Arc. tang. $(\delta + \varepsilon x)$. Quare cum hic differentia inter binas formulas integrales similes occurrat, ex singulis partibus logarithmicis orietur talis forma $-\,i f\,l\,\frac{\alpha + \beta x + \gamma xx}{\alpha + \beta y + \gamma yy}$, ubi tam x quam y imaginaria involvit, hanc ob rem ponamus brevitatis gratia $x = r + is$ et $y = r - is$, ubi erit

$$r = \frac{\cos.\,\Phi}{\sqrt[n]{\cos.\,n\Phi}} \text{ et } s = \frac{\sin.\,\Phi}{\sqrt[n]{\cos.\,n\Phi}};$$

his igitur valoribus substitutis, quaelibet pars logarithmica erit

$$-\,i f\,l\,\frac{\alpha + \beta r + \gamma rr - \gamma ss + i(\beta s + 2\gamma rs)}{\alpha + \beta r + \gamma rr - \gamma ss - i(\beta s + 2\gamma rs)}.$$

§. 6. Loco hujus expressionis prolixioris scribamus brevitatis gratia $-\,i f\,l\,\frac{t + iu}{t - iu}$, ita ut sit

$$t = \alpha + \beta r + \gamma rr - \gamma ss \text{ et } u = \beta s + 2\gamma rs,$$

sicque etiam hi valores per angulum Φ innotescunt. Quoniam igitur jam saepius est demonstratum, esse

$$l\,\frac{t + u\sqrt{-1}}{t - u\sqrt{-1}} = 2\sqrt{-1}\,.\ \text{Arc. tang. } \frac{u}{t},$$

ista portio integralis erit $= + 2 f$ Arc. tang. $\frac{u}{t}$, quae ergo penitus est realis, dum imaginaria se mutuo sustulerunt, ita ut quaelibet portio logarithmica imaginaria producat arcum circularem realem.

§. 7. Simili modo conjungamus in genere binos arcus circulares per integrationem prodeuntes, qui ex forma assumta erunt

$$-ig\text{ Arc. tang. }(\delta+\varepsilon x)+ig\text{ Arc. tang. }(\delta+\varepsilon y),$$

quae forma ita in unum arcum contrahetur, qui erit

$$-ig\text{ Arc. tang. }\frac{\varepsilon(x-y)}{1+(\delta+\varepsilon x)(\delta+\varepsilon y)}:$$

quae introductis valoribus assumtis $x=r+is$ et $y=r-is$, induet hanc formam

$$-ig\text{ Arc. tang. }\frac{2i\varepsilon s}{1+\delta\delta+2\varepsilon\delta r+\varepsilon\varepsilon(rr+ss)}.$$

Cum igitur in genere sit

$$\text{Arc. tang. }v\sqrt{-1}=\frac{\sqrt{-1}}{2}l\frac{1+v}{1-v},$$

ista pars circularis transformabitur in sequentem logarithmum realem

$$\frac{g}{2}l\frac{1+\delta\delta+2\delta\varepsilon r+\varepsilon\varepsilon(rr+ss)+2\varepsilon s}{1+\delta\delta+2\delta\varepsilon r+\varepsilon\varepsilon(rr+ss)-2\varepsilon s};$$

hoc ergo modo sumendis omnium integralium partibus, tandem obtinebitur integrale quaesitum per meros logarithmos et arcus circulares realiter expressum.

Problema. 2.

§. 8. *Proposita formula differentiali* $\frac{\partial\Phi\sin.\Phi}{\sqrt[n]{\cos.n\Phi}}$, *ejus integrale per logarithmos et arcus circulares investigare.*

Solutio.

