1794

De curvis hyperbolicis quae intra suas assymtotas spatium finitum includunt

Leonhard Euler

DE

CVRVIS HYPERBOLICIS

QVAE INTRA SVAS ASSYMTOTAS SPATIVM FINITVM INCLVDVNT.

Auctore

L. EVLERO.

Conuent. exhib. die 13 *Febr.* 1777.

1.

Tab. II. Fig. 4. Considerabo hic eiusmodi curuas hyperbolicas, quarum assymtotae inter se sunt normales, quoniam, quae de his reperientur, omnia facile ad Hyperbolas obliquangulas accommodari possunt. Sit igitur fYe eiusmodi Hyperbola, cuius assymtotae CF et CE sint inter se normales, atque nobis hic est propositum eas huius generis curuas inuestigare, quae vtrinque in infinitum continuatae intra suas assymtotas spatium finitum includant. Ad hoc igitur requiritur, vt, positis coordinatis $CX = x$ et $XZ = y$, formula integralis $\int y\partial x$ ita sit comparata, vt a termino $x = 0$ vsque ad terminum $x = \infty$ extensa, valorem finitum obtineat. Notum autem est, nullam talium curuarum aequatione binomia expressarum, veluti $x^m y^n = 1$, hac proprietate praeditam esse, sed semper spatium ad alterutram assymtotam relatum infinite magnum prodire, atque adeo in Hyperbola conica vtrumque spatium euadere infinitum.

§. 2.

§. 2. Hic igitur potiſſimum contemplabor eiusmodi Hyperbolas, quarum aequationes inter coordinatas x et y ſunt trinomiales, cuiusmodi generatim haec eſt aequatio:

$$A x^{\alpha} y^{\beta} + B x^{\gamma} y^{\delta} = C,$$

vbi quidem omnes exponentes α, β, γ, δ, poſitiui, ſeu nihilo maiores eſſe debent, quia alioquin, poſito $x = 0$, applicata y non fieret infinita, vel non euaneſceret poſito $x = \infty$. Praeterea etiam ad inſtitutum noſtrum requiritur, vt ambo coëfficientes A et B ſint poſitiui. Si enim alter foret negatiuus, curua non vniformi tractu intra aſſymtotas protenderetur, ſed alicubi extra eas euagaretur; vnde nihil impedit, quominus ſtatuamus $B = A$, atque adeo etiam $C = A$, ita vt habeamus $x^{\alpha} y^{\beta} + x^{\gamma} y^{\delta} = 1$. Quae enim ſymtomata pro his curuis fuerint inuenta, eadem facile transferentur ad caſus, quibus iſti coëfficientes ſunt inaequales.

§. 3. Inter has autem curuas imprimis notatu dignae ſunt eae, in quibus binas coordinatas x et y permutare inter ſe licet, id quod euenit quando $\gamma = \beta$ et $\delta = \alpha$, vt aequatio ſit $x^{\alpha} y^{\beta} + x^{\beta} y^{\alpha} = 1$. Hoc enim modo ambo rami huius curuae ad ſuas aſſymtotas pariter conuergent; ita vt ſi ſpatium ad alterutram aſſymtotam relatum fuerit vel finitum, vel infinitum, etiam alterum eandem legem ſequatur. Nunc igitur inueſtigari conueniet, quemadmodum ambo exponentes α et β comparati eſſe debeant, vt valor formulae integralis $\int y \partial x$, a termino $x = 0$ vsque ad $x = \infty$ extenſus, quantitati finitae aequalis euadat. Quoniam igitur iſtos exponentes α et β tanquam incognitos ſpectamus, euidens eſt ex hac aequatione neque y per x, neque x per y definiri poſſe.

§. 4. Interim tamen determinatio areae huius curuae facile ſuccedet, ſi nouam variabilem in calculum introducamus,

mus, per quam tam x quam y commode exprimere liceat, id quod succedet, si ponamus $y = u\,x$, tum enim nostra aequatio fiet: $(u^\alpha + u^\beta) + x^{\alpha+\beta} = 1$; vnde posito br. gr. $\alpha + \beta = \lambda$, erit $x = \frac{1}{\sqrt[\lambda]{(u^\alpha + u^\beta)}}$, hincque $y = \frac{u}{\sqrt[\lambda]{(u^\alpha + u^\beta)}}$. Hic notetur abscissam x euanescere casu $u = \infty$, quo casu simul y in infinitum crescere debet. Quia enim numerator u ita exhiberi potest, vt sit $u = \sqrt[\lambda]{u^\lambda} = \sqrt[\lambda]{u^{\alpha+\beta}}$, erit

$$y = \sqrt[\lambda]{\frac{u^{\alpha+\beta}}{u^\alpha + u^\beta}},$$

vbi, quia exponens ipsius u in numeratore maior est quam in denominatore, necesse est vt posito $u = \infty$ tota expressio euadat infinita; contra autem, sumto $u = 0$, valor ipsius x manifesto fit infinitus; at ipsius $y = 0$, ob rationem modo allegatam. Hanc ob rem sequentes integrationes a termino $u = \infty$ vsque ad $u = 0$ extendi oportebit.

§. 5. Hinc autem differentiando reperiemus:

$$\partial x = -\frac{\partial u\,(\alpha\,u^{\alpha-1} + \beta\,u^{\beta-1})}{\lambda\,(u^\alpha + u^\beta)^{1+\frac{1}{\lambda}}},$$

quae expressio ducta in y dabit elementum areae:

$$y\,\partial x = -\frac{\partial u\,(\alpha\,u^\alpha + \beta\,u^\beta)}{\lambda\,(u^\alpha + u^\beta)^{1+\frac{2}{\lambda}}},$$

cuius integrale ab $u = \infty$ vsque ad $u = 0$ extendi debet. Quia autem hic potestates ipsius u tam in numeratore quam in denominatore reperiuntur, hanc formulam vlterius reducere licet. Sumamus igitur esse $\beta > \alpha$, ac ponamus $\beta = \alpha + \varepsilon$, atque formula denominatoris ita referri poterit: $u^\alpha\,(1 + u^\varepsilon)$, sicque

ſicque denominator erit

$$\lambda u^{\alpha+\frac{2\alpha}{\lambda}}(1+u^{\varepsilon})^{1+\frac{2}{\lambda}}.$$

