1794

Memorabile genus formularum differentialium maxime irrationalium quas tamen ad rationalitatem perducere licet

Leonhard Euler

quae concinnior redditur ponendo $xx = y$, reperietur enim

$$\frac{\partial y \sqrt[3]{(a + cy^3)}}{a - cy^3} = - \frac{2 p^3 \partial p}{p^6 - 4 ac}.$$

II. Sumto autem $\lambda = 1$, ista prodit expressio

$$\frac{\partial x \sqrt[3]{(a + cx^6)^2}}{a - cx^6} = - \frac{p^4 \partial p}{p^6 - 4 ac}.$$

Scholion.

§. 75. Ex his exemplis satis intelligitur, quam egregie reductiones ex nostris formulis generalibus deduci queant, quarum resolutio, nisi methodus nostra adhibeatur, omnes vires analyseos superare videatur.

4.) Memorabile genus formularum differentialium maxime irrationalium, quas tamen ad rationalitatem perducere licet. *M. S. Academiae exhib. d. 15. Maii 1777.*

§. 76. Cum nuper hanc formulam differentialem

$$\frac{\partial x}{(1 - xx) \sqrt[4]{(2 xx - 1)}}$$

tractassem eamque singulari modo ad rationalitatem perduxissem, mox vidi eandem methodum succedere in hac generaliori $\frac{\partial x}{(a + b xx) \sqrt[4]{(a + 2 b xx)}}$, atque adeo in hac multo generaliori

$\frac{\partial x}{(a+bx^n)\sqrt[2n]{(a+2bx^n)}}$, ubi irrationalitas ad ordinem quantumvis altum assurgere potest, cujus resolutio sequenti modo instituitur.

§. 77. Utor scilicet hac substitutione $\frac{x}{\sqrt[2n]{(a+2bx^n)}} = Z$, ut formula nostra integranda, quam per ∂V indicemus, fiat $\partial V = \frac{\partial x}{x} \cdot \frac{Z}{a+bx^n}$, sumtis ergo logarithmis erit

$$lZ = lx - \frac{1}{2n} l(a + 2bx^n),$$

unde differentiando fit

$$\frac{\partial Z}{Z} = \frac{\partial x}{x} - \frac{bx^{n-1}\partial x}{a+2bx^n} = \frac{\partial x(a+bx^n)}{x(a+2bx^n)},$$

erit ergo

$$\frac{\partial x}{x} = \frac{\partial Z(a+2bx^n)}{Z(a+bx^n)},$$

hinc ergo nostra formula erit

$$\partial V = \frac{\partial Z(a+2bx^n)}{(a+bx^n)^2}.$$

Cum igitur sit

$$Z^{2n} = \frac{x^{2n}}{a+2bx^n}, \text{ erit } a + 2bx^n = \frac{x^{2n}}{Z^{2n}},$$

ideoque

$$\partial V = \frac{x^{2n}\partial Z}{Z^{2n}(a+bx^n)^2}.$$

Cum porro sit $aa + 2abx^n = \frac{ax^{2n}}{Z^{2n}}$, addatur utrinque bbx^{2n}, et prodibit

$$(a+bx^n)^2 = \frac{ax^{2n}}{Z^{2n}} + bbx^{2n} = \frac{x^{2n}(a+bbZ^{2n})}{Z^{2n}},$$

quo valore substituto nostra formula evadet

$$\partial V = \frac{\partial Z}{a + bbZ^{2n}},$$

quae ergo formula est rationalis, ideoque per logarithmos et arcus circulares integrari poterit.

