



1793

Supplementum ad dissertationem praecedentem circa
integrationem vormalae $\int (z^{m-1} dz)/(1-z^n)$ casu quo ponitur $z =$
 $\sqrt{\cos(\phi) + \sqrt{-1} \cdot \sin(\phi)}$

Leonhard Euler

Follow this and additional works at: <https://scholarlycommons.pacific.edu/euler-works>

Record Created:

2018-09-25

Recommended Citation

Euler, Leonhard, "Supplementum ad dissertationem praecedentem circa integrationem vormalae $\int (z^{m-1} dz)/(1-z^n)$ casu quo ponitur $z = \sqrt{\cos(\phi) + \sqrt{-1} \cdot \sin(\phi)}$ " (1793). *Euler Archive - All Works*. 657.

<https://scholarlycommons.pacific.edu/euler-works/657>

This Article is brought to you for free and open access by the Euler Archive at Scholarly Commons. It has been accepted for inclusion in Euler Archive - All Works by an authorized administrator of Scholarly Commons. For more information, please contact mgibney@pacific.edu.

SUPPLEMENTVM

ad dissertationem praecedentem, circa integrationem for-

mulae $\int \frac{z^{m-1} \partial z}{1-z^n}$, casu quo ponitur

$$z = v (\cos. \Phi + \sqrt{-1} \sin. \Phi).$$

§. 1. Resolutio formulae $\int \frac{z^{m-1} \partial z}{1+z^n}$, quam supra in problemate, pro casu quo $z = v (\cos. \Phi + \sqrt{-1} \sin. \Phi)$ dedimus, eximia et notatu dignissima artificia complectitur, quae animo firmiter imprimere haud inutile erit. Cum igitur formula, quam hic tractandam suscipimus, non minore attentione sit digna quam ea quam supra tractauimus, eius integrale per eandem methodum exhibere constitui; ubi simul occasionem inueniemus nouum compendium in calculo adhibendi.

Problema.

Si ponatur $z = v (\cos. \Phi + \sqrt{-1} \sin. \Phi)$, inuestigare integrale huius formulae: $\int \frac{z^{m-1} \partial z}{1-z^n}$.

Solutio.

§. 2. Cum ob valorem ipsius z imaginarium integrale quaesitum etiam esse debeat imaginarium, id sub forma $P + Q\sqrt{-1}$ complectamur, ita vt P et Q sint quantitates reales. Hanc ob rem erit facta substitutione

$$\int \frac{z^{m-1} \partial z}{1-z^n} = P + Q\sqrt{-1}.$$

§. 3. Cum porro fit $z = v (\cos. \Phi + \sqrt{-1} \sin. \Phi)$, hincque $\frac{\partial z}{z} = \frac{\partial v}{v} + \partial \Phi \sqrt{-1}$, erit numerator

$$z^{m-1} \partial z = v^m (\text{cof. } m \Phi + \sqrt{-1} \text{ fin. } m \Phi) \left(\frac{\partial v}{v} + \partial \Phi \sqrt{-1} \right),$$

denominator vero erit

$$1 - v^n (\text{cof. } n \Phi + \sqrt{-1} \text{ fin. } n \Phi),$$

qui ergo euanescit ponendo

$$v^n = \frac{1}{\text{cof. } n \Phi + \sqrt{-1} \text{ fin. } n \Phi} = \text{cof. } n \Phi - \sqrt{-1} \text{ fin. } n \Phi.$$

§. 4. Iam vt imaginaria ex denominatore tollantur, supra et infra multiplicemus per

$$1 - v^n (\text{cof. } n \Phi - \sqrt{-1} \text{ fin. } \Phi),$$

ficque fractio nostra euoluenda erit

$$\partial V = \frac{z^{m-1} \partial z (1 - v^n \text{cof. } n \Phi + v^n \sqrt{-1} \text{ fin. } n \Phi)}{1 - 2 v^n \text{cof. } n \Phi + v^{2n}}$$

Quod si iam hic loco $z^{m-1} \partial z$ valor modo assignatus substituat et partes reales ab imaginariis segregentur, ob

$$\begin{aligned} & (\text{cof. } m \Phi + \sqrt{-1} \text{ fin. } m \Phi) (\text{cof. } n \Phi - \sqrt{-1} \text{ fin. } n \Phi) \\ & = \text{cof. } (m - n) \Phi + \sqrt{-1} \text{ fin. } (m - n) \Phi \end{aligned}$$

prodibit pars realis ita expressa:

$$\begin{aligned} & v^{m-1} \partial v [\text{cof. } m \Phi - v^n \text{cof. } (m - n) \Phi] \\ & - v^m \partial \Phi [\text{fin. } m \Phi - v^n \text{fin. } (m - n) \Phi], \end{aligned}$$

pars vero imaginaria per $\sqrt{-1}$ diuisa:

$$\begin{aligned} & v^{m-1} \partial v [\text{fin. } m \Phi - v^n \text{fin. } (m - n) \Phi] \\ & + v^m \partial \Phi (\text{cof. } m \Phi - v^n \text{cof. } (m - n) \Phi). \end{aligned}$$

