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Characterization of digraphs with equal domination graphs and underlying graphs

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Characterization of digraphs with equal domination graphs and underlying graphs

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Abstract

A domination graph of a digraph D, dom(D), is created using the vertex set of D and edge $\{u, v\} \in E[\text{dom}(D)]$ whenever $(u, z) \in A(D)$ or $(v, z) \in A(D)$ for every other vertex $z \in V(D)$. The underlying graph of a digraph D, UG(D), is the graph for which D is a biorientation. We completely characterize digraphs whose underlying graphs are identical to their domination graphs, UG$(D) = \text{dom}(D)$. The maximum and minimum number of single arcs in these digraphs, and their characteristics, is given.
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1. Introduction

Let D be a digraph with nonempty vertex set $V(D)$ and arc set $A(D)$. If $(u, v) \in A(D)$, then u is said to *dominate* v. A vertex is also considered to dominate itself. The *domination graph of D*, dom(D), is the graph associated with D where $V[\text{dom}(D)] = V(D)$ and $\{u, v\} \in E[\text{dom}(D)]$ whenever (u, z) or (v, z) are arcs in D for all vertices $z \neq u, v$. Fisher et al. [9–14] pioneered the research in the area of domination graphs, and concentrated most of their attention on the domination graphs of tournaments. Since that time, a variety of modifications to tournaments have been the focus of this research. These include domination graphs of regular tournaments [3,4], semicomplete digraphs [7], compressed tournaments [8], and the use of the domination graph as an operator [5].

The complete characterization of digraphs and their associated domination graphs is elusive. In this paper, we provide another piece of the characterization by examining those digraphs with underlying graphs that are equal to their domination graphs. The *underlying graph of D*, UG(D), is the graph for which D is a biorientation. D is considered a *biorientation* of G if for every $\{u, v\} \in E(G)$, $(u, v) \in A(D)$ or $(v, u) \in A(D)$ or both (u, v) and (v, u) are arcs in D. If both (u, v) and (v, u) are arcs in D for every edge $\{u, v\}$ in G, then D is referred to as the *complete biorientation* of G, denoted by $D = \overset{\leftrightarrow}{G}$. Any complete biorientation is therefore a *symmetric digraph*. We refer to an arc (u, v) where there is no companion arc (v, u) as a *single arc* or a *single outgoing arc* from u to v.

To begin, we connect the work done by Brigham and Dutton [2] on neighborhood graphs to the problem of characterizing digraphs where UG$(D) = \text{dom}(D)$. A *neighborhood graph* is the intersection graph of the neighborhoods of the vertices of a graph G. An *intersection graph* is any graph that represents the intersection of a family of sets, where

each vertex is a set, and there is an edge between any two sets with common elements. The *neighborhood* of a vertex u is the set of all vertices adjacent to u. The concept of a neighborhood graph is first used to consider digraphs that are complete biorientations of their underlying graphs and then generalize this to digraphs that are biorientations of their underlying graphs. In so doing, we characterize all digraphs where $UG(D) = \text{dom}(D)$.

2. Consequences of neighborhood graphs

Research done on the neighborhood graph of a graph G has results that relate directly to the focus of this paper. Before examining those, however, we must build the structure that makes such comparison possible. We use the relationship between a domination graph and a competition graph. Recall that in a domination graph, an edge $\{u, v\}$ is formed whenever vertices u and v dominate in D. Fig. 1 shows a digraph D and its domination graph.

The *competition graph* of a digraph D, $C(D)$, has the same vertex set as D, with edge $\{u, v\} \in E[C(D)]$ if and only if there exists a vertex z in D such that (u, z) and (v, z) are both arcs in D. Thus, u and v *compete* for z. The *complement* D^c of a digraph D is the loopless digraph with vertex set $V(D)$ in which arc (u, v) is in D^c if and only if it is not in D. In D^c, neither (u, z) nor (v, z) are arcs, so $\{u, v\}$ cannot be an edge in $\text{dom}(D^c)$. It follows that the domination graph of a digraph D is the complement of the competition graph of D^c, $\text{dom}(D) = [C(D^c)]^c$ [13]. That is to say, every edge in the competition graph is not in the domination graph, and vice versa.

To illustrate the relationship between $\text{dom}(D)$ and $[C(D^c)]^c$, Fig. 2 is the digraph D^c obtained from D in Fig. 1 and its associated competition graph. Although the digraphs shown are both complete biorientations, this relationship holds true for all digraphs.

