1794

Exercitatio analytica

Leonhard Euler

EXERCITATIO ANALYTICA.

Auctore

L. EVLERO.

Conuent. exhib. die 3 *Octobr.* 1776.

§. 1.

Consideranti productum infinitum cosinum cuiusque anguli exprimens, quod est

$$\cos.\frac{\pi}{2n} = \left(1-\frac{1}{nn}\right)\left(1-\frac{1}{9nn}\right)\left(1-\frac{1}{25nn}\right)\left(1-\frac{1}{49nn}\right) \text{ etc.}$$

in mentem venit methodum inuestigare, cuius ope vicissim ex indole istius producti eius valor, quem nouimus esse $=\cos.\frac{\pi}{2n}$, erui queat, in quo negotio plura se obtulere artificia, quorum explicationem Geometris haud ingratam fore confido.

§. 2. Pono igitur

$$S = \left(1-\frac{1}{nn}\right)\left(1-\frac{1}{9nn}\right)\left(1-\frac{1}{25nn}\right) \text{ etc.}$$

et sumtis logarithmis prodit mihi:

$$lS = l\left(1-\frac{1}{nn}\right) + l\left(1-\frac{1}{9nn}\right) + l\left(1-\frac{1}{25nn}\right) + \text{etc.}$$

et cum sit

$$l\left(1-\frac{1}{x}\right) = -\frac{1}{x} - \frac{1}{2xx} - \frac{1}{3x^3} - \frac{1}{4x^4} - \text{etc.}$$

erit his seriebus ordine dispositis signisque mutatis:

$$-lS = \frac{1}{nn} + \frac{1}{2n^4} + \frac{1}{3n^6} + \frac{1}{4n^8} + \text{etc.}$$
$$+ \frac{1}{9nn} + \frac{1}{2.9^2n^4} + \frac{1}{3.9^3n^6} + \frac{1}{4.9^4n^8} + \text{etc.}$$
$$+ \frac{1}{25.nn} + \frac{1}{2.25^2.n^4} + \frac{1}{3.25^3.n^6} + \frac{1}{4.25^4.n^8} + \text{etc.}$$
$$+ \frac{1}{49.nn} + \frac{1}{2.49^2.n^4} + \frac{1}{3.49^3.n^6} + \frac{1}{4.49^4.n^8} + \text{etc.}$$
etc.

§. 3. Quodſi iam ſingulas columnas verticales in ordinem diſponamus, ſequentes ſeries pro $-lS$ obtinebimus:

$$-lS = \frac{1}{nn}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \text{etc.}\right)$$
$$+ \frac{1}{2.n^4}\left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \text{etc.}\right)$$
$$+ \frac{1}{3.n^6}\left(1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \text{etc.}\right)$$
$$+ \frac{1}{4.n^8}\left(1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \frac{1}{9^8} + \text{etc.}\right)$$
etc.

Sicque negotium perductum eſt ad ſummationem ſerierum poteſtatum parium progreſſionis harmonicae $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}$, etc.

§. 4. Oſtendi autem olim, poſito breuitatis gratia $\frac{\pi}{2} = \varrho$, ſi harum poteſtatum ſummae repraeſententur ſequenti modo:

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \text{etc.} = A\varrho^2,$$
$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \text{etc.} = B\varrho^4,$$
$$1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \text{etc.} = C\varrho^6,$$
etc.

primo eſſe $A = \frac{1}{2}$, tum vero litteras reliquas ſequenti modo per praecedentes determinari:

$$B = \tfrac{2}{3}A^2,\ C = \tfrac{2}{5}.2AB,\ D = \tfrac{2}{7}(2AC + BB),$$
$$E = \tfrac{2}{9}(2AD + 2BC),\ F = \tfrac{2}{11}(2AE + 2BD + CC),\ \text{etc.}$$

cuius

cuius veritas simul ex pulcherrimo consensu huius Analyseos elucebit.