Diuidamus igitur tam numeratorem quam denominatorem per $u^{\alpha+\frac{2\alpha}{\lambda}}$, et habebimus

$$y\,\partial x = -\frac{\partial u\,(\alpha u^{-\frac{2\alpha}{\lambda}} + \beta u^{\varepsilon-\frac{2\alpha}{\lambda}})}{\lambda(1+u^{\varepsilon})^{1+\frac{2}{\lambda}}},$$

hinc igitur integrale noſtrum conſtat ex ſequentibus duobus membris:

$$\int y\,\partial x = -\frac{\alpha}{\lambda}\int\frac{u^{-\frac{2\alpha}{\lambda}}\partial u}{(1+u^{\varepsilon})^{1+\frac{2}{\lambda}}} - \frac{\beta}{\lambda}\int\frac{u^{\varepsilon-\frac{2\alpha}{\lambda}}\partial u}{(1+u^{\varepsilon})^{1+\frac{2}{\lambda}}}.$$

Quoniam vero neutra harum formularum, integrationem, in genere quidem, admittit, id tantum nobis inquirendum relinquitur: vtrum haec duo integralia a termino $u=\infty$ vsque ad terminum $u=0$ extenſa, valores adipiſcantur finitos, an infinitos, ad quod diiudicandum ſequens Lemma praemitti oportet: *Iſta formula integralis:* $\int\frac{u^m\,\partial u}{(1+u^n)^k}$, *a termino* $u=0$ *vsque ad terminum* $u=\infty$ *extenſa, valorem habebit finitum, quoties fuerit* $m+1>0$, *ſimulque* $m+1<kn$.

Demonſtratio.

§. 6. Quaeramus primo tantum huius formulae valorem ab $u=0$ vsque ad $u=1$ extenſum, quem vocemus $=P$, atque vt integrale per ſeriem exhibeamus, quia eſt

$$\frac{1}{(1+u^n)^k} = 1 - \frac{k}{1}.u^n + \frac{k(k+1)}{1.2}u^{2n} - \frac{k(k+1)(k+2)}{1.2.3}u^{3n} + \text{etc.}$$

erit

erit iſtud integrale in genere:

$$P = \frac{u^{m+1}}{m+1} - \frac{k}{1} \cdot \frac{u^{n+m+1}}{n+m+1} + \frac{k(k+1)}{1.2} \frac{u^{2n+m+1}}{2n+m+1} - \text{etc.}$$

qui valor vtique euaneſcit poſito $u = 0$, ſi modo fuerit $m + 1 > 0$, quae eſt conditio primo commemorata. Hinc igitur poſito $u = 1$ erit valor quem quaerimus:

$$P = \frac{1}{m+1} - \frac{k}{1(n+m+1)} + \frac{k(k+1)}{1.2(2n+m+1)} - \frac{k(k+1)(k+2)}{1.2.3(3n+m+1)} + \text{etc.}$$

cuius feriei ſumma, quoniam terminorum ſigna alternantur, certe eſt finita, ſicque littera P valorem habebit finitum.

§. 7. Huic valori igitur inſuper addere debemus eum qui ex integratione eiusdem formulae naſcitur, ſiquidem a termino $u = 1$ vsque ad terminum $u = \infty$ extendatur, quem valorem indicemus littera Q, ita vt ſit

$$Q = \int \frac{u^m \partial u}{(1+u^n)^k} \begin{bmatrix} \text{ab } u = 1 \\ \text{ad } u = \infty \end{bmatrix}.$$

Hunc in finem ſtatuamus $u = \frac{1}{v}$, et nunc termini integrationis erunt a $v = 1$ vsque ad $v = 0$. Facta autem ſubſtitutione formula noſtra euadet:

$$Q = -\int \frac{v^{kn-m-2} \partial v}{(v^n+1)^k} \begin{bmatrix} \text{a } v = 1 \\ \text{ad } v = 0 \end{bmatrix}.$$

Sin autem terminos integrationis permutemus, erit

$$Q = +\int \frac{v^{kn-m-2} \partial v}{(v^n+1)^k} \begin{bmatrix} \text{a } v = 0 \\ \text{ad } v = 1 \end{bmatrix}.$$

§. 8. Iam denominatorem vt ante in feriem reſoluamus, quae erit

$$1 - \frac{k}{1} v^n + \frac{k(k+1)}{1.2} v^{2n} - \frac{k(k+1)(k+2)}{1.2.3} v^{3n} + \text{etc.}$$

quae ducta in $v^{nk-m-2} \partial v$ et integrata dabit:

$$\frac{v^{nk-m-1}}{nk-m-1} - \frac{k}{1}\cdot\frac{v^{nk+n-m-1}}{nk+n-m-1} + \frac{k(k+1)}{1.2}\cdot\frac{v^{nk+2n-m-1}}{nk+2n-m-1} - \text{etc.}$$

quae feries euanefcit cafu $v = 0$, fi modo fuerit $nk - m - 1 > 0$, hoc eft $nk > m + 1$, quae eft altera conditio praefcripta. Statuatur $v = 1$, eritque

$$Q = \frac{1}{nk-m-1} - \frac{k}{1(nk+n-m-1)} + \frac{k(k+1)}{1.2(nk+2n-m-1)} - \text{etc.}$$

cuius feriei valor, quoniam figna terminorum alternantur, certe eft finitus, confequenter formulae propofitae $\int \frac{u^m \partial u}{(1+u^n)^k}$, a termino $u = 0$ vsque ad $u = \infty$ extenfae, valor erit $= P + Q$, ideoque finitus, fi modo fuerit tam $m + 1 > 0$, quam $m + 1 < kn$.

Alia demonftratio eiusdem Lemmatis.

§. 9. Statuamus $u^n = \frac{t^n}{1-t^n}$ fietque $u = 0$, fi $t = 0$; at fiet $u = \infty$, facto $t = 1$, ficque termini integrationis nunc erunt a $t = 0$ ad $t = 1$. Tum autem erit $1 + u^n = \frac{1}{1-t^n}$ et denominator $= \frac{t}{(1-t^n)^k}$. Deinde vero ob $u = \frac{t}{\sqrt[n]{(1-t^n)}}$, erit $u^m = \frac{t^m}{(1-t^n)^{\frac{m}{n}}}$, denique $\partial u = \frac{\partial t}{(1-t^n)^{2+\frac{1}{n}}}$. His igitur valoribus fubftitutis formula integranda erit:

$$\int \frac{t^m \partial t}{(1-t^n)^{\frac{m+1}{n}} - k + 1} \begin{bmatrix} \text{a} & t = 0 \\ \text{ad} & t = 1 \end{bmatrix}.$$

Hic primum obferuaffe iuuabit, vt integrale pofito $t = 0$ euanefcere poffit, requiri vt fit $m + 1 > 0$. Deinde vero, quia denominator euanefcit pofito $t = 1$, ne etiam integrale hoc cafu in infinitum excrefcat, neceffe eft vt exponens denominatoris $\frac{m+1}{n} - k + 1$ fit vnitate minor; vnde fequitur conditio $m + 1 < k n$, quae funt eaedem conditiones in Lemmate allatae.

