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De integrationibus maxime memorabilibus ex calculo imaginariorum oriundis

Leonhard Euler

DE INTEGRATIONIBVS MAXIME MEMORABILIBVS EX CALCVLO IMAGINARIORVM ORIVNDIS.

Auctore

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§. 1.

Considero hic in genere formulam differentialem quamcunque $Z\partial z$, cuius integrale saltem per logarithmos et arcus circulares exhibere liceat, quod per characterem $\Delta : z$ designo, ita vt sit $\int Z\partial z = \Delta : z$. Iam loco z scribo quantitatem quamcunque imaginariam, scilicet $z = x + y\sqrt{-1}$, vnde functio Z transmutetur in formam $M + N\sqrt{-1}$. Hoc modo formа differentialis euadet $(\partial x + \partial y\sqrt{-1})(M + N\sqrt{-1})$, cuius producti pars realis ergo erit $M\partial x - N\partial y$, imaginaria vero $(N\partial x + M\partial y)\sqrt{-1}$. Tum vero ipsum integrale, quod est $\Delta : (x + y\sqrt{-1})$, transmutari poterit in similem formam $P + Q\sqrt{-1}$. Quare cum quantitates reales et imaginariae seorsim inter se conferri debeant, hinc duplex integratio orietur:

$$\text{I. } P = \int(M\partial x - N\partial y),$$

$$\text{II. } Q = \int(N\partial x + M\partial y),$$

 quae

quae ergo duae formulae ſemper erunt integrabiles, etiamſi binas variabiles x et y inuoluant. Erit ſcilicet per notum integrabilitatis criterium tam $\left(\frac{\partial M}{\partial y}\right) = -\left(\frac{\partial N}{\partial x}\right)$, quam $\left(\frac{\partial N}{\partial y}\right) = \left(\frac{\partial M}{\partial x}\right)$. Vnde intelligitur, ex qualibet formula differentiali propoſita binas deduci poſſe integrationes eo magis notatu dignas et arduas, quo magis integrale fuerit complicatum, quam ob rem plures caſus euoluiſſe operae erit pretium.

I. Euolutio
formulae differentialis $z^n \partial z$.

§. 2. Cum igitur ſit $\int z^n \partial z = \frac{z^{n+1}}{n+1}$, ſi loco z ſcribamus $x + y\sqrt{-1}$, hae poteſtates binomii, in vſum vocando characteres, quibus iam ſaepius vncias deſignaui, euolutae dabunt

$$(x+y\sqrt{-1})^n = x^n + \left(\tfrac{n}{1}\right) x^{n-1} y\sqrt{-1} - \left(\tfrac{n}{2}\right) x^{n-2} yy$$
$$- \left(\tfrac{n}{3}\right) x^{n-3} y^3 \sqrt{-1} + \text{etc.}$$

Hinc colligitur fore

$$M = x^n - \left(\tfrac{n}{2}\right) x^{n-2} yy + \left(\tfrac{n}{4}\right) x^{n-4} y^4 - \left(\tfrac{n}{6}\right) x^{n-6} y^6 + \text{etc. et}$$
$$N = \left(\tfrac{n}{1}\right) x^{n-1} y - \left(\tfrac{n}{3}\right) x^{n-3} y^3 + \left(\tfrac{n}{5}\right) x^{n-5} y^5 - \text{etc.}$$

Simili modo pro forma integralis erit

$$(n+1)P = x^{n+1} - \left(\tfrac{n+1}{2}\right) x^{n-1} yy + \left(\tfrac{n+1}{4}\right) x^{n-3} y^4$$
$$- \left(\tfrac{n+1}{6}\right) x^{n-5} y^6 + \text{etc.}$$
$$(n+1)Q = \left(\tfrac{n+1}{1}\right) x^n y - \left(\tfrac{n+1}{3}\right) x^{n-2} y^3 + \left(\tfrac{n+1}{5}\right) x^{n-4} y^5 - \text{etc.}$$