§. 5. Quod si iam breuitatis gratia statuamus

$$R = v^{m-1} [\text{cof. } m \Phi - v^n \text{cof. } (m - n) \Phi] \text{ et}$$

$$S = v^{m-1} [\text{fin. } m \Phi - v^n \text{cof. } (m - n) \Phi],$$

ambae litterae quaesitae P et Q per sequentes formulas integrales exprimentur:

$$P =$$

$$P = \int \frac{R \partial v - S v \partial \Phi}{1 - 2 v^n \cos. n \Phi + v^{2n}} \text{ et}$$

$$Q = \int \frac{S \partial v + R v \partial \Phi}{1 - 2 v^n \cos. n \Phi + v^{2n}}.$$

Has igitur duas formulas integrare oportebit, quod fiet, dum denominatoris singulos factores trinomiales inuestigabimus et ex singulis fractiones partiales inde oriundas definiemus.

§. 6. Consideremus igitur in genere hanc fractionem:

N
 $\frac{N}{1 - 2 v^n \cos. n \Phi + v^{2n}}$, et fingamus denominatoris factorem esse $1 - 2 v \cos. \omega + v v$, vbi angulus ω ita debet esse comparatus, vt posito

$$1 - 2 v \cos. \omega + v v = 0, \text{ siue}$$

$$v = \cos. \omega + \sqrt{-1} \sin. \omega,$$

simul quoque denominator euanescat, id quod fit, vti vidimus, quando $v^n = \cos. n \Phi - \sqrt{-1} \sin. n \Phi$. At vero ex factore supposito fit $v^n = \cos. n \omega + \sqrt{-1} \sin. n \omega$, vnde statui debet $\cos. n \omega = \cos. n \Phi$ et $\sin. n \omega = -\sin. n \Phi$, id quod euenit in genere quando $n \omega + n \Phi = i \pi$, denotante i omnes numeros pares, sicque erit $n \omega = i \pi - n \Phi$, ideoque $\omega = \frac{i \pi}{n} - \Phi$, vnde n diuersi valores pro angulo ω deducuntur, dum scilicet loco i scribuntur successive numeri 0, 2, 4, 6, etc. vsque ad $2n$, excluso postremo.

§. 7. Ponamus nunc fractionem partialem ex isto factore oriendam esse $\frac{F}{1 - 2 v \cos. \omega + v v}$, atque ex superioribus patet statui debere

$$F = \frac{N (1 - 2 v \cos. \omega + v v)}{1 - 2 v^n \cos. n \Phi + v^{2n}},$$

vnde

vnde scilicet ope aequationis $v v - 2 v \cos. \omega + 1 = 0$ pro F huiusmodi forma $A v + B$ elici debet. Quoniam vero hoc casu tam numerator quam denominator evanescit, differentialibus in subsidium vocatis fiet

$$F = \frac{N(v - \cos. \omega)}{n v^{n-1} (v^n - \cos. n \Phi)}$$

§. 8. Cum nunc casu quo $v v - 2 v \cos. \omega + 1 = 0$ fit

$$v - \cos. \omega = \sqrt{1 - \sin. \omega} \text{ et}$$

$$v^n - \cos. n \Phi = -\sqrt{1 - \sin. n \Phi}, \text{ erit}$$

$$F = -\frac{N v \sin. \omega}{n v^n \sin. n \Phi},$$

qui valor prorsus convenit cum eo qui supra est repertus. Hic igitur tantum opus est, ut loco N siue R siue S substituaturs, indeque forma praescripta pro isto numeratore F derivetur, in usum vocando lemma supra allatum

Evolutio fractionis

$$\frac{R v \sin. \omega}{n v^n \sin. n \Phi} \text{ siue } \frac{v^m \sin. \omega [\cos. m \Phi - v^n \cos. (m-n) \Phi]}{n v^n \sin. n \Phi}$$

§. 9. Hinc ergo erit

$$F = -\frac{v^{m-n} \sin. \omega \cos. m \Phi + v^m \sin. \omega \cos. (m-n) \Phi}{n \sin. n \Phi}$$

Per lemma autem memoratum habebitur

$$\sin. \omega v^{m-n} = v \sin. (m-n) \omega - \sin. (m-n-1) \omega.$$

Cum igitur sit $n \omega = i \pi - n \Phi$, erit

$$\sin. (m-n) \omega = \sin. (m \omega + n \Phi) \text{ et}$$

$$\sin. (m-n-1) \omega = \sin. (m-1) \omega + n \Phi].$$

Deinde vero est

$$\sin. \omega. v^m = v \sin. m \omega - \sin. (m - 1) \omega,$$

quibus valoribus substitutis erit

$$F = - \frac{1}{n \sin. n \Phi} \left[\frac{v \cos. m \Phi \sin. (m \omega + n \Phi) - \cos. m \Phi \sin. [(m-1) \omega + n \Phi]}{v \sin. m \omega \cos. (m-n) \Phi} + \sin. (m-1) \omega \cos. (m-n) \Phi \right].$$