§. 10. Applicemus nunc hoc Lemma ad binas formulas integrales, ex quibus aream totam $\int y \partial x$ componi inuenimus, quod quo facilius fieri poffit, immutemus quoque terminos integrationis, vt fit:

$$\int y \partial x = \frac{\alpha}{\lambda} \int \frac{u^{-\frac{2\alpha}{\lambda}} \partial u}{(1+u^{\varepsilon})^{1+\frac{2}{\lambda}}} + \frac{\beta}{\lambda} \int \frac{u^{\varepsilon - \frac{2\alpha}{\lambda}} \partial u}{(1+u^{\varepsilon})^{1+\frac{2}{\lambda}}} \begin{bmatrix} \text{ab } u = 0 \\ \text{ad } u = \infty \end{bmatrix},$$

atque applicatio prioris partis ad noftrum Lemma dabit $m = -\frac{2\alpha}{\lambda}$, $n = \varepsilon$ et $k = 1 + \frac{2}{\lambda}$, vnde prior conditio $m + 1 > 0$ praebet $\lambda > 2\alpha$. Quia igitur eft $\lambda = \alpha + \beta$, debet effe $\beta > \alpha$, quae conditio iam fponte eft adimpleta; altera vero conditio $m + 1 < n k$ pro noftro cafu dat $\lambda - 2\alpha < (\lambda + 2)\varepsilon$, ideoque $\lambda + 1 > 0$, quod etiam vltro euenit, quoniam bini exponentes α et β neceffario funt pofitiui, vnde prior formula perpetuo habet valorem finitum, quicunque valores litteris α et β tribuantur.

§. 11. Applicemus pari modo noftrum Lemma ad alteram formulam, pro qua erit $m = \varepsilon - \frac{2\alpha}{\lambda}$, $n = \varepsilon$ et $k = 1 + \frac{2}{\lambda}$; hinc prior conditio $m + 1 > 0$ multo magis fponte adimpletur quam ante. At vero altera conditio $m + 1 < k n$ praebet $\lambda - 2\alpha < 2\varepsilon$; quia autem $\lambda - 2\alpha = \beta - \alpha = \varepsilon$, haec conditio pariter fponte adimpletur, nifi fit $\varepsilon = 0$, hoc eft $\beta = \alpha$.

§. 12.

§. 12. Hinc igitur patet, omnes Hyperbolas in hac aequatione: $x^\alpha y^\beta + x^\beta y^\alpha = 1$, contentas intra ſuas aſſymtotas ſemper ſpatia finita includere, quicunque numeri poſitiui exponentibus α et β tribuantur, ſolo caſu $\beta = \alpha$ excepto, quo aequatio noſtra abit in $2 x^\alpha y^\alpha = 1$, ideoque $x y =$ Conſt. quae aequatio eſt pro Hyperbola conica, cuius ſpatium vtique eſt infinitum, id quod eo magis eſt mirandum, quia in Hyperbolis binomialibus nulla plane noſtro ſcopo ſatisfacit.

§. 13. Hactenus in aequatione tractata omnes coëfficientes vnitati aequales aſſumſimus: facile autem apparet, demonſtrationem pari modo eſſe ſucceſſuram, ſi coëfficientes quicunque adiungantur, dummodo fuerint poſitiui, quandoquidem etiam Lemma ſupra allatum omnem vim retinet, etiamſi formula integralis hoc modo proponatur: $\int \frac{u^m \partial u}{(a + b u^n)^k}$. Hanc ob rem ſequens Theorema generalius in medium afferre licet.

Theorema.

Omnes curuae hyperbolicae in hac aequatione contentae:

$$a x^\alpha y^\beta + b x^\beta y^\alpha = c,$$

intra aſſymtotas ſuas ſpatium finitum includent, 1°.) *ſi omnes coëfficientes* a, b, c, *fuerint poſitiui;* 2°.) *ſi exponentes* α *et* β *fuerint pariter ambo poſitiui;* 3°.) *ſi fuerint inter ſe inaequales.*

§. 14. Iam enim obſeruauimus, ſi coëfficientium a et b alteruter euaneſcat, quo caſu aequatio fit binomialis, tum ſpatium inter has Hyperbolas et ſuas aſſymtotas incluſum ſemper eſſe infinite magnum. Deinde, ſi exponentes α et β inter

ter ſe eſſent aequales, curua abiret in Hyperbolam conicam, ideoque etiam memoratum ſpatium haberet infinitum, ex quo intelligitur, quo magis hi exponentes α et β a ratione aequalitatis recedant, eo minus eſſe futurum ſpatium inter curuas et aſſymtotas contentum. Porro etiam hoc ſpatium eo magis diminuetur, quo propius ambo coëfficientes a et b ad aequalitatem acceſſerint.

§. 15. Caſus autem iſte tractatus maxime eſt particularis reſpectu aequationis generalis ex tribus terminis conſtantis, quae eſt:

$$a x^{\alpha} y^{\beta} + b x^{\gamma} y^{\delta} = c,$$

quam autem aequationem non eodem modo, vti praecedentem, tractare licet. Interim tamen circa hanc aequationem generaliſſimam ſequens theorema rigoroſe demonſtrare licet.

Theorema generale.

Omnes curuae hyperbolicae in hac aequatione generali contentae: $a x^{\alpha} y^{\beta} + b x^{\gamma} y^{\delta} = c$, *intra ſuas aſſymtotas ſpatium finitum includent ſub ſequentibus conditionibus:* 1°.) *ſi ſinguli coëfficientes* a, b, c, *fuerint poſitiui ſiue nihilo maiores;* 2°.) *ſi etiam omnes exponentes* α, β, γ, δ, *fuerint poſitiui, quandoquidem, ſi vnicus eſſet negatiuus, vel ſaltem nihilo aequalis, curuae ne quidem forent Hyperbolae;* 3°.) *requiritur vt harum duarum fractionum* $\frac{\alpha}{\beta}$ *et* $\frac{\gamma}{\delta}$, *altera ſit vnitate maior, altera vero minor; ſi enim vel ambae eſſent vnitate maiores vel ambae minores, vel alterutra ſaltem* $=1$, *tum ſpatium de quo loquimur, ſemper foret infinite magnum.*

Demonſtratio.