§. 78. Observavi porro, cum hic post signum radicale tantum binomium involvatur, ejus loco quoque trinomia, atque adeo polynomia introduci posse. Pro trinomiis autem formula differentialis talem habebit formam

$$\partial V = \frac{\partial x}{(a + bx^n)\sqrt[3n]{(aa + 3\,abx^n + 3\,bbx^{2n})}},$$

ubi ergo irrationalitas ad ordinem multo altiorem ascendit. Nihilo vero minus etiam ista formula ab irrationalitate liberari poterit ope similis substitutionis

$$Z = \frac{x}{\sqrt[3n]{(aa + 3\,abx^n + 3\,bb\,x^{2n})}};$$

hinc enim sumtis logarithmis per differentiationem nanciscemur

$$\frac{\partial Z}{Z} = \frac{\partial x}{x} - \frac{abx^{n-1}\,\partial x - 2\,bbx^{2n-1}\,\partial x}{aa + 3\,abx^n + 3\,bbx^{2n}}, \quad \text{seu}$$

$$\frac{\partial Z}{Z} = \frac{\partial x\,(a + bx^n)^2}{x\,(aa + 3\,abx^n + 3\,bbx^{2n})},$$

ideoque

$$\frac{\partial x}{x} = \frac{\partial Z}{Z} \cdot \frac{aa + 3\,abx^n + 3\,bbx^{2n}}{(a + bx^n)^2}.$$

Cum igitur nostra formula jam sit $\partial V = \frac{\partial x}{x} \cdot \frac{Z}{a + bx^n}$, introducto elemento ∂Z, obtinebimus

$$\partial V = \frac{\partial Z\,(aa + 3\,abx^n + 3\,bbx^{2n})}{(a + bx^n)^3}.$$

§. 79. Cum igitur vi substitutionis sit

$$\sqrt[3n]{(aa + 3abx^n + 3bbx^{2n})} = \frac{x}{Z}, \quad \text{erit}$$

$$aa + 3abx^n + 3bbx^{2n} = \frac{x^{3n}}{Z^{3n}}.$$

Multiplicetur utrinque per a, et addatur utrinque $b^3 x^{3n}$, eritque

$$(a + bx^n)^3 = \frac{x^{3n}(a + b^2 Z^{3n})}{Z^{3n}}:$$

hoc igitur valore substituto ex formula nostra littera x penitus excludetur, prodibitque $\partial V = \frac{\partial Z}{a + b^3 Z^n}$. Cujus ergo integrale semper per logarithmos et arcus circulares reperire licebit.

§. 80. Pro quadrinomiis autem ponamus brevitatis gratia

$$\sqrt[4n]{(a^3 + 4aabx^n + 6abbx^{2n} + 4b^3x^{3n})} = S,$$

ac formula ad rationalitatem reducenda proponatur haec

$$\partial V = \frac{\partial x}{(a + bx^n)S},$$

id quod simili modo succedet ope hujus substitutionis $\frac{x}{S} = Z$, unde formula nostra erit $\partial V = \frac{\partial x}{x} \cdot \frac{Z}{a + bx^n}$. Cum nunc sit

$$\frac{\partial S}{S} = \frac{aabx^{n-1}\partial x + 3abbx^{2n-1}\partial x + 3b^3x^{3n-1}\partial x}{a^3 + 4aabx^n + 6abbx^{2n} + 4b^3x^{3n}},$$

sive

$$\frac{\partial S}{S} = \frac{\partial x}{x} \cdot \frac{bx^n(aa + 3abx^n + 3bbx^{2n})}{S^{4n}},$$

erit $\frac{\partial Z}{Z} = \frac{\partial x}{x} - \frac{\partial S}{S}$; consequenter

$$\frac{\partial Z}{Z} = \frac{\partial x}{x} \cdot \frac{(a + bx^n)^3}{S^{4n}}, \quad \text{hincque} \quad \frac{\partial x}{x} = \frac{S^{4n}\partial Z}{Z(a + bx^n)^3},$$

quo valore substituto formula nostra erit

$$\partial V = \frac{S^{4n}\,\partial Z}{(a + bx^n)^4}.$$

§. 81. Cum autem sit

$$S^{4n} = a^3 + 4\,aabx^n + 6\,abbx^{2n} + 4\,b^3x^{3n}, \quad \text{erit}$$
$$a\,S^{4n} + b^4\,x^{4n} = (a + bx^n)^4,$$

quo valore substituto erit

$$\partial V = \frac{S^{4n}\,\partial Z}{a\,S^{4n} + b^4\,x^{4n}}:$$

quia igitur posuimus $Z = \frac{x}{S}$, erit $S = \frac{x}{Z}$, ideoque $S^{4n} = \frac{x^{4n}}{Z^{4n}}$, qui valor surrogatus dabit

$$\partial V = \frac{\partial Z}{a + b^4\,Z^{4n}},$$

sicque itidem ad rationalitatem est perducta.

