De vero valore formulae integralis $\int \partial x \cdot (\log(1/x))^n$ a termino $x=0$ usque ad terminum $x=1$ extensae

Leonhard Euler

DE VERO VALORE FORMVLAE INTEGRALIS

$\int \partial x \, (l \frac{1}{x})^n$

A TERMINO $x = 0$ VSQVE AD TERMINVM $x = 1$ EXTENSAE.

Auctore
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§. 1.

Cum haec formula exprimat aream curuae transcendentis, cuius abscissae x respondet applicata $= (l \frac{1}{x})^n$; quaestio huc redit, vt eadem area, quatenus abscissae $x = 1$ conuenit, vel per numeros absolutos, vel saltem per quadraturas curuarum algebraicarum exhibeatur. Ac primo quidem manifestum est, quoties exponens n fuerit numerus integer, hanc formulam integralem sistere terminum generalem progressionis hypergeometricae, cum sit

$$\int \partial x \, (l \tfrac{1}{x})^0 = 1,$$
$$\int \partial x \, (l \tfrac{1}{x})^1 = 1$$
$$\int \partial x \, (l \tfrac{1}{x})^2 = 1 . 2,$$
$$\int \partial x \, (l \tfrac{1}{x})^3 = 1 . 2 . 3$$
$$\int \partial x \, (l \tfrac{1}{x})^4 = 1 . 2 . 3 . 4,$$

atque adeo in genere

$$\int \partial x \, (l \tfrac{1}{x})^n = 1 . 2 . 3 . 4 \ldots\ldots n$$

quem autem valorem cognoscere non datur, nisi exponens n fuerit

fuerit numerus integer poſitiuus; praeterea vero ſi exponens n fuerit numerus integer negatiuus, ex indole ſeriei hyper- geometricae facile perſpicitur valores noſtrae formulae omnes fieri infinite magnos. Quaeſtio igitur hic potiſſimum complec- tetur caſus, quibus exponens n eſt numerus fractus, quibus vtique valor noſtrae formulae neutiquam per numeros abſolu- tos aſſignari poteſt, ſed potius quadraturas curuarum algebrai- carum eo altiorum ordinum poſtulat, quo maior fuerit de- nominator fractionis pro n aſſumtae, quemadmodum iam olim fuſius monſtraui. Nuper autem ſe mihi obtulit alia methodu eosdem valores transcendentes inueſtigandi, quam ergo hi explicare conſtitui; cum inde haud contemnenda increment in Analyſin redundare videantur.

§. 2. Ante omnia igitur ſtatuamus breuitatis grati $l\frac{1}{x}=u$, vt ſit $\partial u = -\frac{\partial x}{x}$, ideoque $\partial x = -x\,\partial u$. Hinc ſt tim inſignes reductiones ope lemmatis vulgatiſſimi, quo e $\int P\,\partial Q = PQ - \int Q\,\partial P$, deriuari poſſunt. Sumto enim $P=$ et $\partial Q = \partial x$, ob $\partial P = n u^{n-1}\partial u = -\frac{n u^{n-1}\partial x}{x}$ et $Q=$ hoc Lemma nobis praebet

$$\int u^n \partial x = x u^n + n \int u^{n-1}\partial x.$$

Deinde cum ſit $u^n \partial x = -x^n \partial u$, ſi hic capiatur $P = -x$ $\partial Q = u^n \partial u$, ob $\partial P = -\partial x$ et $Q = \frac{1}{n+1} u^{n+1}$, habebimus

$$\int u^n \partial x = -\frac{1}{n+1} x u^{n+1} + \frac{1}{n+1}\int u^{n+1}\partial x.$$

Quare, quoniam haec integralia ita capi debent, vt euaneſcant poſito $x=0$, tum vero ſtatui debet $x=1$, notum eſt tum membra abſoluta in his reductionibus in nihilum abire, ita v pro hoc caſu, de quo hic vnice agitur, fit $\int u^n \partial x = n\int u^{n-1}\partial x$ tum vero etiam $\int u^n \partial x = \frac{1}{n+1}\int u^{n+1}\partial x$, quae quidem po ſterior reductio ſponte ex priori fluit.

§. 3

§. 3. Quod autem formula $x u^n$ casu $x = 0$ semper euanescat, vulgo non satis directe demonstrari solet, atque adeo dubium videri queat, propterea quod posito $x = 0$ fiat $u^n = \infty$; at vero haec veritas sequenti modo rigorose ostendi potest: Namque pro casu $x = 0$ statuamus $x u^n = v$, ita vt valor huius litterae v nobis sit explorandus, quem ergo ita per fractionem repraesentemus: $v = \frac{x}{u^{-n}}$, cuius tam numerator quam denominator casu $x = 0$ euanescit, vnde per regulam communem tam loco numeratoris quam denominatoris eorum differentialia scribantur, et quia valor ipsius v idem prodire debet, erit quoque

$$v = \frac{\partial x}{-n u^{-n-1} \partial u} = \frac{+x}{n u^{-n-1}} \left(\text{ob } \partial u = -\frac{\partial x}{x}\right).$$

Cum igitur ex priore valore sit $v = x u^n$, ex posteriori vero $v = \frac{1}{n} x u^{n+1}$, inde fiet $v^{n+1} = x^{n+1} u^{n(n+1)}$, hinc vero

$$v^n = \left(\frac{x}{n}\right)^n u^{n(n+1)},$$

quorum valorum ille per hunc diuisus dabit $v = n^n x$, haecque expressio pariter verum valorem ipsius v pro casu $x = 0$ exhibere debet, hic autem posito $x = 0$ manifesto fit $v = 0$.