§. 16. Quia coëfficientes a, b, c in iudicio circa infinitum vel finitum non in computum ingrediuntur, eorum loco ca, commoditatis gratia vnitatem ſcribamus. Deinde quia fractionum $\frac{\alpha}{\beta}$ et $\frac{\gamma}{\delta}$ altera debet eſſe vnitate maior, altera minor, vt huius conditionis rationem in calculum inferamus, ponamus eſſe $\frac{\alpha}{\beta} > 1$ et $\frac{\gamma}{\delta} < 1$, huncque in finem ſtatuamus $\alpha = \beta + \mu$ et $\delta = \gamma + \nu$, vt aequatio noſtra ſit:

$$x^{\beta+\mu} . y^{\beta} + x^{\gamma} y^{\gamma+\nu} = 1.$$

§. 17. Hic autem ſtatim patet, ſubſtitutionem ante vſurpatam $y = ux$ hic nullum plane vſum eſſe allaturam. Ponamus autem modo generaliori $y = ux^{\theta}$, et aequatio reſultans erit:

$$x^{\beta+\mu+\beta\theta} u^{\beta} + x^{\gamma+\gamma\theta+\nu\theta} u^{\gamma+\nu} = 1.$$

Iam ambos ipſius x exponentes ſtatuamus aequales, vt ſit

$$\beta + \mu + \beta\theta = \gamma + \gamma\theta + \nu\theta,$$

vnde reperitur $\theta = \frac{\beta+\mu-\gamma}{\gamma+\nu-\beta}$. Hinc autem fiet exponens ipſius $x = \frac{\beta\nu+\gamma\mu+\mu\nu}{\gamma+\nu-\beta}$. Hunc autem exponentem br. gr. ponemus $= \lambda$, vt ſit $\lambda = \frac{\beta\nu+\gamma\mu-\mu\nu}{\gamma+\nu-\beta}$, atque aequatio noſtra iam erit:

$$x^{\lambda}(u^{\beta} + u^{\nu+\gamma}) = 1, \text{ ideoque}$$

$$x = \frac{1}{(u^{\beta} + u^{\gamma+\nu})^{\frac{1}{\lambda}}}; \text{ tum autem erit}$$

$$y = x^{\theta} . u = \frac{u}{(u^{\beta} + u^{\gamma+\nu})^{\frac{\theta}{\lambda}}}, \text{ vbi eſt}$$

$$\frac{\theta}{\lambda} = \frac{\beta+\mu-\gamma}{\beta\nu+\gamma\mu+\mu\nu}.$$

Hic igitur exponens denominatoris ſemper eſt poſitiuus.

§. 18. Nunc vt aream exprimere poſſimus, differentiemus abſciſſam x, et quia coëfficientes ad noſtrum inſtitutum nihil conferunt, eos prorſus negligamus, eritque:

$$\partial x = \frac{(u^{\beta - 1} + u^{\gamma + \nu - 1})\,\partial u}{(u^\beta + u^{\gamma + \nu})^{\frac{\theta}{\lambda} + 1}}.$$

Hinc autem porro erit elementum areae:

$$y\,\partial x = \frac{\partial u\,(u^\beta + u^{\gamma+\nu})}{(u^\beta + u^{\gamma+\nu})^{\frac{1+\lambda+\theta}{\lambda}}},$$

vbi exponens denominatoris eſt $\frac{\beta\nu + \gamma\mu + \mu\nu + \mu + \nu}{\beta\nu + \gamma\mu + \mu\nu}$, qui ergo ſemper eſt poſitiuus, atque adeo vnitate maior. Vnde ſi br. gr. ponamus $\frac{\mu + \nu}{\beta\nu + \gamma\mu + \mu\nu} = \xi$, iſte exponens erit $1 + \xi$, atque elementum areae nunc erit:

$$y\,\partial x = \frac{\partial u\,(u^\beta + u^{\gamma+\nu})}{(u^\beta + u^{\gamma+\nu})^{1+\xi}}.$$

§. 19. Ante autem quam noſtrum Lemma ſupra datum huc transferre liceat, tres caſus probe a ſe inuicem diſtingui oportet, prouti fuerit vel 1°. $\gamma + \nu > \beta$, vel 2°. $\gamma + \nu < \beta$, vel 3°. $\gamma + \nu = \beta$, vnde a poſtremo, vtpote ſimpliciſſimo, inchoëmus.

Caſus I. quo $\beta = \gamma + \nu$.

§. 20. Hoc caſu ſubſtitutio adhibita locum habere plane nequit, propterea quo tam λ quam θ in infinitum excreſcerent; verum hoc caſu negotium ſine vlla ſubſtitutione expediri poteſt. Cum enim aequatio noſtra fiat:

$$x^{\beta+\mu} y^\beta + x^\gamma y^\beta = 1,$$

hinc ſtatim fit:

$y^\beta =$

$$y^\beta = \frac{1}{x^{\beta+\mu} + x^{\gamma}}, \text{ ideoque}$$

$$y = \frac{1}{(x^{\beta+\mu} + x^{\beta-\nu})^{\frac{1}{\beta}}},$$

consequenter elementum areae:

$$y \partial x = \frac{\partial x}{(x^{\beta+\mu} + x^{\beta-\nu})^{\frac{1}{\beta}}};$$

vbi denominator ita exhiberi potest: $x^{\frac{\beta-\nu}{\beta}}(1 + x^{\mu+\nu})^{\frac{1}{\beta}}$, quo pacto formula nostra integranda erit:

$$y \partial x = \frac{x^{\frac{\nu-\beta}{\beta}} \partial x}{(1 + x^{\mu+\nu})^{\frac{1}{\beta}}}.$$

§. 21. Nunc igitur ad hanc formam nostrum Lemma applicare licebit, vt pateat, vtrum integrale huius formulae a termino $x = 0$ vsque ad $x = \infty$ valorem obtineat finitum, nec ne. Facta autem applicatione erit $m = \frac{\nu - \beta}{\beta}$, $n = \mu + \nu$, $k = \frac{1}{\beta}$, vnde vt spatium quaesitum euadat finitum, primo esse debet $m + 1 > 0$, hoc est $\frac{\nu}{\beta} > 0$, quae conditio sponte adimpletur, quoniam omnes litterae denotant numeros positiuos Altera vero conditio postulat vt sit $m + 1 < nk$, hoc est $\frac{\nu}{\beta} < \frac{\mu+\nu}{\beta}$, quod pariter est manifestum, ita vt iam certum sit, casu $\gamma + \nu = \beta$ spatium, quod consideramus, esse finitae magnitudinis.

Casus II. quo $\gamma + \nu > \beta$.