Prior to reaching our characterization of the digraphs that are complete biorientations of their underlying graphs where $UG(D) = \text{dom}(D)$, we must introduce the concept of neighborhood graphs. Acharya and Vartak [1] introduced the characterization of a neighborhood graph in 1973. Neighborhood graphs have also been referred to as *two-step graphs* by authors such as Exoo and Harary [6] and Greenburg et al. [15]. The definition of a two-step graph is perhaps easier to connect to the concepts of domination and competition graphs, so we include it here. The *two-step graph* $S_2(G)$ of a graph G, has the same vertex set as G and $\{u, v\} \in E[S_2(G)]$ if and only if there exists a vertex $z \in V(G)$ such that $\{u, z\}, \{z, v\} \in E(G)$. Lundgren et al. [16,17] show that the competition graph of the complete biorientation of a graph G, a symmetric digraph, is the two-step graph $S_2(G)$. Therefore, given the complete biorientation of a graph G, $\overleftrightarrow{G}$, $C(\overleftrightarrow{G})$ is a neighborhood graph.

Brigham and Dutton [2] characterize all neighborhood graphs, $N(G)$, that are isomorphic to G. We will use that result in this section to ascertain the circumstances where equality occurs.

Theorem 1 (*Brigham and Dutton [2]*). *$N(G) \cong G$ if and only if every component of G is either an odd cycle or a complete graph having other than two nodes.*

When we take into account that a competition graph is a neighborhood graph, and we are interested in the specific competition graph obtained from D^c, the following corollary is obtained.
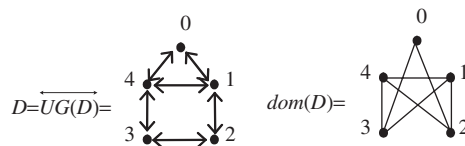

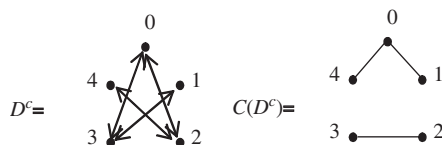Fig. 1. The complete biorientation of a graph and its associated domination graph.

Fig. 2. The complement of digraph D in Fig. 1, and its competition graph.

Corollary 2. $C[\overleftrightarrow{UG}(D^c)] \cong UG(D^c)$ *if and only if every component of the* $UG(D^c)$ *is either an odd cycle or a complete graph having other than two nodes.*

The relationship $C(D^c) = [\text{dom}(D)]^c$ observed earlier will now play an important role in relating Theorem 1 directly to the task of determining what complete biorientations of graphs will yield $UG(D) = \text{dom}(D)$. Toward that end, we define additional terms. First notice that if D is a complete biorientation, then $UG(D^c) = [UG(D)]^c$. Now, let

$$G^c = [UG(D)]^c = UG(D^c) = \bigcup_{i=1}^{p} G_i^c$$

such that all G_i^c, $1 \leqslant i \leqslant p - 1$, are components of G^c where each is either an odd cycle or a complete graph having more than two nodes. It is the convention adopted in this paper to let G_p^c be the subgraph consisting of all copies of K_1 if any exist. The *join* of two graphs G and H, $G + H$, is the graph that consists of $G \cup H$ and all edges joining a vertex in G and a vertex in H. All edges between the components will be in the complement of $UG(D^c)$, $UG(D)$. Thus, we can define the underlying graph of D in terms of the join of the complements of the G_i^c. Note that G_p is a complete graph.

$$G = UG(D) = \sum_{i=1}^{p} G_i.$$

To restate Corollary 2 as a theorem in terms of the domination graph, we need to use K_m^c, the complements of complete graphs for various values of m. Each graph forms an independent set of vertices, and these K_2-free graphs will be referred to as *independent sets*.

Theorem 3. *Let* $D = [\overleftrightarrow{UG}(D)]$. $UG(D) \cong \text{dom}(D)$ *if and only if* $UG(D)$ *is the join of independent sets with other than two vertices and components that are the complements of odd cycles.*

Proof. Since $D = \overleftrightarrow{UG}(D)$, it follows that $D^c = \overleftrightarrow{UG}(D^c)$, and $UG(D^c) = [UG(D)]^c$. So, the relationship $C(D^c) = [\text{dom}(D)]^c$ gives us $C[\overleftrightarrow{UG}(D)] = [\text{dom}(D)]^c$. Therefore, by Corollary 2, $[UG(D)]^c \cong [\text{dom}(D)]^c$ if and only if every component of $[UG(D)]^c$ is either an odd cycle or a complete graph having other than two nodes. The theorem follows. $\square$

Using these results, we now concentrate our attention on equality when D is a complete biorientation of its underlying graph. First, we address equality in neighborhood graphs so that we may use the consequences for our domination graphs.

Lemma 4. $N(G) = G$ *if and only if every component of G is a complete graph having other than two nodes.*

Proof. This follows directly from the work in [2]. If every component of G consists of a complete graph having other than two vertices, then it is clear that $G = N(G)$. By Theorem 1, if $N(G) = G$ and G contains a component different than a complete graph on other than two vertices, it must be a cycle on 5 or more vertices. Such a cycle must contain three vertices u, v, w with u and v adjacent to w but not to each other. Thus, u and v are adjacent in $N(G)$, but not in G, a contradiction. Hence, every component of G must be a complete graph having other than two vertices. $\square$

This leads directly to a characterization of equality for $\text{dom}(D)$ and $UG(D)$ when D is a complete biorientation.