Facta iam evolutione formularum

$$\begin{aligned} \sin. (m \omega + n \Phi) &= \sin. m \omega \cos. n \Phi + \cos. m \omega \sin. n \Phi \text{ et} \\ \cos. (m-n) \Phi &= \cos. m \Phi \cos. n \Phi + \sin. m \Phi \sin. n \Phi, \end{aligned}$$

littera v hic multiplicatur per hanc formam:

$$\begin{aligned} \sin. n \Phi \cos. m \Phi \cos. m \omega - \sin. n \Phi \sin. m \Phi \sin. m \omega \\ = \sin. n \Phi \cos. (m \Phi + m \omega), \end{aligned}$$

reliqui vero termini, quia ab his tantum in eo differunt ut loco $m \omega$ scribi debeat $(m-1) \omega$, erunt:

$$- \sin. n \Phi \cos. [m (\omega + \Phi) - \omega]$$

ficque pro numeratore quem quaerimus erit

$$F = - \frac{1}{n} v \cos. m (\omega + \Phi) + \frac{1}{n} \cos. [m (\omega + \Phi) - \omega].$$

Evolutio fractionis

$$\frac{v \sin. \omega}{n v^n \sin. n \Phi} = \frac{v^m \sin. \omega [\sin. m \Phi - v^n \sin. (m-n) \Phi]}{n v^n \sin. n \Phi}.$$

§. 10. Hoc casu erit

$$F = - \frac{v^{m-n} \sin. \omega \sin. m \Phi + v^m \sin. \omega \sin. (m-n) \Phi}{n \sin. n \Phi}.$$

Hic igitur eodem lemmate in subsidium vocato erit

$$F = - \frac{1}{n \sin. n \Phi} \left[\frac{v \sin. m \Phi \sin. (m \omega + n \Phi) - \sin. m \Phi \sin. [(m-1) \omega + n \Phi]}{v \sin. (m-n) \Phi \sin. m \omega + \sin. (m-n) \Phi \sin. (m-1) \omega} \right];$$

vbi per similem evolutionem quantitas, qua v multiplicatur, inuenitur $= \sin. n \Phi \sin. [m (\omega + \Phi)]$; reliqua vero pars erit

$$- \sin. n \Phi \sin. [m (\omega + \Phi) - \omega],$$

hinc igitur pro littera S valor quaesitus numeratoris erit

$$F = - \frac{i}{n} v \sin. m (\omega + \Phi) + \frac{i}{n} \sin. [m (\omega + \Phi) - \omega].$$

§. 11. Cum igitur sit $\omega + \Phi = \frac{i\pi}{n}$, ponamus breuitatis gratia angulum $m (\omega + \Phi) = \frac{m i \pi}{n} = \zeta$, atque pro littera R erit

$$F = - \frac{i}{n} [v \cos. \zeta - \cos. (\zeta - \omega)]$$

at vero pro S erit

$$F = - \frac{i}{n} [v \sin. \zeta - \sin. (\zeta - \omega)],$$

quibus valoribus inuentis pro denominatoris factore $1 - 2v \cos. \omega + v^2$ partes, ex quibus litterae P et Q componuntur, per sequentes formulas integrales exprimentur:

$$P = - \frac{i}{n} \int \frac{[v \cos. \zeta - \cos. (\zeta - \omega)] \partial v - v \partial \Phi [v \sin. \zeta - \sin. (\zeta - \omega)]}{1 - 2v \cos. \omega + v^2}$$

$$Q = - \frac{i}{n} \int \frac{[v \sin. \zeta - \sin. (\zeta - \omega)] \partial v + v \partial \Phi [v \cos. \zeta - \cos. (\zeta - \omega)]}{1 - 2v \cos. \omega + v^2}$$

§. 12. Quoniam hae formulae prorsus conveniunt cum iis, quas supra sumus nacli, et ne signa quidem sunt immutata, peculiari integratione non indigemus, sed pro quantitibus P et Q sequentes habebimus valores integratos:

$$P = - \frac{\cos. \zeta}{n} \int \sqrt{(1 - 2v \cos. \omega + v^2)} + \frac{\sin. \zeta}{n} A \operatorname{tang.} \frac{v \sin. \omega}{1 - v \cos. \omega} \text{ et}$$

$$Q = - \frac{\sin. \zeta}{n} \int \sqrt{(1 - 2v \cos. \omega + v^2)} - \frac{\cos. \zeta}{n} A \operatorname{tang.} \frac{v \sin. \omega}{1 - v \cos. \omega}.$$

Tales scilicet formulae ex singulis factoribus denominatoris formae $1 - 2v \cos. \omega + v^2$ derivari et in vnam summam colligi debent, ut veri valores pro P et Q obtineantur, ubi tantum recordari oportet esse $\omega = \frac{i\pi}{n} - \Phi$ et $\zeta = \frac{m i \pi}{n}$; pro i autem hic numeros pares accipi oportet.

Exemplum 1.