§. 22. Sit igitur $\gamma + \nu = \beta + \varepsilon$; hoc ergo casu erit

$$\theta = \frac{\beta + \mu - \gamma}{\varepsilon} \text{ et } \lambda = \frac{\beta\nu + \gamma\mu + \mu\nu}{\varepsilon}.$$

Hinc cum iam sit

$x =$

$$x = \frac{1}{u^{\frac{\beta}{\lambda}}(1 + u^{\varepsilon})^{\frac{\xi}{\lambda}}} \text{ et } y = \frac{u^{1 - \frac{\beta\theta}{\lambda}}}{(1 + u^{\varepsilon})^{\frac{\theta}{\lambda}}},$$

primo, ob λ numerum positiuum, euidens est sumto $u = \infty$ prodire $x = 0$; contra vero sumto $u = 0$ fieri $x = \infty$. Casu autem $u = \infty$ fit

$$y = \frac{u^{1 - \frac{\beta\theta}{\lambda}}}{u^{\varepsilon\frac{\theta}{\lambda}}} = u^{1 - \frac{\beta\theta - \varepsilon\theta}{\lambda}},$$

quod erit infinitum, si fuerit $1 > \frac{(\beta + \varepsilon)\theta}{\lambda}$, hoc est $\frac{\lambda}{\theta} > \beta + \varepsilon$. Est vero $\frac{\lambda}{\theta} = \frac{\beta\nu + \gamma\mu + \mu\nu}{\beta + \mu - \gamma}$, ideoque $\gamma + \nu > \beta$, quae est ipsa nostra hypothesis. Eodem modo ostenditur, casu $u = 0$ etiam fieri $y = 0$: erit enim $y = u^{1 - \frac{\beta\theta}{\lambda}}$, vbi ergo exponens multo magis est positiuus quam praecedente casu, ita vt certe sit $y = 0$, facto $u = 0$. Hoc igitur notato formulam integralem $\int y\,\partial x$ a termino $u = \infty$ vsque ad $u = 0$ extendi oportet.

§. 23. Cum igitur θ et λ sint numeri positiui, elementum areae in duas partes discerpatur:

$$y\,\partial x = \frac{u^{\beta}\,\partial u}{u^{\beta + \beta\xi}(1 + u^{\varepsilon})^{1 + \xi}} + \frac{u^{\beta + \varepsilon}\,\partial u}{u^{\beta + \beta\xi}(1 + u^{\varepsilon})^{1 + \xi}}.$$

Nunc igitur Lemma nostrum primo ad formulam priorem accommodemus, eritque $m = -\beta\xi$, $n = \varepsilon$ et $k = 1 + \xi$, vnde prima conditio, quae postulat $m + 1 > 0$, dat $1 > \beta\xi$, hoc est $\gamma + \nu > \beta$, quae est ipsa hypothesis; vnde patet priorem conditionem $m + 1 > 0$ multo magis in altera formula locum habere. E contrario autem altera conditio $m + 1 < nk$ in priore formula certe valebit, si in posteriore locum habeat. Pro altera autem formula est $m = \varepsilon - \beta\xi$ et $kn = \varepsilon(1 + \xi)$. Quare cum esse debeat $m + 1 < nk$, erit nobis $1 - \beta\xi < \varepsilon\xi$,

hinc

hinc substituto loco ξ valore erit $\beta < \gamma + \nu$, quae iterum est ipsa hypothesis praescripta, vnde etiam pro hoc casu Theorema nostrum est euictum.

Casus III. quo $\gamma + \nu < \beta$.

§. 24. Sit igitur $\gamma + \nu = \beta - \varepsilon$, vnde ambo valores θ et λ euadent negatiui, scil. $\theta = \frac{\beta + \mu - \gamma}{-\varepsilon}$ et $\lambda = \frac{\beta\nu + \gamma\mu + \mu\nu}{-\varepsilon}$, vnde erit

$$x = \frac{1}{u^{\frac{\beta - \varepsilon}{\lambda}}(1 + u^{\varepsilon})^{\frac{1}{\lambda}}}.$$

Hinc quia λ valorem habet negatiuum, euidens est, inuerso modo fieri $x = 0$, quando $u = 0$, atque $x = \infty$, si $u = \infty$. Hinc autem iam ex natura rei sequitur, priore casu fieri $y = \infty$, posteriore vero $y = 0$, ideoque nunc formulam integralem $\int y\,\partial x$ ab $u = 0$ vsque ad $u = \infty$ extendi oportet.

§. 25. Quanquam autem hic valores litterarum θ et λ sunt negatiui, tamen exponens principalis ξ semper est positiuus, siquidem est $\xi = \frac{\mu + \nu}{\beta\nu + \gamma\mu + \mu\nu}$, quae expressio ob $\gamma + \nu = \beta - \varepsilon$ transformatur in hanc:

$$\xi = \frac{\mu + \nu}{\beta(\mu + \nu) - \varepsilon\mu};$$

vbi notetur ε non solum esse numerum positiuum sed etiam minorem quam β. Nunc igitur formulam pro area accuratius perpendamus, quae ob $\gamma + \nu = \beta - \varepsilon$ ita se habebit:

$$y\,\partial x = \frac{u^{\beta}\,\partial u + u^{\beta - \varepsilon}\,\partial u}{(u^{\beta} + u^{\beta - \varepsilon})^{1 + \xi}},$$

cuius denominator, quia factorem habet $u^{\beta - \varepsilon}$, ita repraesentetur: $(u^{\beta - \varepsilon})^{1 + \xi}(1 + u^{\varepsilon})^{1 + \xi}$, hocque modo area quaesita dua-

 bus

bus his conſtabit partibus:

$$\int y\,\partial x = \int \frac{u^{-\xi(\beta-\varepsilon)}\,\partial u}{(1+u^{\varepsilon})^{1+\xi}} + \int \frac{u^{\varepsilon-\xi(\beta-\varepsilon)}\,\partial u}{(1+u^{\varepsilon})^{1+\xi}}.$$

§. 26. Nunc vtramque hanc formulam ſecundum noſtrum Lemma examinemus, et facta comparatione pro priore habebimus $m = -\xi(\beta-\varepsilon)$; $n = \varepsilon$ et $k = 1+\xi$, vnde prima conditio $m+1 > 0$ dat $1-\xi(\beta-\varepsilon) > 0$, quae, ſubſtituto valore ipſius ξ, praebet:

$$\beta(\mu+\nu) - \varepsilon\mu - (\mu+\nu)(\beta-\varepsilon) > 0,$$

quae euoluta dat $\varepsilon\nu > 0$, quod ob ε et ν numeros poſitiuos per ſe eſt manifeſtum. Simul vero hinc patet, ſi ν eſſet negatiuum, tum iſtam conditionem non adimpleri, ideoque aream prodituram eſſe infinitam. Altera vero conditio, quae poſtulat $m+1 < nk$, praebet

$$1-\xi(\beta-\varepsilon) < \varepsilon(1+\xi),$$

ſiue $1-\beta\xi < \varepsilon$, ac pro ξ valore ſubſtituto:

$$-\varepsilon\mu < \varepsilon[\beta(\mu+\nu) - \varepsilon\mu],$$

quae per numerum poſitiuum diuiſa praebet hanc conditionem:

$$\varepsilon\mu - \mu < \beta(\mu+\nu),$$

quae conditio etiam manifeſto adimpletur, ob $\varepsilon < \beta$.