Theorem 5. *Let* $D = \overleftrightarrow{UG}(D)$. $UG(D) = \text{dom}(D)$ *if and only if* $UG(D)$ *is the join of independent sets with other than two vertices.*

Proof. This is similar to the proof of Theorem 3 and follows from Lemma 4. $\square$

Fig. 3 illustrates the case where there is isomorphism without equality. Here, D is the complete biorientation of C_5.
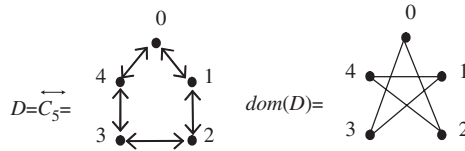
Fig. 3. $\mathrm{UG}(\overleftrightarrow{C_5})$ is isomorphic to $\mathrm{dom}(\overleftrightarrow{C_5})$.

3. *D* is a biorientation of UG(*D*)

To expand the characterization to digraphs that are not complete biorientations, we will establish conditions that must occur. When $\mathrm{UG}(D) = \mathrm{dom}(D) = K_n$, Factor and Factor [7] characterized all semicomplete digraphs with this property. The result is given in terms of *semicomplete digraphs*, where a digraph *D* is semicomplete if it is a biorientation of a complete graph. The theorem focuses on the construction of the oriented edges (single arcs) in *D*. If $d_s^+(u)$ denotes the number of oriented edges outgoing from a vertex *u*, then the result is expressed as follows.

Theorem 6 (*Factor and Factor [7]*). *Let D be a semicomplete digraph on n vertices. D has at most n oriented edges, and $d_s^+(u) \leqslant 1$ for every $u \in V(D)$ if and only if $\mathrm{dom}(D) = K_n$.*

Now we will build the foundations to help us characterize the other digraphs with the property that $\mathrm{UG}(D) = \mathrm{dom}(D)$. Since *D* is no longer a complete biorientation, it is not symmetric and the neighborhood graph results cannot be directly applied. Therefore, we will first explore the structure of the underlying graphs which support biorientations with this property. After these graphs have been identified, the focus will turn to necessary and sufficient conditions for edge (bi)orientation.

To begin, we look to see if, by allowing single arcs, there are more possibilities in the structure of the underlying graph than were available in Theorem 5. We will use the following notation that follows the conventions set forth earlier. Let $G = \mathrm{UG}(D) = \sum_{i=1}^{p} G_i$ where $G^c = \bigcup_{i=1}^{p} G_i^c$ is the union of components, except for G_p^c, which is the collection of all isolated vertices in G^c. If there are no isolated vertices, then $V(G_p^c) = \emptyset$. First, we will prove that under certain conditions, it is possible that $G_i^c = K_2$ for some $i = 1, \dots, p - 1$.

Proposition 7. *If $G = \mathrm{UG}(D) = \mathrm{dom}(D)$ is a graph on $n \geqslant 3$ vertices and $\{u, v\} = K_2 = G_i^c$ for some $i = 1, \dots, p - 1$, then there exists a vertex $w \notin V(G_i^c)$ such that $\{u, w\}$ and $\{v, w\}$ are both edges in G and $(w, u), (w, v)$ are single arcs in D.*

Proof. Since $\{u, v\} \in E(G^c)$, $\{u, v\} \notin E(G)$ and does not dominate in *G*. Since G_i^c is a component in G^c, vertices *u* and *v* are joined to all other vertices in *G*, of which we are guaranteed at least one because $n \geqslant 3$. For them to not dominate, there must exist a vertex *w* such that (w, u) and (w, v) are single arcs in any biorientation of *G*. □

Thus, we may have K_2 as a component of G^c as long as another vertex exists that we are able to use to orient an edge toward each of the vertices in K_2. Not all vertices are appropriate for this use, however, and we address that issue when we begin determining biorientations that lead to our equality. For now, we illustrate Proposition 7 with an underlying graph that has three copies of K_2^c. Fig. 4 shows $G = \mathrm{UG}(D) = \sum_{i=1}^{4} G_i$ where $G_1^c = G_2^c = G_3^c = K_2$ and G_4^c contains three isolated vertices w_1, w_2, w_3. The biorientation of *G* that is shown is actually the only one possible for this underlying graph that results in $\mathrm{UG}(D) = \mathrm{dom}(D)$. To avoid a busy digraph, in the remainder of the paper bidirectional edges are represented using lines, while single arcs are represented traditionally. Vertices forming G_1, G_2 and G_3 have been circled.

