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Variae considerationes circa series hypergeometricas

Leonhard Euler

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VARIAE CONSIDERATIONES

CIRCA
SERIES HYPERGEOMETRICAS.

Auctore
L. EULERO.

Conuent. exhib. die 19 Aug. 1776.

I.

Proposito hoc producto in infinitum excurrente:

$\frac{a(a+2b) \cdot (a+2b)(a+4b) \cdot (a+4b)(a+6b) \cdot (a+6b)(a+8b)}{(a+b)(a+b) \cdot (a+3b)(a+3b) \cdot (a+5b)(a+5b) \cdot (a+7b)(a+7b)} \&c. = \frac{P}{Q},$
constat esse

$$P = \int \frac{x^{a+b-1} \partial x}{\sqrt{(1-x^{2b})}} \text{ et } Q = \int \frac{x^{a-1} \partial x}{\sqrt{(1-x^{2b})}},$$

his integralibus ab $x = 0$ ad $x = 1$ extensis; vbi notetur illius producti membrum indici i respondens esse

$$\frac{[a + (2i - 2)b] (a + 2ib)}{[a + (2i - 1)b] [a + (2i - 1)b]}.$$

§. 2. Iam occasione istius producti consideremus sequens productum indefinitum, in quo factorum numerus sit $= n$, ac ponatur

$a(a+2b)(a+4b)(a+6b) \dots [a + (2n-2)b] = \Delta : n,$
siquidem hoc productum, ob a & b numeros datos, spectari potest tanquam certa functio ipsius n ; ex eius igitur natura perspicuum est fore

$$\Delta : (n+1) = \Delta : n \cdot (a + 2nb),$$

similique modo

$$\Delta : (n+2) = \Delta : (n+1) \cdot [a + (2n+2)b],$$

et ita porro.

A 2

Hinc

(4)

Hinc si i denotet numerum infinite magnum, erit

$$\Delta : i = a(a + 2b)(a + 4b)(a + 6b) \dots [a + (2i - 2)b]$$

vnde pariter colligitur fore

$$\Delta : (i + 1) = \Delta : i \cdot (a + 2ib),$$

$$\Delta : (i + 2) = \Delta : i (a + 2ib) [a + (2i + 2)b],$$

$$\Delta : (i + 3) = \Delta : i (a + 2ib) [a + (2i + 2)b] [a + (2i + 4)b]$$

ubi factores insuper accedentes tanquam inter se aequales spectari poterunt; quamobrem in genere statui poterit $\Delta : (i + n) = \Delta : i (a + 2ib)^n$, vbi cum $(a + 2ib)$ sit factor proxime sequens, eodem iure quilibet sequentium sumi potuisset, quo adhuc generalius statuere poterimus

$$\Delta : (i + n) = (a + 2ib)^n \Delta : i,$$

denotante a numerum quemcunque finitum, quippe qui pro $2ib$ evanescit.

§. 3. In computum nunc ducamus casum producti definiti, quo $n = \frac{1}{2}$, ac vocemus $\Delta : \frac{1}{2} = k$, quem valorem methodi interpolationum semper vero proxime assignare liceat. Hinc igitur per superiora erit

$$\Delta : (1 + \frac{1}{2}) = k(a + b);$$

$$\Delta : (2 + \frac{1}{2}) = k(a + b)(a + 3b),$$

$$\Delta : (3 + \frac{1}{2}) = k(a + b)(a + 3b)(a + 5b),$$

vnde in infinitum progrediendo erit

$$\Delta : (i + \frac{1}{2}) = k(a + b)(a + 3b)(a + 5b) \dots [a + (2i - 1)b]$$

§. 4. Cum igitur supra iam dederimus formulam pro $\Delta : (i + n)$, posito nunc $n = \frac{1}{2}$ habebimus quoque

$$\Delta : (i + \frac{1}{2}) = \Delta : i \sqrt{a + 2ib},$$

sicque pro eadem formula $\Delta : (i + \frac{1}{2})$ nacti sumus duas diversas expressiones, ex iisque conficitur ista aequatio:

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===== (5) =====

$\Delta : i \sqrt{(a+2ib)} = k(a+b)(a+3b)(a+5b) \dots [a+(2i-1)b]$,
 atque hinc concludere poterimus, valorem ipsius producti in-
 finiti

$$(a+b)(a+3b)(a+5b) \dots [a+(2i-1)b] = \frac{\Delta : i \sqrt{(a+2ib)}}{k}$$

sicque innotescit relatio inter hoc productum et id quod su-
 pra per $\Delta : i$ expressimus. Hic autem probe notandum est
 factores huius producti eos ipsos esse, qui denominatorem pro-
 ducti initio propositi constituunt, quamobrem tam numerato-
 rem illius producti quam denominatorem per valores modo
 inuentos $\Delta : i$ et $\frac{\Delta : i \sqrt{(a+2ib)}}{k}$ exprimere poterimus.

§. 5. Numerator autem producti propositi in infini-
 tum expansus ita repraesentari potest:

$$a(a+2b)^2(a+4b)^2 \dots [a+(2i-2)b]^2(a+2ib)$$

vbi factores primus et vltimus sunt solitarii, reliqui vero omnes
 quadrati. Cum igitur sit

$$(\Delta : i)^2 = (a)^2(a+2b)^2(a+4b)^2(a+6b)^2 \dots [a+(2i-2)b]^2,$$

evidens est illum numeratorem esse $\frac{(\Delta : i)^2}{a}(a+2ib)$. Pro de-
 nominatore autem per se manifestum est, eum esse aequalem
 quadrato producti alterius $(a+b)(a+3b) \&c.$, cuius valor
 cum repertus sit $\frac{\Delta : i \sqrt{(a+2ib)}}{k}$, denominator erit $\frac{(\Delta : i)^2(a+2ib)}{k k}$,
 quamobrem his valoribus substitutis pro fractione supra expo-
 sita $\frac{P}{Q}$ affecti sumus hanc aequationem:

$$\frac{P}{Q} = \frac{(\Delta : i)^2(a+2ib)}{a} = \frac{k k(a+2ib)}{a(a+2ib)} = \frac{k k}{a}$$

Ex hac aequatione igitur statim innotescit verus valor formulae
 interpolatae $k = \Delta : \frac{1}{2}$, quandoquidem erit $\Delta : \frac{1}{2} = \sqrt{\frac{aP}{Q}}$, at-
 que hinc porro sequentes:

$$A \ 3$$

$$\Delta : i$$

==== (6) ====

$$\Delta : (1 + \frac{1}{2}) = (a + b) \sqrt{\frac{aP}{Q}},$$

$$\Delta : (2 + \frac{1}{2}) = (a + b) (a + 3b) \sqrt{\frac{aP}{Q}},$$

$$\Delta : (3 + \frac{1}{2}) = (a + b) (a + 3b) (a + 5b) \sqrt{\frac{aP}{Q}}, \text{ etc.}$$

haecque interpolatio eo magis est notatu digna, quod sine approximatione statim verum valorem horum terminorum interpolatorum suppeditat.

§. 6. Quod si insuper istud productum infinitum, in quo utriusque factores coniunguntur, contemplemur, ac statuamus

$$a(a+b)(a+2b)(a+3b)\dots[a+(i-1)b] = \Gamma : i, \text{ erit}$$

$$\Gamma : 2i = a(a+b)(a+2b)(a+3b)\dots[a+(2i-1)b],$$

quod manifesto est productum ex binis superioribus, ita ut fit

$$\Gamma : 2i = \frac{(\Delta : i)^2 \sqrt{a+2ib}}{k}; \text{ vnde si forma } \Gamma : 2i \text{ vti voluerimus,}$$

valores amborum praecedentium ex eo assignare poterimus, cum sit

$$\Delta : i = \sqrt{\frac{k \cdot \Gamma : 2i}{\sqrt{a+2ib}}}, \text{ qui est ipse valor prioris producti}$$

$$a(a+2b)(a+4b)(a+6b), \text{ etc.}$$

alterius vero producti

$$(a+b)(a+3b)(a+5b) \text{ etc. valor erit } \sqrt{\frac{\Gamma : 2i \sqrt{a+2ib}}{k}},$$