§. 3. His valoribus determinatis, binae integrationes, quas hinc adipiſcimur, ita ſe habebunt:

$$P = \int \left\{ \begin{array}{l} \partial x \left[x^n - \left(\tfrac{n}{2}\right) x^{n-2} y^2 + \left(\tfrac{n}{4}\right) x^{n-4} y^4 - \left(\tfrac{n}{6}\right) x^{n-6} y^6 + \text{etc.}\right] \\ -\partial y \left[\left(\tfrac{n}{1}\right) x^{n-1} y - \left(^{n}\right) x^{n-3} y^3 + \left(\tfrac{n}{5}\right) x^{n-5} y^5 - \text{etc.}\right] \end{array} \right\}$$

quae

quae forma quemadmodum ipsi P aequetur per partes videamus. At est

$$\text{I. } \int x^n \partial x = \frac{x^{n+1}}{n+1},$$

quod cum primo termino seriei, §. 2. pro P inuentae, conuenit. Tum vero sumatur

$$\text{II. } -\int \left(\frac{n}{2}\right) x^{n-2} y^2 \partial x - \int \left(\frac{n}{1}\right) x^{n-1} y \partial y,$$

hinc ex parte priore, sumto y constante, oritur integrale

$$-\left(\frac{n}{2}\right) \frac{x^{n-1}}{n-1} yy,$$

ex parte vero posteriore, sumto x constante, orietur $-\left(\frac{n}{1}\right) x^{n-1} \frac{yy}{2}$, quae duae expressiones manifesto sunt inter se aequales, scilicet $= -\frac{n}{2} x^{n-1} yy$. At vero secunda pars ipsius P est

$$-\frac{1}{n+1} \left(\frac{n+1}{2}\right) x^{n-1} yy,$$

quae ob $\left(\frac{n+1}{2}\right) = \frac{n+1}{1} \cdot \frac{n}{2}$ manifesto fit $-\frac{n}{2} x^{n-1} yy$. Sumatur nunc

$$\text{III. } \int \left(\frac{n}{4}\right) x^{n-4} y^4 \partial x + \left(\frac{n}{3}\right) x^{n-3} y^3 \partial y.$$

Hic ex parte priore concluditur integrale $\frac{1}{n-3} \left(\frac{n}{4}\right) x^{n-3} y^4$; ex parte autem posteriore $\frac{1}{4} \left(\frac{n}{3}\right) x^{n-3} y^4$. Quoniam igitur est $\left(\frac{n}{4}\right) = \left(\frac{n}{3}\right) \frac{n-3}{4}$, hae duae formulae manifesto sunt inter se aequales, et integrale erit

$$\frac{1}{4} \left(\frac{n}{3}\right) x^{n-3} y^4 = \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{1}{4} x^{n-3} y^4.$$

Pars tertia autem formulae pro P datae est $\frac{1}{n+1} \cdot \left(\frac{n+1}{4}\right) x^{n-3} y^4$, qua ob $\left(\frac{n+1}{4}\right) = \frac{n+1}{1} \cdot \frac{n}{2} \cdot \frac{n-1}{3} \cdot \frac{n-2}{4}$ manifesto illi est aequalis. Simili modo conuenientia sequentium membrorum ipsius P ostenditur, simulque facile intelligitur pari modo consensum formulae Q ostendi posse.

§. 4. Quoties igitur exponens n est numerus integer positiuus, veritas nostrarum formularum manifesto in oculos incurrit. Verum si n fuerit vel numerus negatiuus vel fractus, tum

tum formulae pro litteris M et N, item P et Q, in infinitum excurrerent; vnde his casibus calculum alio modo instrui oportet. Scilicet loco x et y binas alias variabiles in calculum introduci conueniet, statuendo $\sqrt{(xx+yy)}=v$, et quaerendo angulum Φ, vt sit $\tang.\Phi=\frac{y}{x}$; tum autem erit $x=v\cos.\Phi$ et $y=v\sin.\Phi$, ideoque differentiando

$$\partial x=\partial v\cos.\Phi-v\partial\Phi\sin.\Phi \text{ et}$$
$$\partial y=\partial v\sin.\Phi+v\partial\Phi\cos.\Phi.$$