§. 82. Hinc jam facile intelligitur, quo modo pro omnibus polynomiis formulae differentiales comparatae esse debeant, ut tali substitutione ad rationalitatem perduci queant, id quod in sequente problemate expediamus

Problema 19.

§. 83. *Si proposita fuerit haec formula differentialis*

$$\partial V = \frac{\partial x}{(a + bx^n)\sqrt[\lambda n]{[(a + bx^n)^\lambda - b^\lambda x^{\lambda n}]}},$$

eam ad rationalitatem reducere, quantumvis magni numeri pro n *et* λ *accipiantur.*

Solutio.

Ponamus etiam hic brevitatis gratia

$$\sqrt[\lambda n]{[(a + bx^n)^\lambda - b^\lambda x^{\lambda n}]} = S,$$

ut formula fiat

$$\partial V = \frac{\partial x}{(a + bx^n)\, S},$$

fiatque insuper $\frac{x}{S} = Z$, ut habeamus

$$\partial V = \frac{\partial x}{x} \cdot \frac{Z}{a + bx^n}.$$

Jam logarithmos differentiando reperietur

$$\frac{\partial S}{S} = \frac{bx^{n-1}\partial x\,(a + bx^n)^{\lambda-1} - b^\lambda x^{\lambda n - 1}\partial x}{S^{\lambda n}}, \text{ sive}$$

$$\frac{\partial S}{S} = \frac{\partial x}{x} \cdot \frac{bx^n (a + bx^n)^{\lambda-1} - b^\lambda x^{\lambda n}}{S^{\lambda n}}.$$

Cum igitur sit $\frac{\partial Z}{Z} = \frac{\partial x}{x} - \frac{\partial S}{S}$, hoc valore substituto erit

$$\frac{\partial Z}{Z} = \frac{\partial x}{x} \cdot \frac{a (a + bx^n)^{\lambda-1}}{S^{\lambda n}},$$

hincque vicissim erit

$$\frac{\partial x}{x} = \frac{S^{\lambda n}\partial Z}{a Z (a + bx^n)^{\lambda-1}},$$

quo valore substituto impetramus

$$\partial V = \frac{S^{\lambda n}\partial Z}{a (a + bx^n)^\lambda},$$

quia nunc est $(a + bx^n)^\lambda = S^{\lambda n} + b^\lambda x^{\lambda n}$, erit

$$\partial V = \frac{S^{\lambda n}\partial Z}{a (S^{\lambda n} + b^\lambda x^{\lambda n})}.$$

Denique ob $S = \frac{x}{Z}$, ideoque $S^{\lambda n} = \frac{x^{\lambda n}}{Z^{\lambda n}}$, hoc valore substituto obtinebitur

$$\partial V = \frac{\partial Z}{a(1 + b^{\lambda} Z^{\lambda n})},$$

quae est rationalis unicam variabilem Z involvens, cujus adeo integrale per logarithmos et arcus circulares assignari poterit.

Corollarium 1.

§. 84. Eadem solutio etiam locum habet, si pro λ numeri fracti accipiantur, qua ratione post signum radicale denuo radicalia involvuntur: ita si fuerit $\lambda = \frac{2}{n}$, erit formula radicalis

$$S = \sqrt{\left[(a + bx^n)^{\frac{2}{n}} - b^{\frac{2}{n}} xx\right]},$$

et formulae nostrae

$$\partial V = \frac{\partial x}{(a + bx^n) S}$$

integrale erit

$$V = \frac{1}{a} \int \frac{\partial Z}{1 + b^{\frac{2}{n}} ZZ} = \frac{1}{a b^{\frac{1}{n}}} \text{Arc. tang. } b^{\frac{1}{n}} Z.$$

Corollarium 2.