§. 7. Haecenus igitur tria producta in infinitum excurrentia atque inter se affinia sumus contemplati, quae, quoniam ea accuratius sumus perscrutaturi, hic denuo ob oculos exponamus

I. $a(a+b)(a+2b)(a+3b)\dots[a+(i-1)b] = \Gamma : i,$

II. $a(a+2b)(a+4b)(a+6b)\dots[a+(2i-2)b] = \Delta : i,$

III. $(a+b)(a+3b)(a+5b)\dots[a+(2i-1)b] = \Theta : i,$

atque iam inuenimus esse $\Theta : i = \frac{\Delta : i \sqrt{a+2ib}}{k}$; tum vero tam $\Delta : i$ quam $\Theta : i$ sequenti modo per functionem $\Gamma : 2i$ expressimus:

$$\Delta : i$$

$$\Delta : i = \sqrt{\frac{k \cdot \Gamma : 2i}{\sqrt{(a+2ib)}}} \text{ et } \Theta : i = \sqrt{\frac{\Gamma : 2i \sqrt{(a+2ib)}}{k}},$$

quandoquidem manifestum est esse $\Gamma : 2i = \Delta : i \cdot \Theta : i$; vbi meminisse oportet esse $k = \Delta : \frac{1}{2}$, quod scilicet ex forma secunda definiri debet, considerando seriem

$a, a(a+2b), a(a+2b)(a+4b), a(a+2b)(a+4b)(a+6b), \text{ etc.}$
cuius terminum indici $\frac{1}{2}$ respondentem designauimus hac littera k .

§. 8. Iam ad istas formas accommodemus methodum generalem summandi omnis generis progressionem per terminum earum generalem, quae ita se habet, vt proposita serie quacunq; $A, B, C, D, E, \text{ etc.}$ cuius terminus indici indefinito x respondens fit $= X$, eius summa

$$A + B + C + D + \dots + X,$$

quam vocemus $= S$, fit

$$S = \int X \partial x + \frac{1}{2} X + \frac{1}{1 \cdot 2 \cdot 3} \cdot \frac{1}{2} \frac{\partial X}{\partial x} - \frac{1}{1 \cdot \dots \cdot 5} \cdot \frac{1}{8} \frac{\partial^3 X}{\partial x^3} + \frac{1}{1 \cdot \dots \cdot 7} \cdot \frac{1}{8} \frac{\partial^5 X}{\partial x^5} - \text{etc.}$$

vbi fractiones $\frac{1}{2}, \frac{1}{6}, \frac{1}{8}, \frac{3}{16}, \frac{5}{8}, \text{ etc.}$ sunt numeri Bernoulliani.

Evolutio formae primae.

$$a(a+b)(a+2b)(a+3b) \dots [a+(i-1)b] = \Gamma : i.$$

§. 9. Cum numerus factorum hic consideretur vt infinitus, quo methodum summandi ad eam applicare valeamus, consideremus eandem formam numero terminorum finito $= x$ constantem, ac statuamus simili modo

$$a(a+b)(a+2b)(a+3b) \dots [a+(x-1)b] = \Gamma : x.$$

Nunc vero vt loco huius producti seriem summandam nanciscamur, sumamus logarithmos, eritque