§. 4. Quoniam nostra inuestigatio hic potissimum ad casus, quibus exponens n est fractio, restringitur, ope reductionis $\int u^n \partial x = n \int u^{n-1} \partial x$ omnes fractiones loco n assumtae, quantumuis fuerint magnae, continuo vnitate diminui ideoque tandem adeo infra vnitatem deprimi poterunt, ita vt intra limites 0 et 1 contineantur. Deinde vero ope alterius reductionis: $\int u^n \partial x = \frac{1}{n+1} \int u^{n+1} \partial x$, si forte exponens n fuerit fractio negatiua, tandem eius valor pariter ad fractionem inter limites 0 et 1 redigi poterit; vnde nobis hoc loco sufficiet eos

eos tantum casus euoluisse, quibus fractiones pro n assumtae intra limites o et 1 consistunt; haeque fractiones commode in varias classes distribuuntur, prouti denominatores harum fractionum fuerint vel 2, vel 3, vel 4, vel 5, etc.

§. 5. Cum nuper series, quae ex vnciis potestatum Binomii formantur, essem contemplatus, ostendi, si ponatur $(1+z)^m = 1 + A z + B z^2 + C z^3$, etc. tum vero etiam $(1+z)^n = 1 + \alpha z + \beta z^2 + \gamma z^3$, etc. tum summam huius seriei $1 + A\alpha + B\beta + C\gamma +$ etc. $= s$, ita exprimi posse, vt sit $s = \int \frac{u^{m+n}\,\partial x}{\int u^m \partial x . \int u^n \partial x}$, quae ergo summa per valores formulae integralis propositae definitur; deinde vero etiam monstraui, eandem summam quoque hoc modo exprimi posse:

$$s = \frac{m+n}{m n \int x^{m-1} \partial x\,(1-x)^{n-1}},$$

vnde ergo sequitur semper fore

$$\frac{m+n}{m n} \int u^m \partial x . \int u^n \partial x = \int u^{m+n} \partial x . \int x^{m-1} \partial x\,(1-x)^{n-1}$$

siquidem singula haec integralia a termino $x = 0$ vsque ad terminum $x = 1$ extendantur.

§. 6. Quoniam autem praesens nostrum institutum circa fractiones, easque vnitate minores, versatur, ponamus in genere $m = \frac{\mu}{\lambda}$ et $n = \frac{\nu}{\lambda}$, ita vt sit

$$\frac{\lambda(\mu+\nu)}{\mu\,\nu} \int u^{\frac{\mu}{\lambda}} \partial x . \int u^{\frac{\nu}{\lambda}} \partial x = \int u^{\frac{\mu+\nu}{\lambda}} \partial x \int x^{\frac{\mu-\lambda}{\lambda}} \partial x\,(1-x)^{\frac{\nu-\lambda}{\lambda}}$$

Nunc vero vt postremam formulam integralem ab exponentibus fractis liberemus, statuamus $x = z^\lambda$, et ob $\partial x = \lambda z^{\lambda-1} \partial z$ erit

$$\int x^{\frac{\mu-\lambda}{\lambda}} \partial x\,(1-x)^{\frac{\nu-\lambda}{\lambda}} = \lambda \int z^{\mu-1} \partial z\,(1-z^\lambda)^{\frac{\nu-\lambda}{\lambda}},$$

quae

quae formula, ob $\nu - \lambda$, ita referri poteſt: $\lambda \int \frac{z^{\mu-1}\partial z}{\sqrt[\lambda]{(1-z^\lambda)^{\lambda-\nu}}}$, quod integrale pariter a $z=0$ vsque ad $z=1$ eſt extendendum. Facta igitur hac ſubſtitutione aequatio noſtra principalis ita ſe habebit:

$$\tfrac{\mu+\nu}{\mu\nu}\int \partial x \sqrt[\lambda]{u^\mu} \cdot \int \partial x \sqrt[\lambda]{u^\nu} = \int \partial x \sqrt[\lambda]{u^{\mu+\nu}} \int \frac{z^{\mu-1}\partial z}{\sqrt[\lambda]{(1-z^\lambda)^{\lambda-\nu}}},$$

vbi ambo numeri μ et ν perpetuo nobis erunt poſitiui et minores quam λ. Imprimis autem hic obſeruari meretur, caſu quo $\mu+\nu=\lambda$ poſtremum integrale ſemper ad quadraturam circuli ita reduci poſſe, vt ſit

$$\int \frac{z^{\mu-1}\partial z}{\sqrt[\lambda]{(1-z^\lambda)^{\mu}}} = \frac{\pi}{\lambda \operatorname{ſin.} \frac{\mu\pi}{\lambda}}.$$