§. 13. Sit $m = 1$ et $n = 1$, ita ut quaeri debeat $\int \frac{\partial z}{1-z^2} = P + Q\sqrt{-1}$, posito scilicet $z = v(\cos. \Phi + \sqrt{-1} \sin. \Phi)$.

Quia hic $n = 1$, vnicus valor pro ω locum habet, resultans ex $i = 0$, eritque ergo $\omega = -\Phi$ et $\zeta = 0$, vnde statim colligimus

$$P = -l\sqrt{(1 - 2v \cos. \Phi + vv)} \text{ et } Q = -A \text{ tang. } \frac{v \sin. \Phi}{1 - v \cos. \Phi}.$$

Exemplum 2.

§. 14. Sit $m = 1$ et $n = 2$, ideoque formula integranda $\int \frac{\partial z}{1-z^2} = P + Q\sqrt{-1}$, posito $z = v(\cos. \Phi + \sqrt{-1} \sin. \Phi)$.

Quia hic est $n = 2$, pro ω duos habebimus valores ex $i = 0$ et $i = 2$ oriundos, vnde

$$\text{Si } i = 0, \text{ erit } \omega = -\Phi \text{ et } \zeta = 0$$

$$\text{Si } i = 2, \text{ erit } \omega = \pi - \Phi \text{ et } \zeta = \pi.$$

Hinc igitur statim colligemus

$$P = \begin{cases} -\frac{1}{2}l\sqrt{(1 - 2v \cos. \Phi + vv)} + 0 \\ +\frac{1}{2}l\sqrt{(1 + 2v \cos. \Phi + vv)} + 0. \end{cases}$$

$$Q = \begin{cases} 0 + \frac{1}{2}A \text{ tang. } \frac{v \sin. \Phi}{1 - v \cos. \Phi} \\ 0 + \frac{1}{2}A \text{ tang. } \frac{v \sin. \Phi}{1 + v \cos. \Phi}. \end{cases}$$

Exemplum 3.

§. 15. Sit nunc $m = 2$ et $n = 2$, ideoque formula integranda $\int \frac{z \partial z}{1-z^2} = P + Q\sqrt{-1}$, posito scilicet $z = v(\cos. \Phi + \sqrt{-1} \sin. \Phi)$.

Hic ergo primo sumi debet $i = 0$, tum vero $i = 2$, vnde

$$\text{Si } i = 0, \text{ erit } \omega = -\Phi \text{ et } \zeta = 0$$

$$\text{Si } i = 2, \text{ erit } \omega = \pi - \Phi \text{ et } \zeta = 2\pi$$

vnde valores pro P et Q eruuntur sequentes

$$P =$$

$$P = \begin{cases} -\frac{1}{2} l \sqrt{(1 - 2v \operatorname{cof.} \Phi + vv)} - 0 \\ -\frac{1}{2} l \sqrt{(1 + 2v \operatorname{cof.} \Phi + vv)} - 0. \end{cases}$$

$$Q = \begin{cases} 0 + \frac{1}{2} A \operatorname{tang.} \frac{v \operatorname{fin.} \Phi}{1 - v \operatorname{cof.} \Phi} \\ 0 - \frac{1}{2} A \operatorname{tang.} \frac{v \operatorname{fin.} \Phi}{1 - v \operatorname{cof.} \Phi}. \end{cases}$$

Exemplum 4.

§. 16. Sit $m = 1$ et $n = 3$, ideoque formula integranda
 $\int \frac{dz}{1-z^2} = P + Q \sqrt{-1}$, posito $z = v (\operatorname{cof.} \Phi + \sqrt{-1} \operatorname{fin.} \Phi)$.

Hic igitur ternos valores pro angulo ω habebimus,
 quos sequenti modo repraesentemus:

i	0	2	4
ω	$-\Phi$	$120^\circ - \Phi$	$240^\circ - \Phi$
$\operatorname{fin.} \omega$	$-\operatorname{fin.} \Phi$	$+\operatorname{fin.} (60^\circ + \Phi)$	$-\operatorname{fin.} (60^\circ - \Phi)$
$\operatorname{cof.} \omega$	$+\operatorname{cof.} \Phi$	$-\operatorname{cof.} (60^\circ + \Phi)$	$+\operatorname{cof.} (60^\circ - \Phi)$
ζ	0	120°	240°
$\operatorname{fin.} \zeta$	0	$+\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$
$\operatorname{cof.} \zeta$	$+1$	$-\frac{1}{2}$	$-\frac{1}{2}$