Hic loco sin. Φ scribatur haec forma duabus constans partibus

$$\frac{1}{2i}(\cos.\Phi+i\sin.\Phi)-\frac{1}{2i}(\cos.\Phi-i\sin.\Phi),$$

ac formula proposita resolvatur in has partes

$$\partial p = \frac{\partial \Phi (\cos. \Phi + i \sin. \Phi)}{\sqrt[n]{\cos. n \Phi}} \text{ et } \partial q = \frac{\partial \Phi (\cos. \Phi - i \sin. \Phi)}{\sqrt[n]{\cos. n \Phi}},$$

ita ut ipsa formula proposita jam fiat $\frac{\partial p - \partial q}{2 i}$, ideoque ipsum integrale quaesitum $\frac{p - q}{2 i}$.

§. 9. Quodsi jam rursus ut ante statuamus

$$\frac{\cos. \Phi + i \sin. \Phi}{\sqrt[n]{\cos. n \Phi}} = x \text{ et } \frac{\cos. \Phi - i \sin. \Phi}{\sqrt[n]{\cos. n \Phi}} = y,$$

reperietur ut supra

$$\partial p = - \frac{i \partial x}{2 - x^n} \text{ et } \partial q = \frac{i \partial y}{2 - y^n};$$

unde ergo fiet ipsum integrale quaesitum

$$\frac{p - q}{2 i} = - \tfrac{1}{2} \int \frac{\partial x}{2 - x^n} - \tfrac{1}{2} \int \frac{\partial y}{2 - y^n},$$

ubi coefficientes evaserunt reales.

§. 10. Consideremus nunc ex forma integrali utriusque partis quamlibet portionem logarithmicam, quae sit $f l (\alpha + \beta x + \gamma x x)$, hincque pro integrali quaesito ex utraque parte orietur

$$- \tfrac{1}{2} f l (\alpha + \beta x + \gamma x x) - \tfrac{1}{2} f l (\alpha + \beta y + \gamma y y).$$

Quodsi jam ut supra ponamus brevitatis gratia $x = r + i s$ et $y = r - i s$, tum vero

$$t = \alpha + \beta r + \gamma r r - \gamma s s \text{ et } u = \beta s + 2 \gamma r s,$$

hi ambo logarithmi evadunt

$$= - \tfrac{1}{2} f l (t + i u) - \tfrac{1}{2} f l (t - i u),$$

qui contrahuntur in $- \tfrac{1}{2} f l (t t + u u$, quae expressio jam est realis, neque ulla ulteriori reductione indiget.

§. 11. Eodem modo binae partes circulares ex integratione oriundae

$$-\tfrac{1}{2}g \text{ Arc. tang. } (\delta + \varepsilon x) - \tfrac{1}{2}g \text{ Arc. tang. } (\delta + \varepsilon y),$$

quae per r et s ita repraesentantur

$$-\tfrac{1}{2}g\,[\text{Arc. tang. } (\delta + \varepsilon r + i\varepsilon s) + \text{Arc. tang. } (\delta + \varepsilon r - i\varepsilon s)],$$

qui duo arcus ita in unum contrahuntur

$$-\tfrac{1}{2}g \text{ Arc. tang. } \frac{2\delta + 2\varepsilon r}{1-(\delta+\varepsilon r)^2 - \varepsilon\varepsilon ss},$$

quae expressio jam ultro prodiit realis.

Problema 3.

§. 12. *Proposita formula differentiali* $\frac{\partial\phi \cos.\lambda\phi}{\sqrt[n]{\cos.n\phi^\lambda}}$, *ejus integrale per logarithmos et arcus circulares investigare.*

Solutio.

Cum sit

$$\cos.\lambda\phi = \tfrac{1}{2}(\cos.\phi + i\sin.\phi)^\lambda + \tfrac{1}{2}(\cos.\phi - i\sin.\phi)^\lambda,$$

formula proposita in has duas partes discerpatur

$$\partial p = \frac{\partial\phi(\cos.\phi + i\sin.\phi)^\lambda}{\sqrt[n]{\cos.n\phi^\lambda}} \text{ et } \partial q = \frac{\partial\phi(\cos.\phi - i\sin.\phi)^\lambda}{\sqrt[n]{\cos.n\phi^\lambda}},$$

ita ut integrale quaesitum fiat $\frac{p+q}{2}$.