§. 7. Ex hac iam aequatione principali haud difficulter valores formulae integralis propoſitae pro ſingulis denominatoribus λ elicientur, ſi modo litteris μ et ν quouis caſu omnes numeri denominatore λ minores ſucceſſiue tribuantur, tum enim plures formabuntur aequationes, ex quibus valores formularum $\int \partial x \sqrt[\lambda]{u^\mu}$ et $\int \partial x \sqrt[\lambda]{u^\nu}$ definiri poterunt. Quod autem ad formulam $\int \partial x \sqrt[\lambda]{u^{\mu+\nu}}$ attinet, quae nata eſt ex $\int u^{m+n}\partial x$, quando fuerit $m+n>1$, ſiue $\mu+\nu>\lambda$, quoniam vidimus eſſe $\int u^{m+n}\partial x = (m+n)\int u^{m+n-1}\partial x$, erit

$$\int \partial x \sqrt[\lambda]{u^{\mu+\nu}} = \tfrac{\mu+\nu}{\lambda}\int \partial x \sqrt[\lambda]{u^{\mu+\nu-\lambda}},$$

quae ergo formula valebit, quando $\mu+\nu>\lambda$. Denique vero omnes valores, qui ex poſtrema formula integrali

$$\int \frac{z^{\mu+1}\,\partial z}{\sqrt[\alpha]{(1-z^{\lambda})^{\lambda-\nu}}}$$

nafcuntur, tanquam cogniti fpectari poterunt, vnde eos litteris A, B, C, D, etc. indicabimus. His igitur praenotatis pro denominatore λ ordine numeros 2, 3, 4, 5, etc. accipiamus, ideoque fequentes cafus euoluamus, pro quibus in genere ob- feruaffe iuuabit, numeros μ et ν femper inter fe permutari poffe, ita vt fit

$$\int \frac{z^{\mu-1}\,\partial z}{\sqrt[\lambda]{(1-z^{\lambda})^{\lambda-\nu}}} = \int \frac{z^{\nu-1}\,\partial z}{\sqrt[\lambda]{(1-z^{\lambda})^{\lambda-\mu}}}.$$

I. Euolutio cafus, quo $\lambda = 2$.

§. 8. Pro hoc ergo cafu aequatio noftra principalis erit

$$\tfrac{\mu+\nu}{\mu\nu}\int \partial x\,\sqrt{u^{\mu}}\,.\int \partial x\,\sqrt{u^{\nu}} = \int \partial x\,\sqrt{u^{\mu+\nu}}\int \frac{z^{\mu-1}\,\partial z}{\sqrt{(1-zz)^{2-}}}$$

vbi cum loco μ et ν alios numeros praeter vnitatem accipe- re non liceat, pofito $\mu = 1$ et $\nu = 1$ pro formula poftrema vnica fpecies oritur $\int \frac{\partial z}{\sqrt{(1-zz)}}$, cuius valor, vti conftat, eft $= \frac{\pi}{2}$, quem autem ob analogiam fequentium cafuum littera A defignabimus. Hinc igitur cum fit $\mu + \nu = 2$, erit

$$\int \partial x\,\sqrt{uu} = \int u\,\partial x = 1,$$

aequatio autem principalis induet hanc formam:

$$2\int \partial x\,\sqrt{u}\,.\int \partial x\,\sqrt{u} = \tfrac{\pi}{2} = A,$$

vnde fit

$$\int \partial x\,\sqrt{u} = \sqrt{\tfrac{A}{2}} = \tfrac{1}{2}\sqrt{\pi}.$$

§. 9. Quoniam igitur inuenimus effe $\int u^{\frac{1}{2}}\,\partial x = \frac{1}{2}\sqrt{\pi}$, fi exponentem ipfius u continuo vnitate augeamus, per reductionem

ctionem supra ostensam, $\int u^n \partial x = n \int u^{n-1} \partial x$, impetrabimus sequentes valores:

$$\int u^{\frac{3}{2}} \partial x = \tfrac{1 \cdot 3}{2 \cdot 2} \sqrt{\pi},$$

$$\int u^{\frac{5}{2}} \partial x = \tfrac{1 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 2} \sqrt{\pi},$$

$$\int u^{\frac{7}{2}} \partial x = \tfrac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 2 \cdot 2} \sqrt{\pi},$$

et ita porro. Deinde vero regrediendo per alteram reductionem $\int u^n \partial x = \frac{1}{n+1} \int u^{n+1} \partial x$, reperiemus,

$$\int u^{-\frac{1}{2}} \partial x = \sqrt{\pi}; \text{ hincque porro}$$

$$\int u^{-\frac{3}{2}} \partial x = -2 \sqrt{\pi},$$

$$\int u^{-\frac{5}{2}} \partial x = + \tfrac{2 \cdot 2}{3} \sqrt{\pi},$$

$$\int u^{-\frac{7}{2}} \partial x = - \tfrac{2 \cdot 2 \cdot 2}{3 \cdot 5} \sqrt{\pi},$$

$$\int u^{-\frac{9}{2}} \partial x = \tfrac{2 \cdot 2 \cdot 2 \cdot 2}{3 \cdot 5 \cdot 7} \sqrt{\pi},$$

sicque valores nostrae formulae inuenimus pro omnibus fractionibus, quarum denominator est $= 2$.