~~His autem positis erit~~

$$(x+y\sqrt{-1})^n=v^n(\cos.n\Phi+\sqrt{-1}\sin.n\Phi),$$

vnde colligitur

$$M=v^n\cos.n\Phi \text{ et } N=v^n\sin.n\Phi.$$

Deinde vero pro integrali erit

$$z^{n+1}=v^{n+1}[\cos.(n+1)\Phi+\sqrt{-1}\sin.(n+1)\Phi]$$

vnde habebitur

$$P=\frac{v^{n+1}\cos.(n+1)\Phi}{n+1} \text{ et } Q=\frac{v^{n+1}\sin.(n+1)\Phi}{n+1}.$$

§. 5. Cum nunc inuenerimus

$$P=\int(M\partial x-N\partial y) \text{ et } Q=\int(N\partial x+M\partial y),$$

facta substitutione fiet

$$P=\int[v^n\partial v\cos.(n+1)\Phi-v^{n+1}\partial\Phi\sin.(n+1)\Phi] \text{ et}$$
$$Q=\int[v^n\partial v\sin.(n+1)\Phi-v^{n+1}\partial\Phi\cos.(n+1)\Phi].$$

Ambae autem hae formulae manifesto integrationem admittunt, cum ex priore fiat

$$P=\frac{v^{n+1}}{n+1}\cos.(n+1)\Phi \text{ et}$$
$$Q=\frac{v^{n+1}}{n+1}\sin.(n+1)\Phi,$$

quae cum sint obuia ad maiora progrediamur.

II.

II. Euolutio

formulae differentialis $\frac{\partial z}{1+zz}$, cujus integrale est A tang. z.

§. 6. Cum hic sit $Z = \frac{1}{1+zz}$, posito $z = x + y\sqrt{-1}$ erit $Z = \frac{1}{1+2xy\sqrt{-1}+xx-yy}$. Hic ante omnia denominatorem ab imaginariis liberari oportet, quod fit numeratorem et denominatorem multiplicando per $1 + xx - yy - 2xy\sqrt{-1}$ fietque

$$Z = \frac{1+xx-yy-2xy\sqrt{-1}}{(1+xx-yy)^2+4xxyy},$$

sicque erit

$$M = \frac{1+xx-yy}{(1+xx-yy)^2+4xxyy} \text{ et } N = \frac{-2xy}{(1+xx-yy)^2+4xxyy}.$$

Hinc igitur pro integrali $P + Q\sqrt{-1}$ impetrabimus

$$P = \int \frac{(1+xx-yy)\partial x + 2xy\partial y}{(1+xx-yy)^2+4xxyy} \text{ et}$$

$$Q = \int \frac{(1+xx-yy)\partial y - 2xy\partial x}{(1+xx-yy)^2+4xxyy};$$

hasque ambas formulas jam certo scimus esse integrabiles.

§. 7. Consideremus accuratius denominatorem, qui evolvitur in hanc formam: $(xx+yy)^2 + 2(xx-yy) + 1$, quae porro reducitur ad $(xx+yy+1)^2 - 4yy$, quae ergo est productum ex his duobus factoribus:

$$(xx+yy+1-2y)(xx+yy+1+2y),$$

qui ergo factores sunt $xx+(y+1)^2$ et $xx+(y-1)^2$. Hanc obrem ambae illae fractiones resolvi poterunt in binas fractiones, quarum alterius denominator sit $xx+(y+1)^2$ et alterius $xx+(y-1)^2$. Ad hanc resolutionem faciendam vtamur resolutione generali fractionis $\frac{S}{PQ}$ in has duas fractiones: $\frac{F}{P} + \frac{G}{Q}$; vbi numerator F reperitur ex formula $\frac{S}{Q}$, ponendo $P = 0$, alter vero G ex formula $\frac{S}{P}$, ponendo $Q = 0$.