To conclude discussion on the structure of the possible components of G^c, we must determine if any components other than complete graphs are acceptable. Suppose we have a component $G_i^c \neq K_m$. The following lemma states that this will not result in an underlying graph where an orientation exists giving $\mathrm{UG}(D) = \mathrm{dom}(D)$.

Lemma 8. *If $G = \mathrm{UG}(D) = \mathrm{dom}(D)$ is a graph on $n \geqslant 3$ vertices, and $G^c = \bigcup_{i=1}^{p} G_i^c$ is the union of components, except for G_p^c, which is the collection of all isolated vertices in G^c, then all G_i^c for $i = 1, \dots, p - 1$ are complete graphs.*
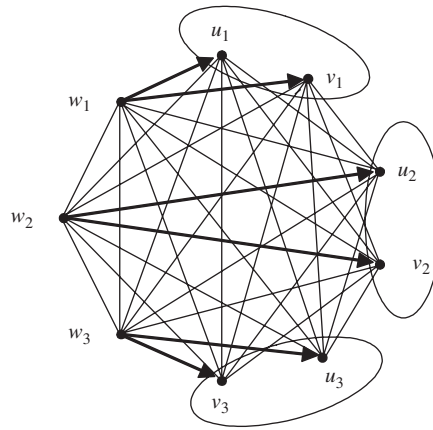
Fig. 4. Biorientation D of the join of 3 copies of K_2^c and K_3, where $\mathrm{UG}(D) = \mathrm{dom}(D)$.

Proof. Without loss of generality, say that G_1^c on vertices $1, \ldots, m$, is not a complete graph. Then there exist two vertices $u, v \in \{1, \ldots, m\}$ such that $\{u, v\} \notin E(G_1^c)$. Therefore, $\{u, v\} \in E(G_1)$. Since $\mathrm{UG}(D) = \mathrm{dom}(D)$, u and v dominate. Consider u and v in G_1^c. There are at least three vertices in G_1^c, and it is connected, so there is a shortest simple uv-path in G_1^c. Suppose that it is u, u_1, v. Then $\{u, u_1\}$ and $\{u_1, v\}$ are edges in G_1^c, but not in G_1. Thus, u and v do not dominate u_1, and $\{u, v\}$ is not an edge in $\mathrm{dom}(D)$, which contradicts $\mathrm{UG}(D) = \mathrm{dom}(D)$. The uv-path must then be longer, and of the form $u, u_1, \ldots, u_k, v$ for some $k \geqslant 2$. This also implies that the path u_{k-1}, u_k, v is the shortest path from u_{k-1} to v. But then $\{u_{k-1}, u_k\}$ and $\{u_k, v\}$ are not edges in G_1, but $\{u_{k-1}, v\}$ is. Vertices u_{k-1} and v do not dominate u_k, so $\{u_{k-1}, v\}$ is not an edge in $\mathrm{dom}(D)$, contradicting $\mathrm{UG}(D) = \mathrm{dom}(D)$. Therefore, no component can have fewer edges than a complete graph. $\quad\square$

Now we rephrase Lemma 8 in terms of the structure necessary for our underlying graph.

Theorem 9. *If* $\mathrm{UG}(D) = \mathrm{dom}(D)$ *is a graph on* $n \geqslant 3$ *vertices, then* $\mathrm{UG}(D)$ *is the join of independent sets.*

What now remains is to characterize the biorientations of these underlying graphs and to enumerate the number of vertices that must be present for the $\mathrm{UG}(D)$ to equal $\mathrm{dom}(D)$. To begin, we will examine vertices w that Proposition 7 states must be in any graph containing an independent set K_2^c where there exists a biorientation yielding $\mathrm{UG}(D) = \mathrm{dom}(D)$. The previous Fig. 4 illustrates an underlying graph G, where G^c has three copies of K_2. Each $K_2^c = G_i$ has a separate vertex w_i associated with it where (w_i, u_i) and (w_i, v_i) are single arcs, and all remaining edges are bidirectional in that example. The following results guarantee that there must always be a distinct vertex w_i for each copy of K_2^c in $\mathrm{UG}(D)$ in any biorientation where $\mathrm{UG}(D) = \mathrm{dom}(D)$.

Lemma 10. *If* $G = \sum_{i=1}^{p} G_i$ *has a biorientation* D *such that* $\mathrm{UG}(D) = \mathrm{dom}(D)$, *and* $G_1, \ldots, G_m = K_2^c$ *for some* m, $2 \leqslant m \leqslant p - 1$, *then there exist vertices* $w_1, \ldots, w_m$, $w_i \neq w_j$, *such that* (w_i, u_i) *and* (w_i, v_i) *are single arcs in* D, *where* $\{u_i, v_i\} = V(G_i)$, *and* $i = 1, \ldots, m$.