Hinc ergo inueniemus

$$P = \begin{cases} -\frac{1}{2} l \sqrt{(1 - 2v \operatorname{cof.} \Phi + vv)} + 0 \\ +\frac{1}{2} l \sqrt{[1 + 2v \operatorname{cof.} (60^\circ + \Phi) + vv]} + \frac{1}{2\sqrt{3}} A \operatorname{tang.} \frac{v \operatorname{fin.} (60^\circ + \Phi)}{1 + v \operatorname{cof.} (60^\circ + \Phi)} \\ +\frac{1}{2} l \sqrt{[1 + 2v \operatorname{cof.} (60^\circ - \Phi) + vv]} + \frac{1}{2\sqrt{3}} A \operatorname{tang.} \frac{v \operatorname{fin.} (60^\circ - \Phi)}{1 + v \operatorname{cof.} (60^\circ - \Phi)}. \end{cases}$$

$$Q = \begin{cases} 0 & +\frac{1}{3} A \operatorname{tang.} \frac{v \operatorname{fin.} \Phi}{1 - v \operatorname{cof.} \Phi} \\ -\frac{1}{2\sqrt{3}} l \sqrt{[1 + 2v \operatorname{cof.} (60^\circ + \Phi) + vv]} + \frac{1}{3} A \operatorname{tang.} \frac{v \operatorname{fin.} (60^\circ + \Phi)}{1 + v \operatorname{cof.} (60^\circ + \Phi)} \\ +\frac{1}{2\sqrt{3}} l \sqrt{[1 + 2v \operatorname{cof.} (60^\circ - \Phi) + vv]} - \frac{1}{3} A \operatorname{tang.} \frac{v \operatorname{fin.} (60^\circ - \Phi)}{1 + v \operatorname{cof.} (60^\circ - \Phi)}. \end{cases}$$

Exemplum 5.

§. 17. Sumatur nunc $m = 2$, manente $n = 3$, ut formula integranda sit $\int \frac{z dz}{1-z^3} = P + Q\sqrt{-1}$, posito $z = v(\cos.\Phi + \sqrt{-1}\sin.\Phi)$.

Hic notetur, valores ipsius ω prorsus eosdem manere ut ante, sicque etiam logarithmi et arcus circulares iidem manebunt; valores autem pro ζ erunt sequentes:

$$\begin{aligned} \text{Si } i = 0, \text{ erit } \zeta = 0, \sin.\zeta = 0 \quad \text{et } \cos.\zeta = +1. \\ \text{Si } i = 2, \text{ erit } \zeta = \frac{2}{3}\pi, \sin.\zeta = -\frac{\sqrt{3}}{2} \quad \text{et } \cos.\zeta = -\frac{1}{2}. \\ \text{Si } i = 4, \text{ erit } \zeta = \frac{4}{3}\pi, \sin.\zeta = +\frac{\sqrt{3}}{2} \quad \text{et } \cos.\zeta = -\frac{1}{2}. \end{aligned}$$

Hinc igitur fiet

$$\begin{aligned} P = & \begin{cases} -\frac{1}{3}l\sqrt{(1-2v\cos.\Phi+vv)} + 0 \\ +\frac{1}{3}l\sqrt{[1-2v\cos.(60^\circ+\Phi)+vv]} - \frac{1}{2\sqrt{3}}A \operatorname{tang}.\frac{v\sin.(60^\circ+\Phi)}{1+v\cos.(60^\circ+\Phi)} \\ +\frac{1}{3}l\sqrt{[1+2v\cos.(60^\circ-\Phi)+vv]} - \frac{1}{2\sqrt{3}}A \operatorname{tang}.\frac{v\sin.(60^\circ-\Phi)}{1+v\cos.(60^\circ-\Phi)}. \end{cases} \\ Q = & \begin{cases} 0 & +\frac{1}{3}A \operatorname{tang}.\frac{v\sin.\Phi}{1-v\cos.\Phi} \\ +\frac{1}{2\sqrt{3}}l\sqrt{[1+2v\cos.(60^\circ+\Phi)+vv]} + \frac{1}{2}A \operatorname{tang}.\frac{v\sin.(60^\circ+\Phi)}{1+v\cos.(60^\circ+\Phi)} \\ -\frac{1}{2\sqrt{3}}l\sqrt{[1+2v\cos.(60^\circ-\Phi)+vv]} + \frac{1}{2}A \operatorname{tang}.\frac{v\sin.(60^\circ-\Phi)}{1+v\cos.(60^\circ-\Phi)}. \end{cases} \end{aligned}$$

Exemplum 6.

§. 18. Sit nunc $m = 1$ et $n = 4$, ut formula integranda fiat $\int \frac{z dz}{1-z^4} = P + Q\sqrt{-1}$, posito $z = v(\cos.\Phi + \sqrt{-1}\sin.\Phi)$.

Quia hic $n = 4$, pro angulis ω et ζ quaternos valores adipiscimur, scilicet

<i>i</i>	0	2	4	6
ω	$-\Phi$	$\frac{1}{2}\pi - \Phi$	$\pi - \Phi$	$\frac{3}{2}\pi - \Phi$
fin. ω	$-\text{fin. } \Phi$	$+\text{cof. } \Phi$	$-\text{fin. } \Phi$	$-\text{cof. } \Phi$
cof. ω	$+\text{cof. } \Phi$	$+\text{fin. } \Phi$	$-\text{cof. } \Phi$	$-\text{fin. } \Phi$
ζ	0	90°	180°	270°
fin. ζ	0	$+1$	0	-1
cof. ζ	$+1$	0	-1	0