§. 8. Pro formula priori erit

S =

$$S = (1 + xx - yy)\partial x + 2xy\partial y,$$
$$P = xx + (y+1)^2 \text{ et } Q = xx + (y-1)^2;$$

quamobrem pro priore fractione $\frac{F}{P}$ littera F definiri debet ex fractione $\frac{(1+xx-yy)\partial x + 2xy\partial y}{xx+(y-1)^2}$, ponendo $xx + (y+1)^2 = 0$. Quare cum hinc fit $xx = -(y+1)^2$, hoc valore tam in numeratore quam in denominatore substituto, vbi quidem xx occurrit, reperietur $\frac{+2\partial x(yy+y)+2xy\partial y}{4y} = \frac{1}{2}\partial x(y+1) - \frac{1}{2}x\partial y$. Simili modo pro fractione $\frac{G}{Q}$ numerator G definiri debet ex hac fractione: $\frac{(1+xx-yy)\partial x + 2xy\partial y}{xx+(y+1)^2}$, ponendo $xx + (y-1)^2 = 0$, vnde fit $xx = -(y-1)^2$, quo valore substituto reperitur

$$G = \frac{2\partial x(y-yy)+2xy\partial y}{4y} = -\frac{1}{2}\partial x(y-1) + \frac{1}{2}x\partial y.$$

Hinc igitur habebimus

$$P = \tfrac{1}{2}\int\frac{\partial x(y+1) - x\partial y}{xx+(y+1)^2} - \tfrac{1}{2}\int\frac{\partial x(y-1) - x\partial y}{xx+(y-1)^2}.$$

§. 9. Nunc autem integratio harum formularum nulla amplius laborat difficultate. Si enim pro priore statuamus $y + 1 = tx$, erit $\partial y = t\partial x + x\partial t$, vnde haec formula integralis transmutabitur in

$$-\tfrac{1}{2}\int\frac{\partial t}{1+tt} = -\tfrac{1}{2}\text{ A tang. } t = -\tfrac{1}{2}\text{ A tang. }\frac{y+1}{x}.$$

Pro altera formula ponatur $y - 1 = ux$, ut fit $\partial y = u\partial x + x\partial u$, eaque abibit in

$$\tfrac{1}{2}\int\frac{\partial u}{1+uu} = \tfrac{1}{2}\text{ A tang. } u = \tfrac{1}{2}\text{ A tang. }\frac{y-1}{x}$$

quocirca adepti sumus valorem litterae P, qui est

$$P = \tfrac{1}{2}\text{ A tang. }\frac{y-1}{x} - \tfrac{1}{2}\text{ A tang. }\frac{y+1}{x}.$$

Cum nunc sit

$$\text{A tang. } a - \text{A tang. } b = \text{A tang. }\frac{a-b}{1+ab}, \text{ erit}$$

$$P = -\tfrac{1}{2}\text{ A tang. }\frac{2x}{xx+yy-1}.$$

§. 10.

§. 10. Simili modo procedamus pro valore Q inveniendo, eritque

$$S = (1 + xx - yy)\partial y - 2xy\partial x \text{ atque}$$

$$T = xx + (y+1)^2 \text{ et } U = xx + (y-1)^2,$$

vnde pro fractione $\frac{S}{T}$ numerator F aequabitur fractioni

$$\frac{S}{U} = \frac{(1 + xx - yy)\partial y - 2xy\partial x}{xx + (y-1)^2},$$

si quidem statuatur

$$xx + (y+1)^2 = 0, \text{ siue } xx = -(y+1)^2.$$

Erit igitur

$$F = \frac{+2\partial y(yy+y) + 2xy\partial x}{4y} = \tfrac{1}{2}\partial y(y+1) + \tfrac{1}{2}x\partial x.$$

Tum vero erit numerator G ex fractione $\frac{(1+xx-yy)\partial y - 2xy\partial x}{xx+(y+1)^2}$, statuendo $xx = -(y-1)^2$, hoc modo expressus:

$$G = \frac{-2\partial y(yy-y) - 2xy\partial x}{4y} = -\tfrac{1}{2}\partial y(y-1) - \tfrac{1}{2}x\partial x.$$

Hinc ergo fiet

$$Q = \tfrac{1}{2}\int \frac{\partial y(y+1) + x\partial x}{xx+(y+1)^2} - \tfrac{1}{2}\int \frac{\partial y(y-1) + x\partial x}{xx+(y-1)^2},$$

vbi in utraque formula valor est dimidium differentiale denominatoris, sicque valor quaesitus

$$Q = \tfrac{1}{4}l[xx+(y+1)^2] - \tfrac{1}{4}l[xx+(y-1)^2] = \tfrac{1}{4}l\frac{xx+(y+1)^2}{xx+(y-1)^2}.$$