Euolutio casus, quo $\lambda = 3$.

§. 10. Quoniam hic litterae μ et ν binos valores recipere possunt, scilicet 1 et 2, formula integralis postrema quatuor nobis suppeditat valores, quos sequenti modo indicemus:

$$\int \frac{\partial z}{\sqrt[3]{(1 - z^3)}} = A, \quad \int \frac{z \, \partial z}{\sqrt[3]{(1 - z^3)}} = B,$$

$$\int \frac{\partial z}{\sqrt[3]{(1 - z^3)^2}} = A', \quad \int \frac{z \, \partial z}{\sqrt[3]{(1 - z^3)^2}} = B'.$$

In

In harum formularum prima et quarta est $\mu+\nu=\lambda=3$ vnde per quadraturam circuli habebimus

$$A=\frac{\pi}{3\,\text{fin.}\frac{1}{3}\pi}=\frac{2\pi}{3\sqrt{3}} \text{ et } B'=\frac{\pi}{3\,\text{fin.}\frac{2}{3}\pi}=\frac{2\pi}{3\sqrt{3}},$$

vnde patet esse $B'=A$, quod etiam inde sequitur, quod litterae μ et ν sunt permutabiles. Praeterea vero notetur casu $\mu+\nu=3$ fore

$$\int\partial x\sqrt[3]{u^{\mu+\nu}}=\int u\,\partial x=1,$$

at vero casu $\mu+\nu=4$, erit

$$\int\partial x\sqrt[3]{u^4}=\int u^{\frac{4}{3}}\partial x=\tfrac{4}{3}\int\partial x\sqrt[3]{u}.$$

§. 11. His praemonitis omnes casus aequationis nostrae principalis ordine euoluamus sequenti modo:

I. Si $\mu=1$ et $\nu=2$, erit $\frac{3}{2}\int\partial x\sqrt[3]{u}\cdot\int\partial x\sqrt[3]{u^2}=A$.

II. Si $\mu=2$ et $\nu=2$, erit $\frac{4}{4}\int\partial x\sqrt[3]{u^2}\cdot\int\partial x\sqrt[3]{u^2}=\frac{4}{3}B\int\partial x\sqrt[3]{\ldots}$

III. Si $\mu=1$ et $\nu=1$, erit $2\int\partial x\sqrt[3]{u}\cdot\int\partial x\sqrt[3]{u}=A'\int\partial x\sqrt[3]{\ldots}$

IV. Si $\mu=2$ et $\nu=1$, erit $\frac{3}{2}\int\partial x\sqrt[3]{u^2}\cdot\int\partial x\sqrt[3]{u}=B'$.

Sicque quatuor nacti sumus aequationes pro determinandis binis valoribus incognitis, scilicet $\int\partial x\sqrt[3]{u}$ et $\int\partial x\sqrt[3]{u^2}$, quos ergo pluribus modis definire licebit, quandoquidem ad hoc duae tantum aequationes sufficiunt.

§. 12. Quo autem hic calculus facilior reddatur, statuamus breuitatis gratia $\int\partial x\sqrt[3]{u}=p$ et $\int\partial x\sqrt[3]{u^2}=q$, et combinemus primo aequationem I et II, quae erunt

$$\tfrac{3}{2}pq=A \text{ et } qq=\tfrac{4}{3}Bp,$$

quarum posterior dat $p=\frac{3qq}{4B}$, qui valor in priore substitutus dat

dat $\frac{9q^3}{8B} = A$, vnde reperitur $q = 2\sqrt[3]{\frac{AB}{9}}$, ex quo porro colligitur $p = \frac{3}{B}\sqrt[3]{\frac{A^2 B^2}{81}}$, ſiue etiam $p = \sqrt[3]{\frac{AA}{3B}}$, ſicque pro p et q reſtitutis valoribus iam nacti ſumus has determinationes:

$$\int \partial x \sqrt[3]{u} = \sqrt[3]{\frac{AA}{3B}} \text{ et } \int \partial x \sqrt[3]{u^2} = 2\sqrt[3]{\frac{AB}{9}}.$$

§. 13. Combinemus nunc primam aequationem cum tertia, et habebimus $\frac{3}{2} p q = A$ et $2pp = A' q$. Ex poſteriore fit $q = \frac{2pp}{A'}$, qui valor in priore ſubſtitutus dat $\frac{3p^3}{A'} = A$, vnde reperitur $p = \sqrt[3]{\frac{AA'}{3}}$, hincque

$$q = \frac{2}{A'}\sqrt[3]{\frac{A^2 A' A'}{9}} = 2\sqrt[3]{\frac{A^2}{9A'}},$$

ſicque haec combinatio nos perducit ad hos valores:

$$\int \partial x \sqrt[3]{u} = \sqrt[3]{\frac{AA'}{3}} \text{ et } \int \partial x \sqrt[3]{u^2} = 2\sqrt[3]{\frac{A^2}{9A'}}.$$