$$l\Gamma : x = la + l(a+b) + l(a+2b) + l(a+3b) \dots l[a+(x-1)b],$$

cuius ergo summa cum fuerit explorata, dabit logarithmum formulae

mulae $\Gamma : x$, ideoque ipsam formulam $\Gamma : x$, in qua si deinceps statuatur $x = i$, obtinebitur formula $\Gamma : i$, quem valorem in superioribus potissimum spectauimus. Hinc igitur, comparatione cum serie generalissima instituta, erit $X = l[a + (x - 1)b]$, atque ipsa summa $S = l\Gamma : x$, siue erit $X = l(a - b + bx)$, vnde colligitur $\int X \partial x = \int \partial x l(a - b + bx)$.

§. 10. Cum igitur sit $\int \partial z l z = z l z - z$, atque $\int \partial y l(a + y) = (a + y)l(a + y) - (a + y)$, nunc loco y scribendo bx erit

$$\int b \partial x l(a + bx) = (a + bx)l(a + bx) - a - bx,$$

ideoque

$$\int \partial x l(a + bx) = \frac{(a + bx)}{b} l(a + bx) - \frac{a}{b} - x,$$

vnde colligitur pro nostro casu fore:

$$\int X \partial x = \frac{(a - b + bx)}{b} l(a - b + bx) - \frac{a}{b} + 1 - x,$$

vbi in vltima parte membrum constans $\frac{a}{b} - 1$ omittere licet, quia expressio constantem quantitatem indefinitam per se possit lat, quam deinceps ex indole seriei definiendi oportet. Deinde vero erit $\frac{\partial X}{\partial x} = \frac{b}{a - b + bx}$; tum vero porro

$$\frac{\partial^2 X}{\partial x^2} = \frac{-2b^2}{(a - b + bx)^2}; \frac{\partial^3 X}{\partial x^3} = \frac{2 \cdot 3 \cdot 4 b^3}{(a - b + bx)^3}; \text{ etc.}$$

quibus valoribus in vsum vocatis erit

$$l\Gamma : x = A + \left(\frac{a}{b} - \frac{1}{2} + x\right) l(a - b + bx) - x + \frac{1}{1 \cdot 2 \cdot 3} \frac{1}{2} \frac{b}{a - b + bx} - \frac{1}{3 \cdot 4 \cdot 5} \cdot \frac{1}{8} \cdot \frac{b^3}{(a - b + bx)^3} + \frac{1}{5 \cdot 6 \cdot 7} \cdot \frac{1}{8} \cdot \frac{b^5}{(a - b + bx)^5} - \frac{1}{7 \cdot 8 \cdot 9} \cdot \frac{3}{10} \cdot \frac{b^7}{(a - b + bx)^7} + \frac{1}{9 \cdot 10 \cdot 11} \cdot \frac{5}{8} \cdot \frac{b^9}{(a - b + bx)^9} - \text{etc.}$$

vbi littera A constantem denotat ex indole seriei definiendam

§. 11. Constans autem ista A ex casu quo summa seriei est cognita determinari debet, quod ergo fieri posset casu

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casu $x = 0$, quippe summa etiam nihilo aequalis prodire debet; foret igitur hinc

$$A = \left(\frac{a}{b} - \frac{1}{2}\right) l(a-b) + \frac{1}{1 \cdot 2 \cdot 3} \cdot \frac{1}{2} \cdot \frac{b}{a-b} - \frac{1}{3 \cdot 4 \cdot 5} \cdot \frac{1}{6} \cdot \frac{b^3}{(a-b)^3} + \frac{1}{5 \cdot 6 \cdot 7} \cdot \frac{1}{2} \cdot \frac{b^5}{(a-b)^5} - \text{etc.}$$

Quoniam autem haec series parum conuergit, atque adeo casu $b = a$ omnes termini fierent infiniti, hinc nihil plane lucrari licet. Sin autem vellemus sumere $x = 1$, ista summa prodire deberet $= la$; vnde pariter vix quicquam pro instituto nostro concludere liceret, quia semper ad seriem infinitam perueniretur, cuius summam demum explorare oporteret, in quo quidem negotio forsitan ea, quae olim de seriebus numeros Bernoullianos inuoluentibus sum commentatus, aliquem usum praestare possent, cui autem labori nunc immorari non vacat.