§. 27. Simili modo alteram formulam tractemus, in qua $m = \varepsilon - \xi(\beta-\varepsilon)$, $n = \varepsilon$ et $k = 1+\xi$. Cum igitur hic ſit m maius quam ante, prior conditio multo magis implebitur; pro altera autem conditione hic habebimus:

$$m+1 = 1+\varepsilon-\xi(\beta-\varepsilon) = 1+\varepsilon(1+\xi) - \beta\xi.$$

At vero nk eſt $\varepsilon(1+\xi)$, vnde ſecunda conditio poſtulat $1-\beta\xi < 0$, ſiue $\beta\xi > 1$, hoc eſt

$$\beta(\mu+\nu) > \beta(\mu+\nu) - \varepsilon\mu$$

ſiue

siue $e\mu > 0$, quod vtique euenit, quia μ supponitur positiuum. Simul vero hinc patet, si μ esset negatiuum, tum spatium quaesitum futurum esse infinitum, quae circumstantia etiam in casu primo locum habet.

§. 28. His igitur tribus casibus coniunctis summo vigore euictum est, spatium inter has Hyperbolas et suas assymtotas inclusum semper fore finitae magnitudinis, si modo litterae μ et ν fuerint positiuae, vti quidem assumsimus; tum autem fractio $\frac{\alpha}{\beta}$ vnitate erit maior, altera vero fractio $\frac{\gamma}{\delta}$ vnitate minor, quae fractiones cum permutationes patiantur, sequitur, quoties ambarum harum fractionum $\frac{\alpha}{\beta}$ et $\frac{\gamma}{\delta}$ altera fuerit vnitate maior, altera minor, toties spatium memoratum finitam habiturum esse magnitudinem, simul vero quoque demonstratum est, si vel ambae hae fractiones fuerint maiores vnitate vel minores, toties istud spatium esse infinitum.

§. 29. Hactenus quidem contenti fuimus eos casus assignare, quibus spatium memoratum habeat finitam quantitatem, neque vero solliciti fuimus de vera eius quantitate, quae plerumque formulas integrales intractabiles postulat, vbi scilicet integratio ad quantitates maxime transcendentes assurgeret. Casum igitur satis memorabilem subiungamus, quo ipsam istam aream adeo algebraice satis simplici modo exprimere licet.

Problema.

Si natura curuae hyperbolicae hac aequatione fuerit expressa: $a x^\alpha y^\beta + b x^\beta y^\alpha = c$, *vbi coëfficientes* a, b, c, *omnes sint positiui, exponentes vero* α et β *ita comparati, vt eorum summa vnitati aequetur, aream inuestigare, quam istae curuae in infinitum productae intra suas assymtotas includunt.*

 Solu-

Solutio.

§. 30. Quoniam assumitur $\alpha + \beta = 1$, ponamus

$$\alpha = \tfrac{1+\lambda}{2} \text{ et } \beta = \tfrac{1-\lambda}{2},$$

vt fit $\alpha - \beta = \lambda$. Nunc ponatur $y = u x$, et aequatio nostra dabit: $(a u^{\beta} + b u^{\alpha}) x = c$, vnde fit

$$x = \frac{c}{a u^{\beta} + b u^{\alpha}}, \text{ ideoque}$$

$$y = \frac{c u}{a u^{\beta} + b u^{\alpha}}; \text{ tum igitur erit}$$

$$\partial x = -\frac{c(a \beta u^{\beta - 1} + b \alpha u^{\alpha - 1}) \partial u}{(a u^{\beta} + b u^{\alpha})^2},$$

ex quo conficitur elementum areae:

$$y \partial x = -\frac{\beta a c c u^{\beta} - \alpha b c c u^{\alpha}}{(a u^{\beta} + b u^{\alpha})^3} \partial u.$$

§. 31. Nunc loco α et β valores ante assumtos scribamus, eritque

$$a u^{\beta} + b u^{\alpha} = u^{\frac{1-\lambda}{2}} (a + b u^{\lambda}),$$

quo valore substituto reperietur:

$$y \partial x = -\frac{\beta a c c u^{\lambda - 1} - \alpha b c c u^{2\lambda - 1}}{(a + b u^{\lambda})^3} \partial u,$$

ideoque

$$\int y \partial x = - c c \int \frac{\beta a u^{\lambda - 1} \partial u + \alpha b u^{2\lambda - 1} \partial u}{(a + b u^{\lambda})^3}.$$

Ponatur nunc $a + b u^{\lambda} = z$, erit $u^{\lambda} = \frac{z-a}{b}$, $u^{\lambda - 1} \partial u = \frac{\partial z}{\lambda b}$ et $u^{2\lambda - 1} \partial u = \frac{(z-a)\partial z}{\lambda b b}$, quibus valoribus substitutis erit:

$$\int y \partial x = -\frac{c c}{\lambda b b} \int \frac{\partial z}{z^3} [(\beta - \alpha) a b + \alpha b z],$$

siue

siue

$$\int y\,\partial x = -\tfrac{cc}{\lambda b}\int\tfrac{(\alpha z - \lambda a)\,\partial z}{z^3},$$

cuius integrale est

$$\int y\,\partial x = C - \tfrac{cc}{\lambda b}\left(\tfrac{\lambda a}{2zz} - \tfrac{\alpha}{z}\right).$$

§. 32. Restituto iam loco z valore assumto erit area quaesita in genere:

$$\int y\,\partial x = C - \frac{cc}{\lambda b}\left(\frac{\lambda a}{2\,(a + b\,u^\lambda)^2} - \frac{\alpha}{a + b\,u^\lambda}\right),$$

vbi, quia x fit $= 0$ sumto $u = \infty$, constans ita definiatur, vt facto $u = \infty$ area euanescat, vnde manifestum est statui debere $C = 0$, ita vt iam sit area indefinita a termino $x = 0$ sumta

$$\int y\,\partial x = \frac{cc}{\lambda b}\left(\frac{\alpha}{a + b\,u^\lambda} - \frac{\lambda a}{2\,(a + b\,u^\lambda)^2}\right).$$

Nunc igitur abscissa x in infinitum vsque extendatur, quod fit ponendo $u = 0$, atque tota nostra area quaesita erit:

$$\tfrac{cc}{ab}\left(\tfrac{\alpha}{\lambda} - \tfrac{1}{2}\right) = \tfrac{cc}{2ab}\cdot\tfrac{\alpha + \beta}{\alpha - \beta} = \tfrac{cc}{2ab\,(\alpha - \beta)},$$

quae ergo area semper est finita, nisi sit $\alpha = \beta$, quem autem casum iam exclusimus, quippe pro Hyperbola conica.