Proof. Let $u_i, v_i \in V(G_i)$ for all $i = 1, \ldots, m$. We know from Proposition 7 that there exists a vertex w_i such that (w_i, u_i) and (w_i, v_i) are single arcs in D. Suppose that, for some G_i and G_j, $w_i = w_j = w$. This implies that (w, u_i), (w, v_i), (w, u_j), and (w, v_j) are single arcs in D. However, $\{u_i, u_j\}$ is an edge in $\mathrm{UG}(D)$, so must be an edge in $\mathrm{dom}(D)$, but neither u_i nor u_j dominates w. Therefore, $w_i \neq w_j$. $\quad\square$

Not only must a unique vertex w exist for each independent set K_2^c, it can only be a vertex from the subgraph G_p.

Lemma 11. *If* $G = \sum_{i=1}^{p} G_i$ *has a biorientation* D *such that* $\mathrm{UG}(D) = \mathrm{dom}(D)$, $G_i = K_2^c = \{u_i, v_i\}$ *for some* i, $1 \leqslant i \leqslant p - 1$, *and* (w, u_i), (w, v_i) *are single arcs in* D, *then* $w \in V(G_p)$, *where* G_p^c *is the collection of all copies of* K_1 *in* G^c.

Proof. Suppose that $w \in V(G_j)$ for some $j \neq i, p$. Consider another vertex $x \in V(G_j)$. By definition, G_i is joined to G_j, so all edges exist between the two subgraphs. For $u \in V(G_i)$, $\{u, x\}$ is an edge in UG(D), so must be an edge in dom(D) but neither u nor x dominates w. Therefore, w cannot be a vertex in any G_i for $i = 1, \ldots, p-1$, and must be in G_p. $\square$

At this time, we are able to completely characterize the structure a graph must have that possesses a biorientation D yielding UG(D) = dom(D).

Theorem 12. *A biorientation D of a graph G on $n \geq 3$ vertices exists such that* UG(D) = dom(D) *if and only if*

(1) $G = \sum_{i=1}^{p} G_i$ *where* G_i, $i = 1, \ldots, p-1$ *are independent sets and* $G_p = K_m$ *for some $m \geq 0$, and*
(2) *if we define the number of $G_i = K_2^c$ to be s, then $s \leq m$.*

Proof. ($\Rightarrow$) Theorem 9 gives us item (1) where $G_p = K_m$ is the join of the copies of K_1 in G^c. We know from Lemmas 10 and 11 that each copy of K_2^c in G must have a unique vertex in G_p from which single arcs will be oriented toward the vertices in K_2^c. Therefore, the number of vertices in G_p must be at least as many as the number of K_2^c in G, which verifies (2).

($\Leftarrow$) Let $G_1, \ldots, G_s$ be the K_2^c in G, and let $u_i, v_i \in G_i$. $s \leq m$ implies that there are at least s vertices in G_p, $w_1, \ldots, w_s$. Create single arcs (w_i, u_i) and (w_i, v_i) for all $i = 1, \ldots, s$. Between all other pairs of adjacent vertices, orient the edges in both directions. This biorientation creates a digraph D such that UG(D) = dom(D). To verify this claim, we see that $u_i, v_i \in V(K_2^c)$ is not an edge in UG(D) and will not be an edge in dom(D) since neither u_i nor v_i dominates w_i. For u_j, v_j vertices in a larger independent set, there is also a vertex x_j in that same set, so neither u_j nor v_j dominate x_j. Thus, $\{u_j, v_j\}$ is not an edge in UG(D) and is not an edge in dom(D). For any vertex $w \in V(G_p)$, w is adjacent to all other vertices in the join, it dominates all other vertices in D, and thus, $\{w, x\} \in E[\text{UG}(D)]$ and $\{w, x\} \in E[\text{dom}(D)]$ for any vertex $x \neq w$. Finally, suppose $u_i \in V(G_i)$ and $u_j \in V(G_j)$ for some $i < j < p$. The vertex u_i dominates all other vertices except for those others in G_i and possibly one $w_i \in G_p$. The vertex u_j dominates all other vertices except for those others in G_j and possibly one $w_j \in G_p$. Thus, u_i and u_j dominate in D, so $\{u_i, u_j\} \in E[\text{UG}(D)]$ and $\{u_i, u_j\} \in E[\text{dom}(D)]$. $\square$

Now we will focus on the types of biorientations that can be applied to these underlying graphs. We begin this exploration by further examining the possibilities of single arcs outgoing from the vertices $w \in V(G_p)$. Lemmas 10 and 11 state that single arcs must be constructed from a unique vertex in $V(G_p)$ to each copy of $G_i = K_2^c$. We go beyond this, and determine what must be true if w has a single arc to any vertex.