§. 12. Quia enim in praesenti instituto potissimum ad valorem $\Gamma : i$ respicimus, sufficiet statim loco x numerum infinitum statui. Sit igitur $x = i$, denotante i numerum infinite magnum, et aequatio nostra hanc induet formam:

$$l \Gamma : i = A + \left(\frac{a}{b} - \frac{1}{2} + i\right) l(a - b + bi) - i,$$

vnde constans ista A sponte determinatur, quam idcirco quasi iam cognitam spectabimus. Hinc ergo ad numeros regrediendo, vbi quidem loco A scriptum intelligamus lA , peruenimus ad hanc expressionem:

$$\Gamma : i = A (a - b + bi)^{\frac{a}{b} - \frac{1}{2} + i} \cdot e^{-i}.$$

Hic quidem conueniet potestatem exponentis i seorsim repraesentare hoc modo:

$$\Gamma : i = A (a - b + bi)^{\frac{a}{b} - \frac{1}{2}} (a - b + bi)^i \cdot e^{-i}.$$

Evolutio binarum reliquarum formularum.

§. 13. Secunda forma a prima in eo tantum discrepat, quod loco b hic scribi oportet $2b$, vnde noua euolutione carere possumus; at vero loco constantis A hic scribamus B , quandoquidem nondum constat, quemadmodum littera b constantem A ingrediatur. Hoc modo igitur statim habebimus

$$\Delta : i = B (a - 2b + 2bi)^{\frac{a}{2b} - \frac{1}{2}} (a - 2b + 2bi)^i e^{-i}$$

Simili modo euidens est, ex hac secunda forma oriri tertiam si modo loco a scribatur $a + b$, vnde loco B constantem introducendo, habebimus

$$\Theta : i = C (a - b + 2bi)^{\frac{a}{2b}} (a - b + 2bi)^i e^{-i}$$

Vbi notetur litteram e hic positam esse pro numero cuius logarithmus hyperbolicus = 1.

Conclusiones hinc oriundae.

§. 14. Videamus nunc quomodo istae nouae determinationes se respectu relationum supra inuentarum sint habent; quare cum ex his nouis valoribus sit

$$\Gamma : 2i = A (a - b + 2bi)^{\frac{a}{b} - \frac{1}{2}} (a - b + 2bi)^{2i} e^{-i}$$

quia inuenimus:

$$\Gamma : 2i = \Delta : i \times \Theta : i,$$

si vbique hic valores modo inuentos substituamus, habebimus pro ista aequatione primo productum:

$$\Delta : i \cdot \Theta : i = B C (a - 2b + 2bi)^{\frac{a}{2b} - \frac{1}{2}} (a - b + 2bi)^i (a - 2b + 2bi)^i (a - b + 2bi)^i e^{-2i},$$

quod cum esse debeat illi valori $\Gamma : 2i$ aequale, si per factores quos habeat communes vtrinque diuidamus, prodibit aequatio:

==== (11) =====

$$A (a - b + 2bi)^{\frac{a}{2b} - \frac{1}{2}} (a - b + 2bi)^i \\ = BC (a - 2b + 2bi)^{\frac{a}{2b} - \frac{1}{2}} (a - 2b + 2bi)^i.$$