§. 13. Jam statuamus, ut ante fecimus,

$$\frac{\cos.\phi + i\sin.\phi}{\sqrt[n]{\cos.n\phi}} = x \text{ et } \frac{\cos.\phi - i\sin.\phi}{\sqrt[n]{\cos.n\phi}} = y,$$

quo facto fiet $\partial p = x^\lambda\partial\phi$ et $\partial q = y^\lambda\partial\phi$. Calculo autem ut supra expedito obtinebimus:

$$\partial \Phi = -\frac{i x^{n-1} \partial x}{2 x^n - x^{2n}}, \text{ hincque } \partial p = -\frac{i x^{\lambda-1} \partial x}{2 - x^n};$$

similique modo erit $\partial q = \frac{i y^{\lambda-1} \partial y}{2 - y^n}$, sicque totum integrale quaesitum erit

$$= -\tfrac{i}{2} \int \frac{x^{\lambda-1} \partial x}{2 - x^n} + \tfrac{i}{2} \int \frac{y^{\lambda-1} \partial y}{2 - y^n}.$$

§. 14. Quoniam haec duo integralia sibi sunt similia, ideoque similes partes tam logarithmicas quam circulares complectuntur, ex parte logarithmica, quae sit $f\, l\,(\alpha + \beta x + \gamma x x)$, ponendo ut supra $x = r + i s$ et $y = r - i s$, tum vero

$$t = \alpha + \beta r + \gamma r r - \gamma s s \text{ et } u = \beta s + 2 \gamma r s,$$

hinc primo ista pars logarithmica colligitur $-\, i f\, l \frac{t + i u}{t - i u}$, quae cum sit imaginaria reducitur ad hunc arcum circularem realem $= 2 f$ Arc. tang. $\frac{u}{t}$: simili modo si forma arcus circularis ex integratione oriunda fuerit $-\, g$ Arc. tang. $(\delta + \varepsilon x)$, ex partibus circularibus primo oritur sequens arcus imaginarius

$$-\, i g \text{ Arc. tang. } \frac{2 i \varepsilon s}{1 + \delta\delta + 2 \varepsilon \delta r + \varepsilon\varepsilon (r r + s s)},$$

qui denique ad hunc logarithmum realem revocatur

$$\frac{g}{2}\, l\, \frac{1 + \delta\delta + 2 \delta \varepsilon r + \varepsilon\varepsilon (r r + s s) + 2 \varepsilon s}{1 + \delta\delta + 2 \delta \varepsilon r + \varepsilon\varepsilon (r r + s s) - 2 \varepsilon s}.$$

Problema 4.

§. 15. *Proposita formula differentiali* $\frac{\partial \Phi \sin. \lambda \Phi}{\sqrt[n]{\cos. n \Phi^{\lambda}}}$, *ejus integrale per logarithmos et arcus circulares investigare.*

Solutio.

Cum sit

$$\sin.\lambda\,\phi = \tfrac{1}{2i}(\cos.\phi + i\sin.\phi)^\lambda - \tfrac{1}{2i}(\cos.\phi - i\sin.\phi)^\lambda,$$

constituamus ut hactenus has duas partes

$$\partial p = \frac{\partial\phi(\cos.\phi + i\sin.\phi)^\lambda}{\sqrt[n]{\cos.n\phi^\lambda}} \text{ et } \partial q = \frac{\partial\phi(\cos.\phi - i\sin.\phi)^\lambda}{\sqrt[n]{\cos.n\phi^\lambda}},$$

ita ut integrale quaesitum sit $\frac{p-q}{2i}$. Statuamus nunc iterum

$$\frac{\cos.\phi + i\sin.\phi}{\sqrt[n]{\cos.n\phi}} = x \text{ et } \frac{\cos.\phi - i\sin.\phi}{\sqrt[n]{\cos.n\phi}} = y,$$