Lemma 13. *If $G = \sum_{i=1}^{p} G_i$ with $p \geq 2$, such that G_i is an independent set of two or more vertices for $i = 1, \ldots, p-1$, G_p is a complete graph, and D is a biorientation of G such that* UG(D) = dom(D), *then for any $w \in V(G_p)$ when (w, u) is a single arc in D, then (w, v) can be a single arc in D only if $u, v \in V(G_i)$ for some $i = 1, \ldots, p-1$.*

Proof. Let (w, u) be a single arc in D for some biorientation of G such that UG(D) = dom(D) and $w \in V(G_p)$. If (w, v) is a single arc from w to v, then u and v fail to be a dominating pair. Since UG(D) = dom(D), there can be no arcs between u and v. Therefore, by construction, u and v must be elements of $V(G_i)$ for some $i = 1, \ldots, p-1$. $\square$

Although each independent set K_2^c requires a separate vertex in G_p whose single arcs oriented toward the two vertices prevents them from dominating, the same is not true of the independent sets with more than two vertices. The mere fact that we have at least vertices u, v, and w that do not dominate each other assures us that any biorientation of the edges in G will not coincidentally provide a structure where the domination graph will have an edge between one of the pairs of vertices. This is an important distinction in the characterization process of the biorientations, and also in the determination of the maximum and minimum number of single arcs that a biorientation may have.

Fig. 5 illustrates the relationships described in Lemma 13, and includes an independent set of size larger than two. Subgraphs $G_1 = K_2^c$, $G_2 = K_3^c$ and $G_3 = G_p = K_3$. Vertex w_i has outgoing arcs to u_1 and v_1 so $\{u_1, v_1\} \notin E[\text{dom}(D)]$, and cannot be incident with another single outgoing arc. Within G_p, w_2 has a single arc to w_3, and all other arcs incident with w_2 cannot be single outgoing arcs in order to preserve domination with the other vertices and w_3. The independent
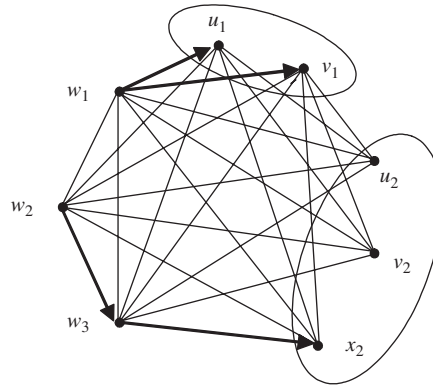
Fig. 5. Orientation D of a graph with a single arc in G_p and a single arc oriented to K_3^c.

set K_3^c does not require a single vertex dominating its three vertices. Arc (w_3, x_2) is an example of a single arc that is not necessary to preserve equality in the domination graph, but whose existence does not alter any dominating pairs. Note that vertex w_3 is incident with two single arcs, but only one is outgoing. The only other possible outgoing arcs from w_3 would be to vertices u_2 and v_2. Unlike Fig. 4, there are many biorientations that can be obtained from the underlying graph such that $\mathrm{UG}(D) = \mathrm{dom}(D)$.

The final construction question involving single arcs concerns the vertices in G_i for $i = 1, \ldots, p - 1$. We find that no vertex in these subgraphs can be incident with any single outgoing arcs.

Lemma 14. *If $G = \sum_{i=1}^{p} G_i$ for $p \geqslant 2$, G_i for $i = 1, \ldots, p - 1$ are each independent sets on 2 or more vertices, G_p is a complete graph, D is a biorientation of G such that $\mathrm{UG}(D) = \mathrm{dom}(D)$, and $u_i \in V(G_i)$, then u_i is not incident with any single outgoing arc.*

Proof. If (u_i, x) is an orientation of an edge in G and v_i is another vertex in G_i, then neither v_i nor x dominates u_i. Since $\{v_i, x\} \in E(G)$, they must also be a dominating pair, and (u_i, x) cannot be a single arc. $\square$

Lemma 14 gives us the final piece that we need to now characterize all digraphs D where $\mathrm{UG}(D) = \mathrm{dom}(D)$. Since D is a biorientation of its underlying graph, the conditions, which are stated regarding single arcs, implicitly dictate that some of the underlying edges be bidirectional. Other edges, where there is no explicit or implicit mention, can be either single arcs or bidirectional. These issues are addressed explicitly as corollaries in the next section, where they will be used to determine the maximum and minimum number of single arcs in a biorientation.