Hinc jam litterae P et Q sequenti modo exprimentur:

$$P = \begin{cases} -\frac{1}{2}l\sqrt{(1-2v\text{cof.}\Phi+vv)} + 0 & +\frac{1}{4}A \text{ tang. } \frac{v\text{cof.}\Phi}{1-v\text{fin.}\Phi} \\ 0 & \\ +\frac{1}{2}l\sqrt{(1+2v\text{cof.}\Phi+vv)} + 0 & +\frac{1}{4}A \text{ tang. } \frac{v\text{cof.}\Phi}{1+v\text{fin.}\Phi} \\ 0 & \end{cases}$$

$$Q = \begin{cases} 0 & +\frac{1}{4}A \text{ tang. } \frac{v\text{fin.}\Phi}{1-v\text{cof.}\Phi} \\ -\frac{1}{2}l\sqrt{(1-2v\text{fin.}\Phi+vv)} + 0 & \\ 0 & +\frac{1}{4}A \text{ tang. } \frac{v\text{fin.}\Phi}{1+v\text{cof.}\Phi} \\ +\frac{1}{2}l\sqrt{(1+2v\text{fin.}\Phi+vv)} + 0. & \end{cases}$$

§. 19. Super hoc exemplo notasse juuabit esse

$$\int \frac{\partial z}{1-z^2} = \frac{1}{2} \int \frac{\partial z}{1-zz} + \frac{1}{2} \int \frac{\partial z}{1+zz}.$$

Modo autem vidimus pro formula $\int \frac{\partial z}{1-zz}$ esse

$$P = -\frac{1}{2}l\sqrt{(1-2v\text{cof.}\Phi+vv)} + \frac{1}{2}l\sqrt{(1+2v\text{cof.}\Phi+vv)} \text{ et}$$

$$Q = +\frac{1}{2}A \text{ tang. } \frac{v\text{fin.}\Phi}{1-v\text{cof.}\Phi} + \frac{1}{2}A \text{ tang. } \frac{v\text{fin.}\Phi}{1+v\text{cof.}\Phi}.$$

Pro altera vero formula $\int \frac{\partial z}{1+zz}$ in superiore dissertatione §. 30. et feqq. inuenimus

$$P = \frac{1}{2}A \text{ tang. } \frac{v\text{cof.}\Phi}{1-vv} \text{ et } Q = \frac{1}{4}l \frac{1+2v\text{fin.}\Phi+vv}{1-2v\text{fin.}\Phi+vv}$$

quos autem valores ob arcum circuli hic contractum potius ex formulis problematis generalis §. 54. et feqq. deriuemus.

Erit

Erit enim, posito ibi $m = 1$, $n = 2$, pro forma integrali $\int \frac{z^2}{1+z^2}$ valor

$$P = \frac{1}{2} A \operatorname{tang.} \frac{v \operatorname{cof.} \Phi}{1 - v \operatorname{sin.} \Phi} + \frac{1}{2} A \operatorname{tang.} \frac{v \operatorname{cof.} \Phi}{1 + v \operatorname{sin.} \Phi}$$

$$Q = -\frac{1}{2} l \sqrt{(1 + 2v \operatorname{sin.} \Phi + vv)} - \frac{1}{2} l \sqrt{(1 - 2v \operatorname{sin.} \Phi + vv)}.$$

Additis ergo binis P et Q per binarium diuisis prodit pro forma integrali $\int \frac{z^2}{1-z^2}$ valor

$$P = \begin{cases} +\frac{1}{4} l \sqrt{(1 + 2v \operatorname{cof.} \Phi + vv)} + \frac{1}{4} A \operatorname{tang.} \frac{v \operatorname{cof.} \Phi}{1 - v \operatorname{sin.} \Phi} \\ -\frac{1}{4} l \sqrt{(1 - 2v \operatorname{cof.} \Phi + vv)} + \frac{1}{4} A \operatorname{tang.} \frac{v \operatorname{cof.} \Phi}{1 + v \operatorname{sin.} \Phi} \end{cases}$$

$$Q = \begin{cases} +\frac{1}{4} l \sqrt{(1 + 2v \operatorname{sin.} \Phi + vv)} + \frac{1}{4} A \operatorname{tang.} \frac{v \operatorname{sin.} \Phi}{1 - v \operatorname{cof.} \Phi} \\ -\frac{1}{4} l \sqrt{(1 - 2v \operatorname{sin.} \Phi + vv)} + \frac{1}{4} A \operatorname{tang.} \frac{v \operatorname{sin.} \Phi}{1 + v \operatorname{cof.} \Phi} \end{cases}$$

prorsus vti supra inuenimus.

§. 20. Quanquam haec solutio satis est commoda et sine multis ambagibus ad optatum finem perducit, tamen aliam hic subjungam, quae quidem multo simplicior et breuior, ita tamen est comparata, vt ejus bonitas nequidem perspici queat, atque eatenus tantum admitti possit, quatenus ad veritatem jam aliunde cognitam perducit. In eo autem ista solutio a praecedente solutione recedit, quod primo denominatorem $1 - z^n$ ab imaginariis liberare non est opus; deinde etiam numerator ita tractari potest, vt quantitas v inde penitus elidatur, neque permixtio quantitatum realium et imaginariarum vllam moram faceffat.