ut fiat $\partial p = x^\lambda\partial\phi$ et $\partial q = y^\lambda\partial\phi$, hincque calculo ut supra instituto, fiet

$$\partial p = -\frac{i x^{\lambda-1}\partial x}{2 - x^n} \text{ et } \partial q = \frac{i y^{\lambda-1}\partial y}{2 - y^n},$$

sicque integrale quaesitum erit

$$-\tfrac{1}{2}\int\frac{x^{\lambda-1}\partial x}{2 - x^n} - \tfrac{1}{2}\int\frac{y^{\lambda-1}\partial y}{2 - y^n}.$$

§. 16. Quodsi jam ut hactenus est factum, ponamus $x = r + is$ et $y = r - is$, et pro partibus logarithmicis, quarum forma sit $fl(\alpha + \beta x + \gamma xx)$, ponamus

$$t = \alpha + \beta r + \gamma rr - \gamma ss \text{ et } u = \beta\alpha + 2\gamma r s,$$

binae partes logarithmicae imaginariae uti in problemate secundo in unum logarithmum realem contrahentur, qui erit $-\tfrac{1}{2}fl(tt + uu)$. At si pro partibus circularibus, quarum forma sit g Arc. tang. $(\delta + \varepsilon x)$, bini tales arcus imaginarii jungantur, illi coalescent in unum arcum realem

$$-\tfrac{1}{2}g \text{ Arc. tang } \frac{2\delta + 2\varepsilon r}{1 - (\delta + \varepsilon r)^2 - \varepsilon\varepsilon ss}.$$

Problema generale.

§. 17. *Si Φ denotet functionem quamcunque rationalem ipsius tang. $n\,\phi$, ac proposita fuerit haec formula differentialis*

$$\frac{\Phi\,\partial\phi\,(F\sin.\lambda\phi + G\cos.\lambda\phi)}{\sqrt[n]{(a\cos.n\phi + b\sin.n\phi)^{\lambda}}},$$

ejus integrationem ad logarithmos et arcus circulares reducere.

Solutio.

Ex praecedentibus jam facile intelligitur, formulam numeratoris F sin. $\lambda\,\phi$ + G cos. $\lambda\,\phi$ semper ad talem formam revocari posse

$$F'(\cos.\phi + i\sin.\phi)^{\lambda} + G'(\cos.\phi - i\sin.\phi)^{\lambda},$$

atque hinc ipsa forma proposita discerpatur in has duas partes

$$\partial p = \frac{\Phi\,\partial\phi\,(\cos.\phi + i\sin.\phi)^{\lambda}}{\sqrt[n]{(a\cos.n\phi + b\sin.n\phi)^{\lambda}}} \text{ et}$$

$$\partial q = \frac{\Phi\,\partial\phi\,(\cos.\phi + i\sin.\phi)^{\lambda}}{\sqrt[n]{(a\cos.n\phi + b\sin.n\phi)^{\lambda}}};$$

ita ut integrale quaesitum jam futurum sit F′ p + G′ q.

§. 18. Jam pro formula priori ∂p statuatur

$$\frac{\cos.\phi + i\sin.\phi}{\sqrt[n]{(a\cos.n\phi + b\sin.n\phi)}} = x, \text{ et pro posteriori}$$

$$\frac{\cos.\phi - i\sin.\phi}{\sqrt[n]{(a\cos.n\phi + b\sin.n\phi)}} = y.$$

ita ut hinc futurum sit

$$\partial p = \Phi\,x^{\lambda}\,\partial\phi \text{ et } \partial q = \Phi\,y^{\lambda}\,\partial\phi;$$

inde autem fiet

$$x^n = \frac{\cos. n\,\Phi + i \sin. n\,\Phi}{a \cos. n\,\Phi + i \sin. n\,\Phi},$$