Theorem 15. *Let D be a biorientation of its underlying graph on $n \geqslant 3$ vertices. Then $\mathrm{UG}(D) = \mathrm{dom}(D)$ if and only if*

(1) *$\mathrm{UG}(D) = \sum_{i=1}^{p} G_i$ where G_i is an independent set for $i = 1, \ldots, p - 1$, G_p is a complete graph K_m for some $m \geqslant 0$, and the number of $G_i = K_2^c$ is $s \leqslant m$;*
(2) *For any $u_i, v_i \in V(G_i)$ where $G_i = K_2^c$, there exists a vertex $w_i \in V(G_p)$ such that (w_i, u_i) and (w_i, v_i) are single arcs in D;*
(3) *For any $w \in V(G_p)$, if $d_s^+(w) \geqslant 2$, then for any vertices u and v such that (w, u) and (w, v) are single arcs, $u, v \in V(G_i)$ for $i = 1, \ldots, p - 1$; and*
(4) *For any $u \in V(G_i)$ for $i = 1, \ldots, p - 1$, u has no single outgoing arcs.*

Proof. ($\Rightarrow$) Condition (1) follows from Theorem 12. Lemmas 7 and 10 imply part (2), while Lemmas 13 and 14 give us conditions (3) and (4), respectively.

($\Leftarrow$) Let D be a digraph on n vertices with characteristics (1)–(4).

(1) We examine K_2^c. Let $u_i, v_i \in V(G_i)$ where $G_i = K_2^c$. $\{u_i, v_i\} \notin E[\mathrm{UG}(D)]$. Condition (2) says that there exists a vertex w_i that neither u_i nor v_i dominates. Thus, $\{u_i, v_i\} \notin E[\mathrm{dom}(D)]$.

(2) We examine other independent sets. Let $u_i, v_i \in V(G_i)$ where $G_i = K_q^c$ for some $q \geqslant 3$. $\{u_i, v_i\} \notin E[\mathrm{UG}(D)]$. There exists a third vertex $x_i \in V(G_i)$. Vertices u_i and v_i do not dominate x_i, so $\{u_i, v_i\} \notin E[\mathrm{dom}(D)]$.

(3) We examine edges within G_p. For any $w_i, w_j \in V(G_p)$, $\{w_i, w_j\} \in E[\mathrm{UG}(D)]$. Suppose that $\{w_i, w_j\} \notin E[\mathrm{dom}(D)]$. Then there exists a vertex x such that (x, w_i) and (x, w_j) are single arcs in D. Condition (4) says that there are no single arcs from any vertex in G_i where $i \neq p$. Therefore, x must be in G_p. But condition (3) says that x can only have more than one outgoing arc to vertices outside of G_p. So there can be no such x, and $\{w_i, w_j\} \in E[\mathrm{dom}(D)]$.

(4) We examine edges between all G_i, $i = 1, \ldots, p$. Let $u, v \in V(D)$ such that $u \in V(G_i)$ for $i = 1, \ldots, p-1$, and $v \notin V(G_i)$. Thus, $\{u, v\} \in E[\mathrm{UG}(D)]$.

   (a) If $v \in V(G_j)$, $j \neq i, p$, by the construction in (1) v is joined to every vertex except for those in G_j. Condition (3) precludes any vertex in G_k for $k \neq p$ from having a single arc outgoing to v, so v dominates all vertices in these G_k. There may be a set of vertices $w_1, \ldots, w_q$ in G_p that have single outgoing arcs to v. But construction (3) guarantees that these vertices cannot have single outgoing arcs to vertex u, so u dominates them. Thus, all other vertices are dominated by u and v, and $\{u, v\} \in E[\mathrm{dom}(D)]$.

   (b) If $v \in V(G_p)$, then v dominates all vertices in G_i for $i = 1, \ldots, p-1$ since G_p is joined to them by condition (1), and no vertices within them have single outgoing arcs from condition (4). If there exists any vertices in G_p with single outgoing arcs to v, then condition (3) precludes those vertices from having any single outgoing arcs to vertex u. Thus, u dominates any vertices that v does not, and $\{u, v\} \in E[\mathrm{dom}(D)]$.

Therefore, from cases 1–4, $\mathrm{UG}(D) = \mathrm{dom}(D)$. $\square$

4. Minimum and maximum number of single arcs

The characterization of all digraphs with the property $\mathrm{UG}(D) = \mathrm{dom}(D)$ naturally leads to the question of how many single arcs such a digraph may possess. We know where there must be single arcs and where there must be bidirectional arcs. However, circumstances exist where either may occur. The following results follow from Theorem 15, but were not stated explicitly. They are included for use in determining the maximum and minimum number of single arcs that may occur in D. The first corollary concludes that all single arcs must originate from the vertices in G_p.

Corollary 16. *If* $\mathrm{UG}(D) = \mathrm{dom}(D)$, $\mathrm{UG}(D) = \sum_{i=1}^{p} G_i$, *where* G_i *is an independent set for* $i = 1, \ldots, p-1$, G_p *is a complete graph* K_m *for some* $m \geqslant 0$, *and* (w, u) *is a single arc in* D, *then* $w \in G_p$.

Next, we state explicitly what was implicit within our characterization of D. Namely, that different vertices in G_p may actually have single arcs directed toward the same vertex.