§. 14. Combinemus nunc quoque primam aequationem cum quarta, et habebimus $\frac{3}{2} p q = A$ et $\frac{3}{2} p q = B'$, vnde nihil aliud ſequitur, niſi $B' = A$, vti ante inuenimus. Combinemus igitur ſecundam cum tertia, et habebimus $qq = \frac{4}{3} B p$ et $2pp = A' q$, ex quarum poſteriore fit $q = \frac{2pp}{A'}$, qui valor in prima ſubſtitutus dat $\frac{4p^3}{A'A'} = \frac{4}{3} B$, vnde reperitur

$$p = \sqrt[3]{\frac{A'A'B}{3}}, \text{ ex quo fit}$$

$$q = 2\sqrt[3]{\frac{A'BB}{9}},$$

ſicque haec combinatio nobis dat hos valores:

$$\int \partial x \sqrt[3]{u} = \sqrt[3]{\frac{A'A'B}{3}} \text{ et } \int \partial x \sqrt[3]{u^2} = 2\sqrt[3]{\frac{A'BB}{9}}.$$

§. 15.

§. 15. Quoniam aequatio quarta cum prima prorſus conuenit, ſuperfluum foret, ſecundam vel tertiam cum quarta combinare, quoniam eas iam cum prima combinauimus. Sicque pro litteris p et q omnino ternos nacti ſumus valores, quos ita coniunctim ob oculos ponamus:

$$\int \partial x \sqrt[3]{u} = \sqrt[3]{\frac{AA}{3B}} = \sqrt[3]{\frac{A'A}{3}} = \sqrt[3]{\frac{A'A'B}{3}} \text{ et}$$

$$\int \partial x \sqrt[3]{u^2} = 2\sqrt[3]{\frac{AB}{9}} = 2\sqrt[3]{\frac{A^2}{9A'}} = 2\sqrt[3]{\frac{A'BB}{9}}.$$

Hinc igitur ſumtis cubis ſequentes nanciſcimur aequationes:

$$\frac{AA}{B} = AA' = A'A'B \text{ et}$$

$$AB = \frac{AA}{A'} = A'BB.$$

§. 16. At relatione inter hos diuerſos valores facta, omnes hae aequalitates ad vnicam hanc proprietatem reuocantur, qua eſt $A = A'B$. Subſtitutis igitur ipſis formulis integralibus conſequimur hanc veritatem maxime memorabilem:

$$\int \frac{\partial z}{\sqrt[3]{(1 - z^3)}} = \int \frac{\partial z}{\sqrt[3]{(1 - z^3)^2}} \cdot \int \frac{z \partial z}{\sqrt[3]{(1 - z^3)}},$$

et quia A per quadraturam circuli definitur, prodibit valor huius producti:

$$\int \frac{\partial z}{\sqrt[3]{(1 - z^3)^2}} \cdot \int \frac{z \partial z}{\sqrt[3]{(1 - z^3)}} = \frac{2\pi}{3\sqrt{3}},$$

vnde ſi alterius harum duarum formularum valor innoteſceret, ſimul alterius valor foret cognitus; hoc enim modo ex binis valoribus A et B bini reliqui A' et B' ita determinantur, vt ſit $A' = \frac{A}{B}$ et B' et A. Denique etiam operae pretium erit notaſſe hanc relationem

$$\int \partial x \sqrt[3]{u} \cdot \int \partial x \sqrt[3]{u^2} = \frac{2}{3} A = \frac{4\pi}{9\sqrt{3}}.$$

§. 17.

Euolutio casus quo $\lambda = 4$.

§. 17. Hic iam primo breuitatis gratia ponamus

$$\int \partial x \sqrt[4]{u^2} = q \text{ et } \int \partial x \sqrt[4]{u^3} = r;$$

praeterea vero designemus formulam integralem $\int \frac{z^{\mu-1}\partial z}{\sqrt[4]{(1-z^4)^{4-\nu}}}$, per characterem (μ, ν), quandoquidem iam vidimus litteras μ et ν inter se permutari posse. Deinde aequatio principalis hoc modo repraesentetur:

$$\int \partial x \sqrt[4]{u^\mu} . \int \partial x \sqrt[4]{u^\nu} = \frac{\mu\nu}{\mu+\nu} \int \partial x \sqrt[4]{u^{\mu+\nu}} (\mu, \nu);$$

vbi notetur si $\mu + \nu = \lambda = 4$, fore $\int \partial x \sqrt[4]{u^4} = 1$; sin autem $\mu + \nu = \lambda + \alpha = 4 + \alpha$, erit

$$\int \partial x \sqrt[4]{u^{4+\alpha}} = (1 + \tfrac{\alpha}{4}) \int u^{\frac{\alpha}{4}} \partial x = \frac{\mu+\nu}{4} \int \partial x \sqrt[4]{u^\alpha}.$$

§. 18. Tribuamus nunc litteris μ et ν successiue omnes valores minores quam 4, atque aequatio principalis nobis praebebit sequentes aequationes:

1°. Si $\begin{pmatrix}\mu = 1\\ \nu = 1\end{pmatrix}$, erit $pp = \frac{1}{2} q (1, 1)$, vnde fit $\frac{pp}{q} = \frac{1}{2}(1, 1) = A$.

2°. Si $\begin{pmatrix}\mu = 1\\ \nu = 2\end{pmatrix}$, erit $pq = \frac{2}{3} r (1, 2)$, vnde $\frac{pq}{r} = \frac{2}{3}(1, 2) = B$.

