



1794

Exercitatio analytica

Leonhard Euler

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EXERCITATIO ANALYTICA.

Auctore

L. E V L E R O.

Conuent. exhib. die 3 Octobr. 1776.

§. 1.

Consideranti productum infinitum cosinum cuiusque anguli exprimens, quod est

$$\cos. \frac{\pi}{2n} = \left(1 - \frac{1}{n^2}\right) \left(1 - \frac{1}{9n^2}\right) \left(1 - \frac{1}{25n^2}\right) \left(1 - \frac{1}{49n^2}\right) \text{ etc.}$$

in mentem venit methodum inuestigare, cuius ope vicissim ex indole istius producti eius valor, quem nouimus esse $= \cos. \frac{\pi}{2n}$, erui queat, in quo negotio plura se obtulere artificia, quorum explicationem Geometris haud ingrattam fore confido.

§. 2. Pono igitur

$$S = \left(1 - \frac{1}{n^2}\right) \left(1 - \frac{1}{9n^2}\right) \left(1 - \frac{1}{25n^2}\right) \text{ etc.}$$

et sumtis logarithmis prodit mihi:

$$lS = l\left(1 - \frac{1}{n^2}\right) + l\left(1 - \frac{1}{9n^2}\right) + l\left(1 - \frac{1}{25n^2}\right) + \text{ etc.}$$

et cum sit

$$l\left(1 - \frac{1}{x}\right) = -\frac{1}{x} - \frac{1}{2x^2} - \frac{1}{3x^3} - \frac{1}{4x^4} - \text{ etc.}$$

erit his seriebus ordine dispositis signisque mutatis:

$$\begin{aligned}
 -1S &= \frac{1}{n \cdot n} + \frac{1}{2 \cdot n^2} + \frac{1}{3 \cdot n^3} + \frac{1}{4 \cdot n^4} + \text{etc.} \\
 &+ \frac{1}{9 \cdot n \cdot n} + \frac{1}{2 \cdot 9^2 \cdot n^2} + \frac{1}{3 \cdot 9^3 \cdot n^3} + \frac{1}{4 \cdot 9^4 \cdot n^4} + \text{etc.} \\
 &+ \frac{1}{25 \cdot n \cdot n} + \frac{1}{2 \cdot 25^2 \cdot n^2} + \frac{1}{3 \cdot 25^3 \cdot n^3} + \frac{1}{4 \cdot 25^4 \cdot n^4} + \text{etc.} \\
 &+ \frac{1}{49 \cdot n \cdot n} + \frac{1}{2 \cdot 49^2 \cdot n^2} + \frac{1}{3 \cdot 49^3 \cdot n^3} + \frac{1}{4 \cdot 49^4 \cdot n^4} + \text{etc.} \\
 &\text{etc.}
 \end{aligned}$$

§. 3. Quod si iam singulas columnas verticales in ordinem disponamus, sequentes series pro $-1S$ obtinebimus:

$$\begin{aligned}
 -1S &= \frac{1}{n \cdot n} \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \text{etc.} \right) \\
 &+ \frac{1}{2 \cdot n^2} \left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \text{etc.} \right) \\
 &+ \frac{1}{3 \cdot n^3} \left(1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \text{etc.} \right) \\
 &+ \frac{1}{4 \cdot n^4} \left(1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \frac{1}{9^8} + \text{etc.} \right) \\
 &\text{etc.}
 \end{aligned}$$

Sicque negotium perductum est ad summationem serierum potestatum parium progressionis harmonicae $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \text{etc.}$

§. 4. Ostendi autem olim, posito breuitatis gratia $\frac{1}{n} = g$, si harum potestatum summae repraesententur sequenti modo:

$$\begin{aligned}
 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \text{etc.} &= A g^2, \\
 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \text{etc.} &= B g^4, \\
 1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \text{etc.} &= C g^6, \\
 &\text{etc.}
 \end{aligned}$$

primo esse $A = \frac{1}{3}$, tum vero litteras reliquas sequenti modo per praecedentes determinari:

$$\begin{aligned}
 B &= \frac{2}{3} A^2, \quad C = \frac{2}{3} \cdot 2 AB, \quad D = \frac{2}{7} (2 AC + BB), \\
 E &= \frac{2}{9} (2AD + 2BC), \quad F = \frac{2}{11} (2AE + 2BD + CC), \text{ etc.}
 \end{aligned}$$

cuius

cuius veritas simul ex pulcherrimo consensu huius Analyseos
 clucebit.