Corollary 17. *If* $\mathrm{UG}(D) = \mathrm{dom}(D)$, $\mathrm{UG}(D) = \sum_{i=1}^{p} G_i$, *where* G_i *is an independent set for* $i = 1, \ldots, p-1$, G_p *is a complete graph* K_m *for some* $m \geqslant 0$, *and* $u \in V(D)$, *then it is possible that there are two or more single arcs incoming to* u.

The following theorem gives the number of single arcs that can occur in a biorientation D of G resulting in $\mathrm{UG}(D) = \mathrm{dom}(D)$. All numbers between the maximum and minimum are possible by simply changing any number of the unnecessary single arcs to bidirectional arcs.

Theorem 18. *Let* D *be a digraph on* $n \geqslant 3$ *vertices. If* $\mathrm{UG}(D) = \mathrm{dom}(D)$, *and we let*

(1) *s be the number of copies of* $G_i = K_2^c$,
(2) *m be the number of vertices in* $G_p = K_m$,
(3) *M be the size of the largest independent set,* $G_j = K_M^c$ *for* $p \neq 1$, *or* $M = 1$ *for* $p = 1$.

Then there is a minimum of $2s$ *single arcs and a maximum of* $M(m - s) + 2s$ *single arcs in* D.

Proof. To determine the minimum number of arcs, we know that there must be two single arcs as a minimum for every copy of K_2^c in the underlying graph. There are no other mandatory single arcs in Theorem 15. Since there are s copies of K_2^c, we have a minimum of $2s$ single arcs in D.
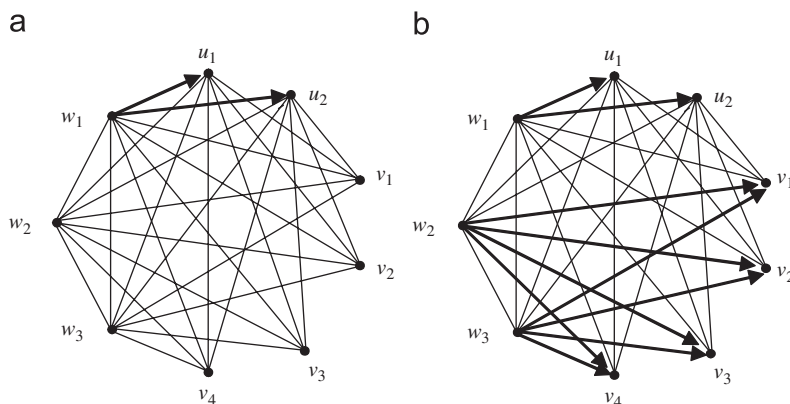
a

b

Fig. 6. Biorientations with (a) minimum number and (b) maximum number of single arcs.

To determine the maximum number of arcs, we know from Corollary 16 that all single arcs must originate from G_p. If $p > 1$, Theorem 15 restricts the number of single outgoing arcs to one if they are directed within G_p, so D has more single arcs if they are directed toward the G_i, $i = 1, \ldots, p - 1$. Since $\mathrm{UG}(D) = \mathrm{dom}(D)$, the restrictions for the K_2^c, must be met. So D must have vertices $w_1, \ldots, w_s$ within G_p, which have single arcs to each copy of K_2^c, giving $2s$ single arcs. That leaves $m - s$ vertices in G_p from which single arcs originate in D. Corollary 17 states that D may have more than one vertex in G_p with an outgoing arc to the same vertex outside of G_p. However, each $w_i \in V(G_p)$ can only have outgoing arcs to one G_i (Theorem 15). Therefore, a digraph D has a maximum number of single arcs if there are single arcs from the remaining vertices $w_{s+1}, \ldots, w_m$ to all of the vertices in K_M^c. This adds an additional $M(m - s)$ single arcs, giving a maximum of $M(m - s) + 2s$ when $M \geqslant 2$. If $p = 1$, so $G_p = G_1 = K_m$, we know from Theorem 6 that there are at most m single arcs. Since $s = 0$ and $M = 1$ in this case, the maximum can be expressed as $M(m - s) + 2s = m$. $\square$

Fig. 6 shows two digraphs with the same underlying graph where $\mathrm{UG}(D) = \mathrm{dom}(D)$. The biorientation in 6(a) contains the fewest number of single arcs, while that in 6(b) has the maximum number. Here, we have $G_1 = K_2^c$, $G_2^c = K_4^c$, and $G_3 = G_p = K_3$. Thus, $s = 1$, $m = 3$, and $M = 4$. The only single arcs in 6(a) are those from w_1 to u_1 and u_2, which are mandated in Theorem 15, giving the minimum number of single arcs $2(1) = 2$. To increase to the maximum number, we orient arcs from w_2 and w_3 to the largest independent set in $\mathrm{UG}(D)$. This produces the maximum number of single arcs, $4(3 - 1) + 2(1) = 10$.

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