unde colligitur

$$\text{tang.}\, n\,\Phi = \frac{1 - a x^n}{b x^n - i};$$

quare cum Φ denotet functionem rationalem ipsius tang. $n\,\Phi$, evadet quoque functio rationalis ipsius x, atque adeo ipsius x^n, quae designetur per X. Praeterea vero etiam differentiale $\partial\,\Phi$ rationaliter determinabitur; cum fiat

$$\partial\,\Phi = \frac{(a - b)\, x^{n-1}\,\partial x}{(a a + b b)\, x^{2n} - 2\,(a - i b)\, x^n},$$

hoc ergo modo habebimus

$$\partial p = \frac{(i a - b)\, \mathrm{X}\, x^{\lambda - 1}\,\partial x}{(a a + b b)\, x^n - 2\,(a + i b)},$$

quae cum sit penitus rationalis, certum est, ejus integrale, quantumcunque etiam laborem postulaverit, semper per logarithmos et arcus circulares expediri posse.

§. 19. Simili modo res se habet in altera formula ∂q, quae ab ista tantum ratione signi litterae i differet, et quoniam hic omnia rationaliter per y prodibunt expressa, quo pacto Φ abeat in Y, atque obtinebitur

$$\partial q = -\frac{(b + i a)\, \mathrm{Y}\, y^{\lambda - 1}\,\partial y}{(a a + b b) y^n - 2 a + 2 i b},$$

cujus integratio omnino similis erit praecedenti, et quasi eodem labore absolvetur.

§. 20. Manifestum autem est, in hujusmodi calculo imaginaria cum realibus multo arctius commisceri, quam in praecedentibus problematibus usu venit, quandoquidem jam statim ab initio coëfficientes derivati F′ et G′ jam imaginaria involvunt; deinde vero etiam utrinque tang. $n\,\phi$ imaginariis inquinatur, unde etiam in valores X et Y imaginaria ingredientur; quamobrem reductio ad realitatem plerumque maximum laborem exigere poterit, proque autem negotio praecepta necessaria jam satis sunt cognita.

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2) Theorema maxime memorabile circa formulam integralem $\int \frac{\partial \phi \cos. \lambda \phi}{(1 + a a - 2 a \cos. \phi)^{n+1}}$. *M. S. Academiae exhib. die* 13. *Augusti* 1778.

§. 21. Haec formula aliam restrictionem non postulat nisi quod littera λ numeros tantum integros designat sive positivos sive negativos. Evidens autem est valores negativos non discrepare a positivis, cum semper sit $\cos. -\phi = \cos. +\phi$. Hoc notato si istius formulae integrale a termino $\phi = 0$ usque ad terminum $\phi = 180^\circ$ sive $\phi = \pi$ porrigatur, ejus valor semper sequenti formula exprimetur $\frac{\pi a}{(1 - a a)^{2n+1}} \cdot V$, existente

$$V = (\tfrac{n-\lambda}{0})(\tfrac{n+\lambda}{\lambda}) + (\tfrac{n-\lambda}{1})(\tfrac{n+\lambda}{\lambda+1})\, a a$$
$$+ (\tfrac{n-\lambda}{2})(\tfrac{n+\lambda}{\lambda+2})\, a^4 + (\tfrac{n-\lambda}{3})(\tfrac{n+\lambda}{\lambda+3})\, a^6$$
$$+ (\tfrac{n-\lambda}{4})(\tfrac{n+\lambda}{\lambda+4})\, a^8 + (\tfrac{n-\lambda}{5})(\tfrac{n+\lambda}{\lambda+5})\, a^{10} \text{ etc.}$$

Ubi formulae uncinulis inclusae non fractiones, sed eos characteres designant, quibus unciae potestatum Binomii designari solent, ita ut sit

$$(\tfrac{\alpha}{\beta}) = \frac{\alpha}{1} \cdot \frac{\alpha - 1}{2} \cdot \frac{\alpha - 2}{3} \cdot \ldots \ldots \frac{\alpha - \beta + 1}{\beta},$$