3°. Si $\begin{pmatrix}\mu = 1\\ \nu = 3\end{pmatrix}$, erit $pr = \frac{3}{4} (1, 3) = C$.

4°. Si $\begin{pmatrix}\mu = 2\\ \nu = 2\end{pmatrix}$, erit $qq = (2, 2) = D$.

5°. Si $\begin{pmatrix}\mu = 2\\ \nu = 3\end{pmatrix}$, erit $qr = \frac{6}{5} . \frac{5}{4} p (2, 3)$, vnde fit $\frac{qr}{p} = \frac{3}{2}(2, 3) = E$.

6°. Si $\begin{pmatrix}\mu = 3\\ \nu = 3\end{pmatrix}$, erit $rr = \frac{9}{6} . \frac{6}{4} q (3, 3)$, vnde fit $\frac{rr}{q} = \frac{9}{4}(3, 3) = F$.

§. 19. Hinc igitur nacti sumus sex aequationes, ex quibus tres nostras incognitas p, q et r definiri oportet, quod igitur pluribus modis fieri potest, siquidem ternae aequationes sufficiunt. Eligamus igitur eas, quae negotium facillime conficiunt, ac primo quidem quarta nobis statim dat $q = \sqrt{D}$, vnde ex prima elicimus $pp = A\sqrt{D}$, ideoque

$$p = \sqrt{A\sqrt{D}} = \sqrt[4]{AAD},$$

denique ex aequatione sexta colligimus $rr = F\sqrt{D}$, ideoque $r = \sqrt[4]{FFD}$, sicque omnes tres formulas transcendentes ita determinauimus, vt sit

$$1^\circ.\ p = \int \partial x \sqrt[4]{u} = \sqrt[4]{AAD} = \sqrt[4]{\tfrac{1}{4}(1,1)^2 (2,2)},$$

$$2^\circ.\ q = \int \partial x \sqrt[4]{u^2} = \int \partial x \sqrt{u} = \sqrt{D} = \sqrt{(2,2)},$$

$$3^\circ.\ r = \int \partial x \sqrt[4]{u^3} = \sqrt[4]{DFF} = \tfrac{3}{2}\sqrt[4]{(2,2)(3,3)^2}.$$

20. Hic iam notasse iuuabit, valorem formulae (μ, ν) casu quo $\mu + \nu = \lambda$, in genere per quadraturam circuli exprimi posse, cum hoc casu sit

$$(\mu, \nu) = \frac{\mu}{\lambda \sin. \frac{\mu\pi}{\lambda}}.$$

Nostro igitur casu, quo $\lambda = 4$, erit

$$(2,2) = \frac{\pi}{4 \sin. \frac{1}{2}\pi} = \frac{\pi}{4},$$

deinde quoque erit

$$(1,3) = \frac{\pi}{4 \sin. \frac{1}{4}\pi} = \frac{\pi}{2\sqrt{2}}.$$

Hinc igitur patet fore $D = \frac{\pi}{4}$ et $C = \frac{3}{4} \cdot \frac{\pi}{2\sqrt{2}} = \frac{3\pi}{8\sqrt{2}}$, ita vt hae duae litterae C et D a sola quadratura circuli pendeant.

§. 21.

§. 21. Quoniam has tres determinationes, nempe:

$$p = \sqrt[4]{AAD}, \quad q = \sqrt{D} \quad \text{et} \quad r = \sqrt[4]{DFF},$$

ex aequationibus 1, 4 et 6^{ta} elicuimus, si eosdem valores in reliquis aequationibus substituamus, reperiemus egregias relationes inter litteras nostras maiusculas. Sic enim secunda aequatio $pq = Br$ dabit $AAD^3 = B^4 DFF$, quae reducitur ad hanc: $AD = BBF$; tertia vero aequatio $pr = C$ dabit $ADF = CC$; denique quinta aequatio $qr = Ep$ praebebit $D^3 FF = A^2 DE$, vnde fit $DF = AEE$. Hoc ergo modo deducti sumus ad tres sequentes relationes:

$$1^\circ.\ AD = BBF, \quad 2^\circ.\ ADF = CC \quad \text{et} \quad 3^\circ.\ DF = AEE,$$

quarum prima ducta in secundam dabit $AD = BC$, at vero secunda ducta in tertiam producit $DF = CE$. Cum igitur sit $AD = BC$, ex prima concluditur quoque fore $C = BF$, ita vt ternae determinationes repertae ad istas ternas reuocentur:

$$1^\circ.\ C = AE, \quad 2^\circ.\ C = BF, \quad 3^\circ.\ AD = BC,$$

quae reducuntur ad istas tres simplicissimas:

$$1^\circ.\ C = AE, \quad 2^\circ.\ C = BF, \quad 3^\circ.\ D = BE.$$

§. 22. Quodsi iam in his postremis aequationibus loco litterarum formulas integrales per nostros characteres designatas introducamus, prouenient sequentes relationes:

$1^\circ.\ (1, 3) = (1, 1)(2, 3)$,

$2^\circ.\ (1, 3) = 2(1, 2)(3, 3)$ et

$3^\circ.\ (2, 2) = (1, 2)(2, 3)$.