Alia solutio Problematis.

§. 21. Cum posito $z = v (\operatorname{cof.} \Phi + \sqrt{-1} \operatorname{sin.} \Phi)$ esse debeat

$$\int \frac{z^{m-1} dz}{1-z^n} = P + Q \sqrt{-1},$$

statim

statim confidero denominatoris factorem $1 - 2v \cos. \omega + vv$, quo ergo posito $= 0$ etiam ipse denominator euanescere debet; inde autem fit $v = \cos. \omega + \sqrt{-1 \sin. \omega}$, et cum fit

$$z = v (\cos. \Phi + \sqrt{-1 \sin. \Phi}), \text{ erit}$$

$$z^n = v^n (\cos. n\Phi + \sqrt{-1 \sin. n\Phi}).$$

Quare cum fit $v^n = \cos. n\omega + \sqrt{-1 \sin. n\omega}$, hinc fiet

$$z^n = \cos. (n\omega + n\Phi) + \sqrt{-1 \sin. (n\omega + n\Phi)},$$

quae expressio cum unitati debeat esse aequalis, erit $\cos. (n\omega + n\Phi) = 1$, unde fit $n\omega + n\Phi = i\pi$, denotante i numerum parem quemcumque, sicque altera pars $\sqrt{-1 \sin. (n\omega + n\Phi)}$ sponte euanescit. Cum igitur hinc fit $n\omega = i\pi - n\Phi$, erit $\omega = \frac{i\pi}{n} - \Phi$, unde n diuersi valores pro ω eliciuntur.

§. 22. Statuamus nunc fractionem partialem ex hoc factore oriundam esse $= \frac{F}{1 - 2v \cos. \omega + vv}$, atque ut supra vidimus, statui debet

$$F = z^{m-1} \partial z \cdot \frac{1 - 2v \cos. \omega + vv}{1 - z^n},$$

unde ope aequationis $vv - 2v \cos. \omega + 1 = 0$ iste valor F penitus a litteris z et v debet liberari. Quoniam autem hinc fractionis illius tam numerator quam denominator euanescit, sumtis differentialibus, ob $\partial. z^n = n z^{n-1} \partial z = n z^{n-1} \frac{\partial z}{v}$, quandoquidem in hac reductione anguli ω et Φ ut constantes spectari possunt, illa fractio induet hanc formam: $\frac{2(v - \cos. \omega)v}{n z^n}$.

Quoniam igitur $v - \cos. \omega = \sqrt{-1 \sin. \omega}$ et $z^n = 1$, erit ista fractio $= -\frac{2v \sqrt{-1 \sin. \omega}}{n}$, sicque habebimus

$$F = -\frac{2v}{n} z^{m-1} \partial z \sqrt{-1 \sin. \omega}.$$

§. 23. Cum nunc, sumto etiam angulo Φ variabili, sit

$$\frac{\partial z}{z} = \frac{\partial v}{v} + \partial \Phi \sqrt{-1}, \text{ ideoque}$$

$$\frac{z v \sqrt{-1}}{n} \cdot \frac{\partial z}{z} = \frac{z}{n} \partial v \sqrt{-1} - z v \frac{\partial \Phi}{n},$$

habebimus

$$F = -\frac{z}{n} z^m \partial v \sqrt{-1} \sin. \omega + \frac{z}{n} v z^m \partial \Phi \sin. \omega, \text{ siue}$$

$$F = \frac{z}{n} z^m \sin. \omega (v \partial \Phi - \partial v \sqrt{-1}).$$

Nunc vero, uti ante euoluimus potestatem z^n , hic simili modo euoluamus potestatem z^m , eritque

$$z^m = \cos. (m \omega + m \Phi) + \sqrt{-1} \sin. (m \omega + m \Phi),$$

quo valore introducto fiet

$$F = \frac{z}{n} \sin. \omega (v \partial \Phi - \partial v \sqrt{-1}) [\cos. (m \omega + m \Phi) + \sqrt{-1} \sin. (m \omega + m \Phi)].$$

Cum denique sit $\omega = \frac{i\pi}{n} - \Phi$, erit $m \omega + m \Phi = \frac{m i \pi}{n}$, quem ergo angulum si vocemus $= \zeta$, valor litterae F quaesitus erit

$$F = \frac{z}{n} \sin. \omega (v \partial \Phi - \partial v \sqrt{-1}) (\cos. \zeta + \sqrt{-1} \sin. \zeta),$$

quem partiamur in has partes:

$$F = +\frac{z}{n} \partial v \sin. \omega (\sin. \zeta - \sqrt{-1} \cos. \zeta)$$

$$+ \frac{z}{n} v \partial \Phi \sin. \omega (\cos. \zeta + \sqrt{-1} \sin. \zeta).$$

§. 24. Quia haec expressio ex partibus realibus et imaginariis constat, videri posset partes reales sumi debere pro valore litterae P, imaginarias pro $Q \sqrt{-1}$; verum hinc in crassissimum errorem illaberemur, quemadmodum ex collatione cum superiore solutione manifestum est. Interim tamen obseruavi, ex hac ipsa formula veros valores pro P et Q elici posse. Scilicet pro valore ipsius P inueniendo haec tota formula ex realibus et imaginariis permixta in valorem realem transformetur; tum enim eius semissis pro littera P valebit. Simili modo pro littera Q eandem expressionem totam in formam simplic-

pliciter imaginariam transfundi oportet, cuius pariter semissis pro valore litterae Q adhiberi debet; scilicet cum valor ipse F coefficientem habeat 2, ex altera semissi littera P, ex altera vero littera Q formari debet.