§. 15. Diuidamus hanc aequationem vtrinque per $(a - 2b + 2bi)^i$, et cum fit

$$\frac{a - b + 2bi}{a - 2b + 2bi} = 1 + \frac{b}{a - 2b + 2bi} = 1 + \frac{1}{2i},$$

ob i numerum infinitum, per resolutionem ordinariam erit $(1 + \frac{1}{2i})^i = e^{\frac{1}{2}}$, vnde aequatio nostra reducitur ad hanc formam:

$$A (a - b + 2bi)^{\frac{a}{2b} - \frac{1}{2}} \times e^{\frac{1}{2}} = BC (a - 2b + 2bi)^{\frac{a}{2b} - \frac{1}{2}},$$

vbi vltimi factores quoque se tollunt, cum fit

$$\left(\frac{a - b + 2bi}{a - 2b + 2bi}\right)^{\frac{a}{2b} - \frac{1}{2}} = \left(1 + \frac{1}{2i}\right)^{\frac{a}{2b} - \frac{1}{2}} = 1,$$

ita vt peruentum fit ad hanc simplicem aequalitatem:

$$A e^{\frac{1}{2}} = BC.$$

§. 16. Deinde cum supra inuenerimus esse

$$\Theta : i = \frac{\Delta i \sqrt{(\alpha + 2ib)}}{k}, \text{ siue } \frac{\Theta : i}{\Delta : i} = \frac{\sqrt{(\alpha + 2ib)}}{k},$$

diuidamus valorem pro $\Theta : i$ inuentum per $\Delta : i$, ac reperiemus:

$$\frac{\Theta : i}{\Delta : i} = \frac{C}{B} \sqrt{(a - 2b + 2bi)} \left(\frac{a - b + 2bi}{a - 2b + 2bi}\right)^i = \frac{C}{B} \sqrt{e(a - 2b + 2bi)}.$$

Erit ergo

$$\frac{\sqrt{(\alpha + 2ib)}}{k} = \frac{C}{B} \sqrt{e(a - 2b + 2bi)}, \text{ siue}$$

$$\frac{1}{k} = \frac{C}{B} \sqrt{\frac{e(a - 2b + 2bi)}{\alpha + 2ib}} = \frac{C}{B} \sqrt{e},$$

siue erit $B = Ck\sqrt{e}$.

§. 17. Nacti ergo fumus eiusmodi binas relationes inter ternas illas constantes A, B, C , ut si earum vnica effect cognita, ex ea binæ reliquæ definiiri possent. Cum enim fit $A = \frac{B C}{\sqrt{e}}$ et $B = C k \sqrt{e}$, si constantem A spectemus ut iam cognitam, binæ reliquæ sequenti modo determinabuntur: Cum fit $B = C k \sqrt{e}$, hic valor in priore aequatione substitutus datur $A = C C k$, vnde elicitur $C = \sqrt{\frac{A}{k}}$, hincque porro $B = \sqrt{k A}$. Interim tamen hinc non patet, quomodo constans A absolute determinari queat; ideoque recurrendum erit ad ipsam illam summationem seriei logarithmicæ, quam littera A supra indicauimus; vbi autem loco A scribendum erit $\log A$. Atque hinc tantum fumus lucrati, ut si binæ reliquæ formæ simili modo euoluantur per series logarithmicas, constantes ibi adhibenda scilicet $\log B$ et $\log C$ simul innotescant.

§. 18. Superest ut adhuc pauca addamus de valore litteræ k , quam per interpolationem inueniri debere iam supra monuimus. Interim tamen hæc littera etiam ex ipsa computatione formularum $\Delta : i$ et $\Theta : i$ absolute per certas quadraturas determinari potest. Cum enim fit

$$k = \frac{\Delta : i}{\Theta : i} \sqrt{(a + 2ib)}, \text{ ideoque}$$

$$k k = \frac{(\Delta : i)^2 (a + 2ib)}{(\Theta : i)^2},$$

si loco $\Delta : i$ et $\Theta : i$ ipsa producta infinita substituamus, quoniam vtrumque ex i factoribus constat, hic autem in numeratore vnus factor $a + 2ib$ insuper accedit, primum numeratoris factorem seorsim exprimamus, hoc pacto peruenientem ad sequens productum determinatum:

$$k k = a \cdot \frac{a(a+2b)(a+2b)(a+4b)(a+4b)(a+6b)}{(a+b)(a+b)(a+3b)(a+3b)(a+5b)(a+5b)} \text{ etc.}$$