Hinc igitur per ipsas formulas integrales habebimus istas tres relationes maxime memorabiles:

$$1^\circ.\ \frac{\pi}{2\sqrt{2}} = \int \frac{\partial z}{\sqrt[4]{(1-z^4)^3}} \cdot \int \frac{z\,\partial z}{\sqrt[4]{(1-z^4)}} = \int \frac{\partial z}{\sqrt[4]{(1-z^4)^3}} \cdot \int \frac{zz\,\partial z}{\sqrt{(1-z^4)}},$$

2°. $\frac{\pi}{4\sqrt{2}} = \int \frac{\partial z}{\sqrt{(1 - z^4)}} \cdot \int \frac{zz\,\partial z}{\sqrt[4]{(1 - z^4)}}$ et

3°. $\frac{\pi}{4} = \int \frac{\partial z}{\sqrt{(1 - z^4)}} \cdot \int \frac{zz\,\partial z}{\sqrt{(1 - z^4)}}$,

quarum vltimam iam dudum in medium attuli.

§. 23. Cum igitur ex ſex formulis integralibus qua[...] hic occurrunt, binae, ſcilicet C et D, a quadratura circuli pen[...] deant, ſi modo ex reliquis vnius valor innoteſcat, valore[...] caeterarum inde aſſignari poterunt. Si enim praeter charactere[...] (1, 3) et (2, 2) inſuper hunc (1, 2) tanquam cognitum ſpec[...] temus, reliqui tres per hos ſequenti modo determinabuntur[...]

$(3, 3) = \frac{(1, 3)}{2(1, 2)}$; $(2, 3) = \frac{(2, 2)}{(1, 2)}$; $(1, 1) = \frac{(1, 2)(1, 3)}{(2, 2)}$.

Euolutio caſus quo $\lambda = 5$.

§. 24. Vocemus hic formulas tranſcendentes quae[...] tas $\int u^{\frac{1}{5}} \partial x = p$, $\int u^{\frac{2}{5}} \partial x = q$, $\int u^{\frac{3}{5}} \partial x = r$, $\int u^{\frac{4}{5}} \partial x = s$. Nu[...] vero character (μ, ν) ſignificet hanc formulam integralem[...] $\int \frac{z^{\mu - 1}\partial z}{\sqrt[5]{(1 - z^5)^{5 - \nu}}}$, quibus poſitis ex aequatione principali [...] cem ſequentes aequationes nanciſcemur:

1°. Si $\binom{\mu = 1}{\nu = 1}$, erit $pp = \frac{1}{2}q(1, 1)$, vnde fit $\frac{pp}{q} = \frac{1}{2}(1, 1) =$[...]

2°. Si $\binom{\mu = 1}{\nu = 2}$, erit $pq = \frac{2}{3}r(1, 2)$, ergo $\frac{pq}{r} = \frac{2}{3}(1, 2) =$[...]

3°. Si $\binom{\mu = 1}{\nu = 3}$, erit $pr = \frac{3}{4}s(1, 3)$, ergo $\frac{pr}{s} = \frac{3}{4}(1, 3) =$[...]

4°. Si $\binom{\mu = 1}{\nu = 4}$, erit $ps = \frac{4}{5}(1, 4) = D$.

5°. Si $\binom{\mu = 2}{\nu = 2}$, erit $qq = s\ (2, 2)$, ergo $\frac{qq}{s} = (2, 2) =$[...]

6°. Si $\begin{pmatrix}\mu=2\\ \nu=3\end{pmatrix}$, erit $q\,r=\frac{6}{5}(2,3)=\mathrm{F}$.

7°. Si $\begin{pmatrix}\mu=2\\ \nu=4\end{pmatrix}$, erit $q\,s=\frac{8}{6}\cdot\frac{6}{5}p\,(2,4)$, ergo $\frac{q\,s}{p}=\frac{8}{5}(2,4)=\mathrm{G}$.

8°. Si $\begin{pmatrix}\mu=3\\ \nu=3\end{pmatrix}$, erit $r\,r=\frac{9}{6}\cdot\frac{6}{5}p\,(3,3)$, ergo $\frac{r\,r}{p}=\frac{9}{5}(3,3)=\mathrm{H}$.

9°. Si $\begin{pmatrix}\mu=3\\ \nu=4\end{pmatrix}$, erit $r\,s=\frac{12}{7}\cdot\frac{7}{5}q\,(3,4)$, ergo $\frac{r\,s}{q}=\frac{12}{5}(3,4)=\mathrm{I}$.

10°. Si $\begin{pmatrix}\mu=4\\ \nu=4\end{pmatrix}$, erit $s\,s=\frac{16}{8}\cdot\frac{8}{5}\nu\,(4,4)$, ergo $\frac{s\,s}{r}=\frac{16}{5}(4,4)=\mathrm{K}$.