§. 85. Quo haec clariora reddantur, capiamus $a = 1$, $b = 1$, et $n = 4$, ut pro postremo casu sit

$$S = \sqrt{[(1 + x^4)^{\frac{1}{2}} - xx]}, \text{ et } \partial V = \frac{\partial x}{(1 + x^4)\sqrt{[(1 + x^4)^{\frac{1}{2}} - xx]}},$$

cujus integrale posito

$$Z = \frac{x}{\sqrt{[(1 + x^4)^{\frac{1}{2}} - xx]}}, \text{ erit}$$

$$V = \text{Arc. tang.} Z, \text{ sive } V = \text{Arc. tang.} \frac{x}{\sqrt{[(1 + x^4)^{\frac{1}{2}} - xx]}}.$$

Sin autem manente $n = 4$ et $a = 1$, fuerit $b = -1$, ideoque

$$S = \sqrt{[(1 - x^4)^{\frac{1}{2}} - xx\sqrt{-1}]},$$

ipsa formula prodiret imaginaria.

Corollarium 3.

§. 86. Pro eodem casu $\lambda = \frac{2}{n}$, sit $n = 6$, $a = 1$ et $b = 1$, eritque

$$S = \sqrt{[(1 + x^6)^{\frac{1}{3}} - xx]}, \text{ ideoque}$$

$$\partial V = \frac{\partial x}{(1 + x^6)\sqrt{[(1 + x^6)^{\frac{1}{3}} - xx]}}.$$

Cujus integrale posito $\frac{x}{S} = Z$, erit

$$V = \text{Arc. tang.} Z = \text{Arc. tang.} \frac{x}{\sqrt{[(1 + x^6)^{\frac{1}{3}} - xx]}}.$$

Similique modo alia hujus generis exempla pro lubitu formari possunt; verum quamquam formula problematis admodum est generalis, tamen adhuc multo magis generalior fieri potest, uti in sequente problemate sumus ostensuri.

Problema 20.

§. 87. *Si proponatur ista formula differentialis multo generalior, quippe in qua tres occurrunt exponentes indeterminati* λ, n, *et* m,

$$\partial V = \frac{x^{m-1}\,\partial x}{(a+bx^n)^\lambda \sqrt[\lambda n]{[(a+bx^n)^\lambda - b^\lambda x^{\lambda n})^m}},$$

eam ab irrationalitate liberare.

Solutio.

Ponatur iterum brevitatis gratia

$$\sqrt[\lambda n]{[(a+bx^n)^\lambda - b^\lambda x^{\lambda n}]} = S,$$

ut formula integranda proposita fiat

$$\partial V = \frac{x^{m-1}\,\partial x}{(a+bx^n)\,S^m} = \frac{\partial x}{x} \cdot \frac{x^m}{(a+bx^n)\,S^m},$$

quae ergo si porro ut ante statuamus $\frac{x}{S} = Z$, fiet

$$\partial V = \frac{\partial x}{x} \cdot \frac{Z^m}{a+bx^n},$$

unde variabilem x penitus eliminari oportet. Quoniam nunc ambae litterae S et Z eosdem habent valores, ut in problemate praecedente atque adeo ipsa formula ∂V oriatur, si praecedens per Z^{m-1} multiplicetur, etiam integrale quaesitum obtinebimus, dum superius integrale per Z^{m-1} multiplicabimus, quo facto erit integrale quaesitum

$$V = \frac{1}{a} \int \frac{Z^{m-1}\,\partial Z}{1 + b^\lambda Z^{\lambda n}}.$$

Corollarium 1.

§. 88. Si exponentem m negativum capiamus, irrationalitas in numeratorem transferetur, ita posita $m = -1$ habebimus

$$\partial V = \frac{\partial x \sqrt[\lambda n]{[(a+bx^n)^\lambda - b^\lambda x^{\lambda n}]}}{xx\,(a+bx^n)},$$

cujus ergo integrale per Z expressum erit

$$V = \frac{1}{a}\int\frac{\partial Z}{ZZ(1+b^\lambda Z^{\lambda n})}.$$

Quin etiam per hunc exponentem m irrationalitas simplicior reddi poterit, veluti si sumamus $m = \lambda$, erit

$$\partial V = \frac{x^{\lambda-1}\partial x}{(a+bx^n)\sqrt[n]{[(a+bx^n)^\lambda - b^\lambda x^{\lambda n}]}}.$$

Cujus integrale posito $Z = \frac{x}{S}$, retinente S superiorem valorem erit

$$V = \frac{1}{a}\int\frac{Z^{\lambda-1}\partial Z}{1+b^\lambda Z^{\lambda n}}.$$

Corollarium 2.