§. 25. Hinc ergo omisso factore formulam pro F inventam primo ad litteram P accomodemus, qui valor cum debeat esse realis, statuatur = A v + B, et loco v valorem cos. ω + √ - 1 sin. ω substituendo habebimus hanc aequationem:

$$\left\{ \begin{array}{l} +\frac{1}{n} \partial v \sin. \omega (\sin. \zeta - \sqrt{-1} \cos. \zeta) \\ +\frac{1}{n} v \partial \Phi \sin. \omega (\cos. \zeta + \sqrt{-1} \sin. \zeta) \end{array} \right\} = A \cos. \omega + B + A \sqrt{-1} \sin. \omega.$$

Hinc iam partibus realibus et imaginariis seorsim aequatis primo ex imaginariis elicitur:

$$A \sin. \omega = \frac{1}{n} \sin. \omega (-\partial v \cos. \zeta + v \partial \Phi \sin. \zeta),$$

vnde fit

$$A = -\frac{1}{n} (\partial v \cos. \zeta - v \partial \Phi \sin. \zeta).$$

Hic iam valor in aequalitate partium realium substitutus dabit

$$\frac{1}{n} \sin. \omega (\partial v \sin. \zeta + v \partial \Phi \cos. \zeta) = -\frac{\cos. \omega}{n} (\partial v \cos. \zeta - v \partial \Phi \sin. \zeta) + B$$

vnde colligitur

$$B = \frac{1}{n} \partial v \cos. (\zeta - \omega) - \frac{1}{n} v \partial \Phi \sin. (\zeta - \omega).$$

Hinc ergo pro littera P erit

$$F = -\frac{1}{n} \partial v [v \cos. \zeta - \cos. (\zeta - \omega)] + \frac{1}{n} v \partial \Phi [v \sin. \zeta - \sin. (\zeta - \omega)],$$

sicque ex factore denominatoris 1 - 2v cos. ω + vv habebimus

$$P = -\frac{1}{n} \int \frac{\partial v [v \cos. \zeta - \cos. (\zeta - \omega) - v \partial \Phi [v \sin. \zeta - \sin. (\zeta - \omega)]]}{1 - 2v \cos. \omega + vv}.$$

§. 26. Pro littera Q altera semissis litterae F aequetur huic quantitati simpliciter imaginariae: (Cv + D) √ - 1, vnde exorietur ista aequatio:

$$\left\{ \begin{aligned} +\frac{1}{n} \partial v \sin. \omega (\sin. \zeta - \sqrt{-1} \cos. \zeta) \\ +\frac{1}{n} v \partial \Phi \sin. \omega (\cos. \zeta + \sqrt{-1} \sin. \zeta) \end{aligned} \right\} = C \cos. \omega \sqrt{-1} + D \sqrt{-1} - C \sin. \omega.$$

Hinc ex partibus realibus concluditur

$$C = -\frac{1}{n} (\partial v \sin. \zeta + v \partial \Phi \cos. \zeta),$$

quo valore substituto ex partibus imaginariis haec emerget aequatio :

$$-\frac{1}{n} \sin. \omega (\partial v \cos. \zeta - v \partial \Phi \sin. \zeta) = -\frac{\cos. \omega}{n} (\partial v \sin. \zeta + v \partial \Phi \cos. \zeta) + D,$$

vnde eruitur

$$D = \frac{1}{n} \partial v \sin. (\zeta - \omega) + \frac{1}{n} v \partial \Phi \cos. (\zeta - \omega).$$

Hinc ergo pro littera Q habemus:

$$F = -\frac{1}{n} \partial v [v \sin. \zeta - \sin. (\zeta - \omega)] \\ - \frac{1}{n} v \partial \Phi [v \cos. \zeta - \cos. (\zeta - \omega)],$$

vnde valor ipsius Q ex factore $1 - 2v \cos. \omega + v v$ oriundus erit:

$$Q = -\frac{1}{n} \int \frac{\partial v [v \sin. \zeta - \sin. (\zeta - \omega)] + v \partial \Phi [v \cos. \zeta - \cos. (\zeta - \omega)]}{1 - 2v \cos. \omega + v v}.$$

§. 27. Quoniam haec solutio tam egregie cum praecedente convenit, id profecto casui fortuito tribui nequit; quam ob rem mihi quidem haec solutio prorsus singularis haud parum in recessu habere videtur, vnde eam Geometris perscrutandam proponere non dubito, vt eius soliditatem ex firmis principiis derivare conentur.