§. 25. Quoniam igitur decem adepti ſumus aequationes, ex quibus quatuor quantitates incognitas definiri oportet: eligamus eas, quibus negotium facillime expedietur. Quarta autem aequatio ſtatim dat $s=\frac{\mathrm{D}}{p}$; ex ſexta autem fit $r=\frac{\mathrm{F}}{q}$, ita vt tantum ſuperſit binas litteras p et q elicere. Deinde vero ex prima deducimus $q=\frac{p\,p}{\mathrm{A}}$, ita vt ſit $r=\frac{\mathrm{AF}}{p\,p}$. Nunc igitur ex ſecunda aequatione fiet $\frac{p^5}{\mathrm{AAF}}=\mathrm{B}$, vnde fit $p=\sqrt[5]{(\mathrm{AABF})}$, quo valore inuento colligitur fore $q=\sqrt[5]{\left(\frac{\mathrm{BBFF}}{\mathrm{A}}\right)}$, $r=\sqrt[5]{\left(\frac{\mathrm{A\,F^3}}{\mathrm{BB}}\right)}$, denique erit $s=\frac{\mathrm{D}}{\sqrt[5]{(\mathrm{AABE})}}$. Sicque omnes quatuor incognitas per quadraturas ordinarias exprimere licebit. Quodſi iam hos valores in reliquis aequationibus ſubſtituamus, orientur ſequentes aequationes: 1°. $\mathrm{C\,D}=\mathrm{A\,F}$, 2°. $\mathrm{B\,F}=\mathrm{E\,D}$, 3°. $\mathrm{D}=\mathrm{A\,G}$, 4°. $\mathrm{F}=\mathrm{B\,H}$, 5°. $\mathrm{D}=\mathrm{B\,I}$, 6°. $\mathrm{D\,D}=\mathrm{A\,F\,K}$, vnde ob $\mathrm{D}=\mathrm{A\,G}$ eruitur $\mathrm{D\,G}=\mathrm{F\,K}$.

§. 26. Ecce ergo ſex nouae prodierunt determinationes, quibus decem noſtrae litterae a ſe inuicem pendent, ita vt e x quatuor pro cognitis aſſumtis reliquae ſex definiri queant,

ant; pro cognitis autem imprimis aſſumi conueniet binas D et F, quippe quae per quadraturam circuli innoteſcunt, cum ſit

$$D = \tfrac{4}{5}(1, 4) = \tfrac{4}{5} \cdot \frac{\pi}{5\ \text{ſin.}\ \frac{1}{5}\pi} \text{ et } F = \tfrac{6}{5}(2, 3) = \tfrac{6}{5} \cdot \frac{\pi}{5\ \text{ſin.}\ \frac{2}{5}\pi}.$$

Dummodo ergo duae reliquarum etiam vt cognitae ſpectentur, caeteras omnes per eas definire licebit. At vero ſex illae relationes rite inter ſe comparatae ternas formulas tam ipſi D quam ipſi F aequales ſuppeditant, quae ſunt $D = AG = BI = CK$ et $F = BH = CG = EI$. Hinc igitur, ſi praeter D et F etiam litterae A et B pro cognitis aſſumantur, reliquae litterae ex iis determinabuntur vt ſequitur: $C = \frac{AF}{D}$, $E = \frac{BF}{D}$, $G = \frac{D}{A}$, $H = \frac{F}{B}$, $I = \frac{D}{B}$ et $K = \frac{DD}{AF}$.

§. 27. Subſtituamus nunc loco harum litterarum characteres formularum integralium, atque ſex ſequentes relationes obtinebuntur:

1°. $(1, 4) = (1, 1)(2, 4)$,
2°. $(1, 4) = 2(1, 2)(3, 4)$,
3°. $(1, 4) = 3(1, 3)(4, 4)$,
4°. $(2, 3) = (1, 2)(3, 3)$,
5°. $(2, 3) = (1, 3)(2, 4)$,
6°. $(2, 3) = 2(2, 2)(3, 4)$;

vnde plura egregia theoremata formari poſſent.

§. 28. Quoniam ambae litterae D et F, ſeu potius characteres (1, 4) et (2, 3) ambo peripheriam circuli inuoluunt, eorum ratio, ſeu fractio $\frac{(1, 4)}{(2, 3)}$, algebraice exhiberi poterit, quippe cuius valor eſt $= \frac{\text{ſin.}\ \frac{2}{5}\pi}{\text{ſin.}\ \frac{1}{5}\pi} = 2\ \text{coſ.}\ \frac{1}{5}\pi$. Hinc etiam ſequentes ratio-

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nes inter binas formulas integrales deriuantur:

$$2 \operatorname{cof.} \tfrac{1}{3}\pi = \frac{(1, 1)}{(1, 3)} = 2\,\frac{(3, 4)}{(3.\ 3)} = \frac{(1, 2)}{(2, 2)} = 3\,\frac{(4, 4)}{(2, 4)},$$

vnde iterum eximia theoremata formari poffent, fi praefens noftrum inftitutum hoc poftularet. Pleniorem autem huius argumenti expofitionem in aliam occafionem fum dilaturus.

§. 29. Simili modo quo hic cafum $\lambda = 5$ euoluimus, etiam tractare liceret fequentes cafus, quibus litterae λ maiores valores tribuuntur. Quoniam autem numerus aequationum continuo fecundum numeros trigonales increfcit, fuperfluum foret iftum laborem hic fufcipere, quoniam omnes operationes analyticae, quibus hae folutiones nituntur, iam fatis dilucide funt expofitae.

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