§. 89. Deinde vero etiam si pro m fractionem assumamus, irrationalitas adhuc magis complicabitur, veluti si sumamus $m = \frac{1}{2}$, formula differentialis jam erit

$$\partial V = \frac{\partial x}{(a+bx^n)\sqrt[2\lambda n]{x^{\lambda n}[(a+bx^n)^\lambda - b^\lambda x^{\lambda n}]}}.$$

Verum hic casus facile ad primum problema revocatur statuendo $x = vv$, ita ut sit

$$\partial V = \frac{2\partial v}{(a+bv^{2n})\sqrt[2\lambda n]{[(a+bv^{2n})^\lambda - b^\lambda v^{2\lambda n}]}},$$

quae formula a primo problemate aliter non discrepat nisi quod hic exponens n duplo sit major.

Scholion.

§. 90. Quamquam binae litterae a et b pro lubitu tam negative, quam positive accipi possunt, tamen occurrunt casus, qui sub hac generali forma non comprehenduntur: veluti si proponatur haec formula $\frac{\partial x}{(1 - xx\sqrt[4]{(2xx - 1)})}$, haec in problemate primo non continetur, quia fieri deberet $aa = -1$, quod cum in genere evenire posset, etiam problema generale ad hunc casum accommodatum subjungamus.

Problema 21.

§. 91. *Si ponatur ista formula differentialis latissime patens tres exponentes indeterminatos involvens*

$$\partial V = \frac{x^{m-1}\partial x}{(fx^n - g)\sqrt[\lambda n]{[f^\lambda x^{\lambda n} - (fx^n - g)^\lambda]^m}},$$

eam ab omni irrationalitate liberare.

Solutio.

Statuamus ut ante brevitatis gratia

$$\sqrt[\lambda n]{[f^\lambda x^{\lambda n} - (fx^n - g)^\lambda]} = S,$$

tum vero $Z = \frac{x}{S}$, ut formula differentialis fiat

$$\partial V = \frac{\partial x}{x} \cdot \frac{Z^m}{fx^n - g}.$$

Nunc autem sumendo differentialia logarithmica est

$$\frac{\partial S}{S} = \frac{\partial x}{x} \cdot \frac{f^\lambda x^{\lambda n} - fx^n(fx^n - g)^{\lambda-1}}{S^{\lambda n}},$$

atque hinc colligitur fore

$$\frac{\partial Z}{Z} = \frac{\partial x}{x} - \frac{\partial S}{S} = \frac{\partial x}{x} \cdot \frac{g(fx^n - g)^{\lambda-1}}{S^{\lambda n}},$$

sicque habebitur

$$\frac{\partial x}{x} = \frac{\partial Z}{Z} \cdot \frac{S^{\lambda n}}{g(fx^n - g)^{\lambda-1}},$$

quo valore substituto nanciscimur

$$\partial V = \frac{Z^{m-1}\partial Z S^{\lambda n}}{g(fx^n - g)^\lambda}.$$

Manifesto autem est $(fx^n - g)^\lambda = f^\lambda x^{\lambda n} - S^{\lambda n}$, ideoque

$$\partial V = \frac{Z^{m-1} S^{\lambda n}\partial Z}{g(f^\lambda x^{\lambda n} + S^{\lambda n})};$$

unde postremo ob $S = \frac{x}{Z}$ concluditur haec forma

$$\partial V = \frac{Z^{m-1}\partial Z}{g(f^\lambda Z^{\lambda n} - 1)},$$

quae formula a praecedentibus tantum signis discrepat.