§. 33. Cum exponentes α et β eiusmodi designent fractiones, quarum summa vnitati aequetur, sumamus duos numeros integros quoscunque μ et ν, quorum sit summa $\lambda = \mu + \nu$, ac statui poterit $\alpha = \frac{\mu}{\lambda}$ et $\beta = \frac{\nu}{\lambda}$, ita vt aequatio pro nostris curuis hyperbolicis sit:

$$a\sqrt[\lambda]{x^\mu y^\nu} + b\sqrt[\lambda]{x^\nu y^\mu} = c,$$

et iam spatium inter has curuas et suas assymtotas contentum

erit

erit $\frac{\lambda c c}{2 a b (\mu - \nu)}$. Hic autem imprimis quaeritur aequatio rationalis, qua harum curuarum natura exprimatur, vt pateat ad quemnam ordinem hae curuae ſint referendae, id quod ope ſequentis Lemmatis facillime praeſtabitur.

Lemma.

§. 34. Quod ſi p et q fuerint radices huius aequationis quadraticae: $zz - fz + g = 0$, ita vt ſit $p + q = f$ et $pq = g$, aggregatum binarum quarumuis poteſtatum ipſarum p et q ſequenti modo exprimitur:

$$p + q = f,$$
$$p^2 + q^2 = ff - 2g,$$
$$p^3 + q^3 = f^3 - 3fg,$$
$$p^4 + q^4 = f^4 - 4ffg + 2gg,$$
$$p^5 + q^5 = f^5 - 5f^3 g + 5fgg,$$
$$p^6 + q^6 = f^6 - 6f^4 g + 9ffgg - 2g^3,$$
$$p^7 + q^7 = f^7 - 7f^5 g + 14f^3 gg - 7fg^3,$$
$$p^8 + q^8 = f^8 - 8f^6 g + 20f^4 gg - 16ffg^3 + 2g^4,$$
$$p^9 + q^9 = f^9 - 9f^7 g + 27f^5 gg - 30f^3 g^3 + 9fg^4,$$
$$p^{10} + q^{10} = f^{10} - 10f^8 g + 35f^6 gg - 50f^4 g^3 + 25ffg^4 - 2g^5,$$
etc.

atque hinc in genere concluditur fore:

$$p^\lambda + q^\lambda = f^\lambda - \lambda f^{\lambda-2} g + \frac{\lambda(\lambda-3)}{1.2} f^{\lambda-4} gg - \frac{\lambda(\lambda-4)(\lambda-5)}{1.2.3} f^{\lambda-6} g^3 + \frac{\lambda(\lambda-5)(\lambda-6)(\lambda-7)}{1.2.3.4} f^{\lambda-8} g^4 - \text{etc.}$$

quae autem formula non valet, niſi exponens λ fuerit numerus integer poſitiuus, ac praeterea iſti termini non vltra exponentes poſitiuos ipſius f continuentur.

§. 35.

§. 35. Quod si iam hoc Lemma ad nostrum casum accommodemus, erit $p = a\sqrt[\lambda]{x^\mu y^\nu}$ et $q = b\sqrt[\lambda]{x^\nu y^\mu}$, tum vero habebimus $p + q = c$ et productum $pq = abxy$, ita vt nunc sit $f = c$ et $g = abxy$, quibus notatis formula generalis in Lemmate data nobis statim suppeditat hanc aequationem rationalem:

$$a^\lambda x^\mu y^\nu + b^\lambda x^\nu y^\mu = c^\lambda - \lambda c^{\lambda-2} abxy + \frac{\lambda(\lambda-3)}{1.2} c^{\lambda-4} a^2 b^2 x^2 y^2$$
$$- \frac{\lambda(\lambda-4)(\lambda-5)}{1.2.3.} c^{\lambda-6} a^3 b^3 x^3 y^3 + \text{etc.}$$

quae ergo aequatio manifesto pertinet ad ordinem λ^{um}.

§. 36. Percurramus nunc aliquot casus simpliciores, sitque 1°. $\mu = 2$, $\nu = 1$ et $\lambda = 3$, ita vt aequatio pro Hyperbolis irrationalis sit $b\sqrt[3]{xxy} + b\sqrt[3]{xyy} = c$, atque aequatio rationalis hinc nata erit:

$$a^3 xxy + b^3 xyy = c^3 - 3c \cdot abxy,$$

et spatium quaesitum inter curuam et suas assymtotas contentum erit $= \frac{3cc}{2ab}$. 2°. Sit $\mu = 3$ et $\nu = 1$, ideoque $\lambda = 4$, vt aequatio pro curuis sit $a\sqrt[4]{x^3 y} + b\sqrt[4]{xy^3} = c$, quae ad rationalem perducta sit:

$$a^4 x^3 y + b^4 xy^3 = c^4 - 4c^2 abxy + 2a^2 b^2 x^2 y^2,$$

et spatium quaesitum hinc erit $= \frac{cc}{ab}$. 3°. Sit $\mu = 3$ et $\nu = 2$, ideoque $\lambda = 5$, vt aequatio sit

$$a\sqrt[5]{x^3 yy} + b\sqrt[5]{xxy^3} = c,$$

quae ad rationalem perducta sit:

$$a^5 x^3 yy + b^5 xxy^3 = c^5 - 5c^3 abxy + 5c a^2 b^2 x^2 y^2,$$

et spatium quaesitum $= \frac{5cc}{2ab}$. 4°. Sit $\mu = 4$ et $\nu = 1$, ideoque $\lambda = 5$, erit aequatio:

a

$$a\sqrt[5]{x^4 y} + b\sqrt[5]{x y^4} = c,$$

et rationaliter

$$a^5 x^4 y + b^5 x y^4 = c^5 - 5c^3 a b x y + 5 c a^2 b^2 x^2 y^2,$$

et ſpatium $= \frac{5cc}{6ab}$. 5°. Sit $\mu = 5$, $\nu = 1$ et $\lambda = 6$, erit aequatio pro curuis: $a\sqrt[6]{x^5 y} + b\sqrt[6]{x y^5} = c$, et rationalis:

$$a^6 x^5 y + b^6 x y^5 = c^6 - 6c^4 a b x y + 9 c c a^2 b^2 x^2 y^2 - 2 a^3 b^3 x^3 y^3,$$

et ſpatium $= \frac{3cc}{4ab}$.