§. 11. His igitur valoribus pro P et Q inventis valor integralis quaesiti erit $P + Q\sqrt{-1}$, vnde cum formulae propositae integrale sit A tang. z, nunc certi sumus, si loco z scribamus $x + y\sqrt{-1}$, tum arcum circuli, cujus tangens est formula imaginaria $x + y\sqrt{-1}$, semper aequari huic formulae:

$$-\tfrac{1}{2}\text{A tang.}\frac{2x}{xx+yy-1} + \frac{\sqrt{-1}}{4}l\frac{xx+(y+1)^2}{xx+(y-1)^2}.$$

§. 12. Neque vero opus fuerat hos valores pro P et Q per integrationem quaerere, sed immediate ex integrali cognito A tang. $(x + y\sqrt{-1})$ deduci possunt. Si enim ponatur

$$P + Q\sqrt{-1} = A\ \text{tang.}\ (x + y\sqrt{-1}),$$

erit figno imaginarii mutato

$$P - Q\sqrt{-1} = A\ \text{tang.}\ (x - y\sqrt{-1}).$$

His jam formulis additis prodit

$$2P = A\ \text{tang.}\ (x + y\sqrt{-1}) + A\ \text{tang.}\ (x - y\sqrt{-1})$$

$$= A\ \text{tang.}\ \frac{2x}{1 - xx - yy}, \text{ ideoque}$$

$$P = \tfrac{1}{2} A\ \text{tang.}\ \frac{2x}{1 - xx - yy} = -\tfrac{1}{2} A\ \text{tang.}\ \frac{2x}{xx + yy - 1}.$$

Deinde fubtractio illarum formularum praebet

$$2Q\sqrt{-1} = A\ \text{tang.}\ (x + y\sqrt{-1})$$

$$- A\ \text{tang.}\ (x - y\sqrt{-1}) = A\ \text{tang.}\ \frac{2y\sqrt{-1}}{1 + xx + yy}.$$

Quia vero eft

$$A\ \text{tang.}\ u\sqrt{-1} = \int \frac{\partial u \sqrt{-1}}{1 - uu} = \sqrt{-1} \int \frac{\partial u}{1 - uu} = \frac{\sqrt{-1}}{2}\, l\, \frac{1+u}{1-u},$$

hinc, cum noftro cafu fit $u = \frac{2y}{1 + xx + yy}$, erit

$$2Q\sqrt{-1} = \frac{\sqrt{-1}}{2}\, l\, \frac{xx + (y+1)^2}{xx + (y-1)^2}, \text{ ergo}$$

$$Q = \tfrac{1}{4}\, l\, \frac{xx + (y+1)^2}{xx + (y-1)^2}$$

prorfus vti invenimus. Hoc autem imprimis pro aliis cafibus eft notandum, vbi, quoties integrale $\int Z\,\partial z$ per logarithmos vel arcus circulares exprimere licet, quoniam, pofito $z = x + y\sqrt{-1}$, hos in partes duas refolvere licet, alteram realem, alteram fimpliciter imaginariam, inde valores quantitatum P et Q affignari poterunt, quantumvis ipfae formulae integrales pro his litteris refultantes fuerint perplexae et abftrufae.

III. Euolutio

formulae differentialis: $\frac{\partial z}{1 + z^3}$, cujus integrale conftat effe

$$\tfrac{1}{3}\, l\,(1 + z) - \tfrac{1}{3}\, l \sqrt{(1 - z + zz)} + \tfrac{1}{\sqrt{3}}\ A\ \text{tang.}\ \frac{z\sqrt{3}}{2 - z}.$$