§. 5. His igitur valoribus substitutis nanciscimur hanc
 seriem:

$$-lS = \frac{A \rho^2}{n^2} + \frac{1}{2} \cdot \frac{B \rho^4}{n^4} + \frac{1}{3} \cdot \frac{C \rho^6}{n^6} + \frac{1}{4} \cdot \frac{D \rho^8}{n^8} + \text{etc.}$$

Quod si igitur ponamus $\frac{\rho}{n} = x$, vt fit $x = \frac{\pi}{2n}$, ista series hanc
 induet formam:

$$-lS = A x x + \frac{1}{2} B x^4 + \frac{1}{3} C x^6 + \frac{1}{4} D x^8 + \text{etc.}$$

Vt fractiones $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc. abigamus, differentiemus, ac facta
 diuisione per $2 \partial x$ consequemur.

$$-\frac{\partial S}{2S \partial x} = A x + B x^3 + C x^5 + D x^7 + \text{etc.}$$

§. 6. Statuamus hic breuitatis gratia $-\frac{\partial S}{2S \partial x} = t$, vt
 habeamus;

$$t = A x + B x^3 + C x^5 + D x^7 + \text{etc.}$$

vnde sumtis quadratis orietur haec series:

$$t t = A^2 x x + 2 A B x^4 + 2 A C x^6 + 2 A D x^8 + 2 A E x^{10} + \text{etc.}$$

$$+ B B \quad + 2 B C \quad + 2 B D \quad + \text{etc.}$$

$$+ C C \quad + \text{etc.}$$

sicque iam pro quauis potestate ipsius x eas nacti sumus for-
 mulas, quibus determinatio litterarum A , B , C , D , contine-
 tur: desunt tantum coëfficientes illi $\frac{2}{3}$, $\frac{2}{5}$, $\frac{2}{7}$, etc.

§. 7. Hos autem coëfficientes. introducemus integran-
 do, postquam per $2 \partial x$ multiplicauerimus. Reperietur enim

$$2 / t t \partial x = \frac{2}{3} A^2 x^3 + \frac{2}{5} \cdot 2 A B x^5 + \frac{2}{7} (2 A C + B B) x^7$$

$$+ \frac{2}{9} (2 A D + 2 B C) x^9 + \frac{2}{11} (2 A E + 2 B D + C C) x^{11} + \text{etc.}$$

Cum nunc fit

$$\frac{2}{3} A^2 = B, \quad \frac{2}{5} \cdot 2 A B = C, \quad \frac{2}{7} (2 A C + B B) = D, \quad \text{etc.}$$

his

his valoribus restitutis perueniemus ad hanc seriem:

$$2 \int t t \partial x = B x^3 + C x^5 + D x^7 + E x^9 + \text{etc.} \quad ($$

§. 8. Cum igitur ante habuiffemus hanc seriem:

$$t = A x + B x^3 + C x^5 + D x^7 + \text{etc.}$$

hinc manifesto fuit ista aequatio:

$$t = A x + 2 \int t t \partial x,$$

quae differentiata dat

$$\partial t = A \partial x + 2 t t \partial x = \frac{1}{2} \partial x + 2 t t \partial x, \text{ ob } A = \frac{1}{2}.$$

Hinc ergo habebimus $2 \partial t = \partial x (1 + 4 t t)$, vnde fit $\partial x = \frac{2 \partial t}{1 + 4 t t}$, cuius integrale in promptu est, scilicet $x = A \text{ tang. } 2 t$, vbi constantis adiectione non est opus, quandoquidem posito $x = 0$ t iam sponte euanescit. Hac ergo aequatione inuenta, si quantitas x vt angulus spectetur, viciffim erit $2 t = \text{tang. } x$. Erat vero $t = -\frac{\partial S}{2 S \partial x}$, vnde colligitur haec aequatio:

$$-\frac{\partial S}{S \partial x} = \text{tang. } x, \text{ ideoque } -\frac{\partial S}{S} = \frac{\partial x \text{ fin. } x}{\text{cof. } x}.$$

§. 9. Cum igitur fit $\partial x \text{ fin. } x = -\partial \text{ cof. } x$, erit $\frac{\partial S}{S} = \frac{\partial \text{ cof. } x}{\text{cof. } x}$, hincque integrando $l S = l \text{ cof. } x + C$, quae constans inde debet definiri, vt posito $x = 0$ fiat $l S = 0$. Hinc ergo erit $C = 0$, ita vt fit $l S = l \text{ cof. } x$, ideoque ad numeros progrediendo fiet $S = \text{cof. } x$.

§. 10. Posueramus autem $x = \frac{\pi}{2 n}$, vnde manifesto valor quaesitus S prodit $S = \text{cof. } \frac{\pi}{2 n}$, prorsus vti iam ante constabat. Haec igitur Analyfis egregie confirmat illam relationem inter litteras A, B, C, D , quam aliunde in calculum introduxi.