§. 37. Quanquam autem ex aequatione irrationali aream facile definire licuit: tamen ſi aequatio rationalis proponeretur, vix vlla via pateret ex ea aream inueſtigandi, quandoquidem ad hoc requireretur reſolutio aequationum cuiuſuis gradus. Interim tamen, quoties talem aequationem reſoluere licet, ex ea area haud difficulter elicietur, id quod vnico exemplo oſtendiſſe ſufficiet.

Exemplum.

Si proponatur linea hyperbolica tertii ordinis, ſub hac aequatione:

$$x x y + x y y = 1 - 3 x y,$$

contenta, eius aream inter aſſymtotas contentam inueſtigare?

§. 38. Primo igitur ex hac aequatione valor applicatae y per abſciſſam x expreſſus inueſtigetur, qui erit:

$$y = -\frac{(x+3)}{2} \pm \sqrt{\left(\frac{(x+3)^2}{4} + \frac{1}{x}\right)},$$

quae ſi comparetur cum ſpeciebus linearum tertii ordinis a *Newtono* enumeratis, deprehenditur pertinere ad eius ſpeciem vigeſimam ſecundam. Scilicet ſi recta ACB ſit axis abſciſſarum et C initium, tum vero per C ducatur normalis DE, praete-

rea

rea vero capiantur interualla $CF = CG = 3$, tum tres rectae ACB, DCE et HFGI, erunt tres assymtotae curuae in nostra aequatione contentae, quae ergo ex tribus Hyperbolis erit composita, quarum prima ad continetur intra angulum ACD, secunda eb intra angulum EFH, ac tertia ib intra angulum IGB. Praeterea dabitur diameter KCL, angulum rectum ACD bisecans, ita vt portiones curuarum vtrinque sint inter se aequales, Imprimis autem hic notandum est, ad has curuas pertinere punctum coniugatum O, in ipso diametro KL, ad distantiam $CO = \sqrt{2}$ situm. Deinde notasse iuuabit si per O ex G producatur recta GM, hanc fore diametrum obliquangulum nostrae curuae, scilicet, si posita abscissa $AX = x$ applicata YX producatur, vt inferiorem Hyperbolam secet in Y', tum interuallum YY' ab hoc diametro GM in V bifariam secabitur, ita vt vbique sit $VY = VY'$. Nam euidens est, hanc rectam GM ipsam CF bisecare in S, sicque fore $CS = \frac{1}{2} GC$: erit ergo etiam Tab. II. Fig. 5.

$$XV = \tfrac{1}{2} GX = \frac{x+3}{2},$$

cui si addatur $XY = y$, ex valore supra inuento prodit:

$$VY = y + \frac{x+3}{2} = \sqrt{\left(\frac{(x+3)^2}{4} + \frac{1}{x}\right)},$$

quae formula cum sit radicalis, sequitur ex altera parte fore pariter $VY' = VY$, ideoque rectam GV diametrum. Caeterum notatu dignum hic occurrit punctum coniugatum O esse centrum grauitatis trianguli $F'CG$, tum vero si haec recta assymptotam GFH secet in puncto X', erit etiam $VX = VX'$, sicque etiam $X'Y' = y$; vnde patet, aream, quam curua ad intra suas assymptotas includit, aequalem fore areae intra assymptotas HF et FE suamque curuam he comprehensae. Simili modo recta ex F per O producta etiam erit diameter obliquangula bisecans omnes ordinatas axi AB parallelas.

 §. 39.

§. 39. Ex valore autem pro applicata y inuento inuestigemus aream curuae indefinitam $D\,d\,C\,x\,y$, cuius elementum est

$$y\,\partial x = -\tfrac{\partial x}{2}(x+3) \pm \tfrac{\partial x}{2}\sqrt{(x\,x+6\,x+9+\tfrac{4}{x})},$$

vbi partis prioris integrale manifesto est $-\frac{1}{4}x\,x - \frac{3}{2}x$; partis vero posterioris integratio merito maxime ardua videretur, nisi commode eueniret vt formula post signum radicale factorem habeat quadratum. Est enim

$$x\,x+6\,x+9+\tfrac{4}{x} = \tfrac{x^3+6\,x\,x+9\,x+4}{x} = \tfrac{x+4}{x}(x+1)^2,$$

vnde pars areae posterior ita exprimetur vt sit

$$\pm\tfrac{1}{2}(x+1)\,\partial x\sqrt{\tfrac{x+4}{x}}.$$

§. 40. Ad hanc formulam integrandam ponamus $x = v\,v$, ac integralis pars posterior erit

$$\pm\int\partial v\,(v\,v+1)\sqrt{(v\,v+4)}.$$

Sit nunc $\sqrt{(v\,v+4)} = v+2\,t$, eritque $v = \frac{1-t\,t}{t}$, ideoque

$$\sqrt{(v\,v+4)} = \tfrac{1+t\,t}{t} \text{ et } v\,v+1 = \tfrac{1-t\,t+t^4}{t\,t},$$

ac denique $\partial v = -\frac{\partial t\,(1+t\,t)}{t\,t}$, quibus substitutis formula proposita induit hanc formam:

$$-\int\tfrac{(1+t\,t)^2(1-t\,t+t^4)\partial t}{t^5} = -\int\tfrac{\partial t}{t^5}(1+t\,t+t^6+t^8),$$

cuius ergo integrale est

$$\tfrac{1}{4\,t^4}+\tfrac{1}{2\,t\,t}-\tfrac{1}{2}\,t\,t-\tfrac{1}{4}\,t^4 = \tfrac{1+2\,t\,t-2\,t^6-t^8}{4\,t^4} = \tfrac{(1-t\,t)(1+t\,t)^3}{4\,t^4},$$

et restitutis valoribus $\dfrac{v\,(v\,v+4)^{\frac{3}{2}}}{4}$, atque adeo per x erit haec pars: $\dfrac{(x+4)^{\frac{3}{2}}\sqrt{x}}{4}$, hincque area quaesita erit:

$\int y\,\partial x$

$$\int y \partial x = -\tfrac{1}{4}xx - \tfrac{3}{2}x + \frac{(x+4)^{\frac{3}{2}}\sqrt{x}}{4},$$

vbi constantis additione non est opus, quia haec area sponte euanescit casu $x = 0$. Faciamus nunc $x = \infty$, et cum sit

$$\sqrt{(xx + 4x)} = x + 2 - \tfrac{2}{x},$$

haec area erit

$$= -\tfrac{1}{4}xx - \tfrac{3}{2}x + \frac{(x + 2 - \frac{2}{x})(x + 4)}{4},$$

quod euolutum et x infinitum positum dat aream $= \tfrac{3}{2}$, prorsus vti supra assignauimus, ob $a = b = c = 1$. Hinc autem facile intelligitur talem calculum pro curuis altioribus neutiquam institui posse.