§. 19. Vt autem huius producti infiniti verum valorem eruamus, recordandum est, si litterae P et Q denotent sequentes formulas integrales:

$$P = \int \frac{x^{p-1} \partial x}{(1-x^n)^{1-\frac{m}{n}}} \quad \text{et} \quad Q = \int \frac{x^{q-1} \partial x}{(1-x^n)^{1-\frac{m}{n}}},$$

quae integralia ab $x=0$ ad $x=1$, extendi sunt intelligenda, tum per productum infinitum fore:

$$\frac{P}{Q} = \frac{q(m+p)}{p(m+q)} \cdot \frac{(q+n)(m+p+n)}{(p+n)(m+q+n)} \cdot \frac{(q+2n)(m+p+2n)}{(p+2n)(m+q+2n)} \quad \text{etc.}$$

quod productum facile ad nostram formam reducitur, sumendo $q=a$, $p=a+b$, $m=b$, $n=2b$, ita vt pro nostro casu fiat

$$P = \int \frac{x^{a+b-1} \partial x}{\sqrt{1-x^{2b}}} \quad \text{et} \quad Q = \int \frac{x^{a-1} \partial x}{\sqrt{1-x^{2b}}},$$

tum vero erit $k = \frac{aP}{Q}$, ideoque $k = \sqrt{\frac{aP}{Q}}$, sicque eundem valorem k alio modo elucimus, quem iam supra attulimus.

§. 20. Quemadmodum autem est $k = \Delta : \frac{1}{2}$, simili modo pro binis reliquis formis poterimus assignare valores $\Gamma : \frac{1}{2}$ et $\Theta : \frac{1}{2}$. Cum enim forma Γ oriatur ex forma Δ , si in hac loco b scribatur $\frac{1}{2}b$, forma autem Θ oriatur ex Δ , si loco a scribatur $a+b$, his obseruatis erit

$$\Gamma : \frac{1}{2} = \sqrt{a} \frac{\int x^{a+\frac{1}{2}b-1} \partial x : \sqrt{1-x^b}}{\int x^{a-1} \partial x : \sqrt{1-x^b}} \quad \text{et}$$

$$\Theta : \frac{1}{2} = \sqrt{(a+b)} \frac{\int x^{a+2b-1} \partial x : \sqrt{1-x^{2b}}}{\int x^{a+b-1} \partial x : \sqrt{1-x^{2b}}}.$$

Facile autem intelligitur, valorem $\Theta : \frac{1}{2}$ aequae in nostros calculos introduci potuisse ac $\Delta : \frac{1}{2} = k$, cum sit $\Delta : \frac{1}{2} \cdot \Theta : \frac{1}{2} = a$.

Ductis enim in se inuicem illis valoribus integralibus prodit

$$\Delta : \frac{1}{2} . \Theta : \frac{1}{2} = \sqrt{\frac{a(a+b) \int x^{a+2b-1} \partial x : \sqrt{(1-x^{2b})}}{\int x^{a-1} \partial x : (1-x^{2b})}}$$

ex notissima autem reductione talium integralium constat esse

$$\int \frac{x^{a+2b-1} \partial x}{\sqrt{(1-x^{2b})}} = \frac{a}{a+b} \int \frac{x^{a-1} \partial x}{\sqrt{(1-x^{2b})}},$$

pro terminis scilicet integrationis $x=0$ et $x=1$, sicque
perspicuum est fore $\Delta : \frac{1}{2} . \Theta : \frac{1}{2} = a$. Quemadmodum autem
valor $\Gamma : \frac{1}{2}$ ad binos reliquos referatur, nullo modo defini
potest.