§. 13. Ponamus igitur hic $z = x + y\sqrt{-1}$, eritque

Z =

$$Z = \frac{1}{1+z^3} = \frac{1}{1+x^3+3xxy\sqrt{-1}-3xyy-y^3\sqrt{-1}},$$

vbi cum denominator ſit

$$1+x^3-3xyy+\sqrt{-1}\,(3xxy-y^3),$$

multiplicetur ſupra et infra per

$$1+x^3-3xyy-\sqrt{-1}\,(3xxy-y^3), \text{ fietque}$$

$$Z = \frac{1+x^3-3xyy-\sqrt{-1}\,(3xxy-y^3)}{1+2x(xx-3yy)+(xx+yy)^3}.$$

Hinc ergo adipiſcimur

$$M = \frac{1+x^3-3xyy}{1+2x(xx-3yy)+(xx+yy)^3} \text{ et}$$

$$N = -\frac{(3xxy-y^3)}{1+2x(xx-3yy)+(xx+yy)^3}.$$

§. 14. Ex his jam valoribus, ſi integrale quaeſitum deſignemus per $P+Q\sqrt{-1}$, pro vtraque quantitate P et Q ſequentes obtinemus formulas integrales:

$$P = \int \frac{(1+x^3-3xyy)\partial x+(3xxy-y^3)\partial y}{1+2x(xx-3yy)+(xx+yy)^3} \text{ et}$$

$$Q = \int \frac{(1+x^3-3xyy)\partial y-(3xxy-y^3)\partial x}{1+2x(xx-3yy)+(xx+yy)^3}$$

quas ambas formulas jam in anteceſſum novimus eſſe integrabiles, etiamſi evolutio harum formularum ſit difficillima, cum factores denominatoris non pateant; interim tamen valores harum litterarum P et Q ex ipſo integrali principali per z expreſſo derivare licebit.

§. 15. Quoniam in his formulis duae variabiles x et y inſunt, pro lubitu alterutram tanquam conſtantem tractare licebit. Ita ſi x pro conſtante ſumamus, ponendo $x=a$ pro literis P et Q has habebimus formulas integrales:

$$P = \int \frac{(3aay-y^3)\partial y}{1+2a(aa-3yy)+(aa+yy)^3} \text{ et}$$

$$Q = \int \frac{(1+a^3-3ayy)\partial y}{1+2a(aa-3yy)+(aa+yy)^3}.$$

Simili modo ſi y pro conſtante accipiatur, ponendo $y=b$ pro iisdem litteris ſequentes valores prodibunt:

 P =

$$P = \int \frac{(1 + x^3 - 3bbx)\partial x}{1 + 2x(xx - 3bb) + (bb + xx)^3} \text{ et}$$
$$Q = \int \frac{(b^3 - 3bxx)\partial x}{1 + 2x(xx - 3bb) + (bb + xx)^3}$$

qui valores, ſi calculus rite inſtituatur, congruere debent. Veruntamen ſemper tutius erit vti formulis principalibus, in quas ambae variabiles x et y ingrediuntur, propterea quod ſi his poſterioribus formulis vteremur, adiectio conſtantis in errorem praecipitare poſſet; ſi ſcilicet in prioribus littera a in poſterioribus vero littera b in conſtantem induceretur.

§. 16. Ob has ſummas difficultates ergo non parum mirandum eſt, valores horum integralium nihilo minus reuera exhiberi poſſe; tantum enim opus eſt, vt in integrali per z expreſſo loco z ſcribatur $x + y\sqrt{-1}$, atque ſingula membra in binas ſuas partes reſoluantur, alteram realem, alteram imaginariam; tum enim partes reales iunctim ſumtae dabunt valorem ipſius P, partes autem imaginariae valorem ipſius Q.

§. 17. Quoniam enim in memorato integrali tantum logarithmi cum arcu circulari occurrunt, ſufficiet duas ſequentes reductiones noſſe:

I. $l(p+q\sqrt{-1}) = l\sqrt{(pp+qq)} + \sqrt{-1}\,\text{A tang.}\frac{q}{p}$ et

II. $\text{A tang.}(p+q\sqrt{-1}) = \frac{1}{2}\text{A tang.}\frac{2p}{1-pp-qq} + \frac{\sqrt{-1}}{4}\, l\,\frac{pp+(q+1)^2}{pp+(q-1)^2}$.