§. 5. His igitur valoribus substitutis nanciscimur hanc seriem:

$$-lS = \frac{A\rho^2}{nn} + \frac{1}{2}\cdot\frac{B\rho^4}{n^4} + \frac{1}{3}\cdot\frac{C\rho^6}{n^6} + \frac{1}{4}\cdot\frac{D\rho^8}{n^8} + \text{etc.}$$

Quod si igitur ponamus $\frac{\rho}{n} = x$, vt sit $x = \frac{\pi}{2n}$, ista series hanc induet formam:

$$-lS = Axx + \tfrac{1}{2}Bx^4 + \tfrac{1}{3}Cx^6 + \tfrac{1}{4}Dx^8 + \text{etc.}$$

Vt fractiones $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc. abigamus, differentiemus, ac facta diuisione per $2\partial x$ consequemur.

$$-\frac{\partial S}{2S\partial x} = Ax + Bx^3 + Cx^5 + Dx^7 + \text{etc.}$$

§. 6. Statuamus hic breuitatis gratia $-\frac{\partial S}{2S\partial x} = t$, vt habeamus;

$$t = Ax + Bx^3 + Cx^5 + Dx^7 + \text{etc.}$$

vnde sumtis quadratis orietur haec series:

$$\begin{aligned} tt = A^2xx + 2ABx^4 + 2ACx^6 &+ 2ADx^8 + 2AEx^{10} + \text{etc.} \\ + BB &+ 2BC \quad + 2BD \quad + \text{etc.} \\ &\qquad\quad + CC \quad + \text{etc.} \end{aligned}$$

sicque iam pro quauis potestate ipsius x eas nacti sumus formulas, quibus determinatio litterarum A, B, C, D, continetur: desunt tantum coëfficientes illi $\frac{2}{3}$, $\frac{2}{5}$, $\frac{2}{7}$, etc.

§. 7. Hos autem coëfficientes introducemus integrando, postquam per $2\partial x$ multiplicauerimus. Reperietur enim

$$\begin{aligned} 2\int tt\partial x = \tfrac{2}{3}A^2x^3 + \tfrac{2}{5}\cdot 2ABx^5 + \tfrac{2}{7}(2AC + BB)x^7 \\ + \tfrac{2}{9}(2AD+2BC)x^9 + \tfrac{2}{11}(2AE+2BD+CC)x^{11} + \text{etc.} \end{aligned}$$

Cum nunc sit

$$\tfrac{2}{3}A^2 = B,\ \tfrac{2}{5}\cdot 2AB = C,\ \tfrac{2}{7}(2AC + BB) = D,\ \text{etc.}$$

his

his valoribus restitutis perueniemus ad hanc seriem:

$$2\int tt\,\partial x = B x^3 + C x^5 + D x^7 + E x^9 + \text{etc.}$$

§. 8. Cum igitur ante habuissemus hanc seriem:

$$t = A x + B x^3 + C x^5 + D x^7 + \text{etc.}$$

hinc manifesto fluit ista aequatio:

$$t = A x + 2\int tt\,\partial x,$$

quae differentiata dat

$$\partial t = A\,\partial x + 2\,tt\,\partial x = \tfrac{1}{2}\,\partial x + 2\,tt\,\partial x, \text{ ob } A = \tfrac{1}{2}.$$

Hinc ergo habebimus $2\partial t = \partial x\,(1+4tt)$, vnde fit $\partial x = \frac{2\partial t}{1+4tt}$, cuius integrale in promptu est, scilicet $x = A$ tang. $2t$, vbi constantis adiectione non est opus, quandoquidem posito $x = 0$ t iam sponte euanescit. Hac ergo aequatione inuenta, si quantitas x vt angulus spectetur, vicissim erit $2t =$ tang. x. Erat vero $t = -\frac{\partial S}{2S\partial x}$, vnde colligitur haec aequatio:

$$-\frac{\partial S}{S\partial x} = \text{tang. } x, \text{ ideoque } -\frac{\partial S}{S} = \frac{\partial x \text{ sin. } x}{\text{cos. } x}.$$

§. 9. Cum igitur sit ∂x sin. $x = -\partial.$ cos. x, erit $\frac{\partial S}{S} = \frac{\partial.\text{ cos. } x}{\text{cos. } x}$, hincque integrando $l\,S = l$ cos. $x + C$, quae constans inde debet definiri, vt posito $x = 0$ fiat $l\,S = 0$. Hinc ergo erit $C = 0$, ita vt sit $l\,S = l$ cos. x, ideoque ad numeros progrediendo fiet $S =$ cos. x.

§. 10. Posueramus autem $x = \frac{\pi}{2n}$, vnde manifesto valor quaesitus S prodit $S = \text{cos. } \frac{\pi}{2n}$, prorsus vti iam ante constabat. Haec igitur Analysis egregie confirmat illam relationem inter litteras A, B, C, D, quam aliunde in calculum introduxi.