Hinc cum prima pars ſit $\frac{1}{3}l(1+z)$, poſito $z = x + y\sqrt{-1}$, erit

$$l(1+x+y\sqrt{-1}) = l\sqrt{[(1+x)^2+yy]} + \sqrt{-1}\,\text{A tang.}\frac{y}{1+x}.$$

Pro ſecunda parte, quae erat $-\frac{1}{6}l(1-z+zz)$, ob

$$1-z+zz = 1-x+xx-yy+\sqrt{-1}(2xy-y)$$

conſequenter

$$p = 1-x+xx-yy \text{ et } q = 2xy-y, \text{ erit}$$

$l(1-$

$$l(1-z+zz)=l\sqrt{[(xx+yy-x)^2+2xx-yy-2x+1]}$$
$$+\sqrt{-1}\,\mathrm{A\,tang.}\frac{2xy-y}{1-x+xx-yy}.$$

Denique tertia pars erat $\frac{1}{\sqrt{3}}\,\mathrm{A\,tang.}\frac{z\sqrt{3}}{2-z}$, vbi ergo

$$\frac{z}{2-z}=\frac{x+y\sqrt{-1}}{2-x-y\sqrt{-1}}=\frac{2x+2y\sqrt{-1}-xx-yy}{(2-x)^2+yy},$$

vnde pro superiori formula erit

$$p=\frac{(2x-xx-yy)\sqrt{3}}{(2-x)^2+yy} \text{ et } q=\frac{2y\sqrt{3}}{(2-x)^2+yy}.$$

Hinc ergo pro hac parte erit

$$\mathrm{A\,tang}\frac{z\sqrt{3}}{2-z}=\tfrac{1}{2}\mathrm{A\,tang.}\frac{2(2x-xx-yy)[(2-x)^2+yy]\sqrt{3}}{(2-x)^4+2yy(2-x)^2-3(2-x)^2xx+6xyy(2-x)-12yy}$$
$$+\frac{\sqrt{-1}}{4}\,l\,\frac{pp+(q+1)^2}{pp+(q-1)^2},$$

quae expressiones cum tantopere sint prolixae, in vltima parte litteras p et q retinere maluimus; quam ob rem multo minus valores pro P et Q hic exhibemus, cum sufficiat nosse, partes reales iunctim sumtas praebere P, imaginarias, per $\sqrt{-1}$ divisas, Q; atque ob hanc caussam manifestum est, cur euolutio actualis superiorum formularum non successerit.

IV. Euolutio

formulae differentialis $\frac{z^{m-1}\partial z}{1+z^n}$

cuius integrale passim euolutum reperitur, si quidem exponentes m et n fuerint numeri integri.

§. 18. Ex hactenus traditis clare intelligitur, longe aliam viam hic esse ineundam. Statim igitur statuamus $x=v\cos.\Phi$ et $y=v\sin.\Phi$, ita vt loco binarum variabilium x et y statim binas alias v et Φ in calculum introducamus; tum enim erit

$$z^m=v^m(\cos.m\Phi+\sqrt{-1}\sin.m\Phi) \text{ et}$$
$$1+z^n=1+v^n(\cos.n\Phi+\sqrt{-1}\sin.n\Phi).$$

 Quare

Quare fractionem propositam supra et infra multiplicemus per $1 + v^n(\cos. n\Phi - \sqrt{-1}\sin. n\Phi)$, hincque prodibit denominator $1 + 2v^n \cos. n\Phi + v^{2n}$.

§. 19. Pro numeratore, cum sit

$$z^{m-1}\partial z = \tfrac{1}{m}\partial . z^m = \tfrac{1}{m}\partial . v^m(\cos. m\Phi + \sqrt{-1}\sin. m\Phi),$$

at vero in genere

$$\partial . (\cos. \omega + \sqrt{-1}\sin. \omega) = \partial\omega\sqrt{-1}(\cos. \omega + \sqrt{-1}\sin. \omega),$$

erit facta euolutione

$$z^{m-1}\partial z = v^{m-1}\partial v(\cos. m\Phi + \sqrt{-1}\sin. m\Phi)$$
$$+ v^m\partial\Phi\sqrt{-1}(\cos. m\Phi + \sqrt{-1}\sin. m\Phi),$$ siue

$$z^{m-1}\partial z = v^{m-1}(\cos. m\Phi + \sqrt{-1}\sin. m\Phi)(\partial v + v\partial\Phi\sqrt{-1}).$$

Hanc ergo formulam insuper multiplicari oportet per

$$1 + v^n(\cos. n\Phi - \sqrt{-1}\sin. n\Phi),$$

pro qua operatione notetur esse

$$(\cos. \alpha + \sqrt{-1}\sin. \alpha)(\cos. \beta - \sqrt{-1}\sin. \beta)$$
$$= \cos. (\alpha - \beta) + \sqrt{-1}\sin. (\alpha - \beta),$$

hinc ergo noster numerator erit

$$v^{m-1}(\cos. m\Phi + \sqrt{-1}\sin. m\Phi)(\partial v + v\partial\Phi\sqrt{-1})$$
$$+ v^{m+n-1}[\cos. (m-n)\Phi + \sqrt{-1}\sin. (m-n)\Phi](\partial v + v\partial\Phi\sqrt{-1})$$

cuius ergo pars realis erit

$$v^{m-1}\partial v\cos. m\Phi + v^{m+n-1}\partial v\cos. (m-n)\Phi - v^m\partial\Phi\sin. m\Phi$$
$$- v^{m+n}\partial\Phi\sin. (m-n)\Phi,$$

pars vero imaginaria erit

$$v^{m-1}\partial v\sqrt{-1}\sin. m\Phi + v^m\partial\Phi\sqrt{-1}\cos. m\Phi$$
$$+ v^{m+n-1}\partial v\sqrt{-1}\sin. (m-n)\Phi$$
$$+ v^{m+n}\partial\Phi\sqrt{-1}\cos. (m-n)\Phi.$$

§. 20.

§. 20. His praeparatis, si formulae nostrae differentialis integrale quaesitum statuamus $= P + Q\sqrt{-1}$, vtramque partem per sequentes formulas integrales reales inueniemus expressam:

$$P = \int \frac{v^{m-1}\partial v\,[\text{cos}.m\Phi + v^n \text{cos}.(m-n)\Phi] - v^m \partial\Phi[\text{sin}.m\Phi + v^n \text{sin}.(m-n)\Phi]}{1 + 2v^n \text{cos}.n\Phi + v^{2n}},$$

$$Q = \int \frac{v^{m-1}\partial v\,[\text{sin}.m\Phi + v^n \text{sin}.(m-n)\Phi] + v^m \partial\Phi[\text{cos}.m\Phi + v^n \text{cos}(m-n)\Phi]}{1 + 2v^n \text{cos}.n\Phi + v^{2n}}.$$

Haec igitur integralia ex ipso integrali principali per z expresso deriuare licebit, vti ante iam obseruauimus, siquidem totum integrale partim ex logarithmis, partim ex arcubus circularibus, quorum tangentes dantur, componitur. Interim tamen videamus, num methodo consueta haec integralia inuestigare liceat.

Inuestigatio formulae integralis:

$$P = \int \frac{v^{m-1}\partial v[\text{cos}.m\Phi + v^n \text{cos}.(m-n)\Phi] - v^m \partial\Phi[\text{sin}.m\Phi + v^n \text{sin}.(m-n)\Phi]}{1 + 2v^n \text{cos}.n\Phi + v^{2n}}.$$

§. 21. Totum ergo negotium huc redit, vt ante omnia denominator in suos factores resoluatur, eosque trinomiales, quandoquidem ad nostrum institutum omnes factores debent esse reales. Ponamus ergo factorem huius denominatoris esse $1 - 2v\,\text{cos}.\,\omega + vv$, atque necesse est, vt posito hoc factore $= 0$ (vnde fit $v = \text{cos}.\,\omega + \sqrt{-1}\,\text{sin}.\,\omega$) etiam ipse denominator euanescat. Quoniam igitur hinc fiet

$$v^n = \text{cos}.\,n\omega + \sqrt{-1}\,\text{sin}.\,n\omega \text{ et}$$
$$v^{2n} = \text{cos}.\,2n\omega + \sqrt{-1}\,\text{sin}.\,2n\omega,$$

his substitutis denominator induet hanc formam:

$$1 + 2\,\text{cos}.\,n\Phi\,\text{cos}.\,n\omega + \text{cos}.\,2n\omega + \sqrt{-1}\,(2\,\text{cos}.\,n\Phi\,\text{sin}.\,n\omega + \text{sin}.\,2n\omega)$$

cuius ergo tam pars realis quam imaginaria seorsim nihilo aequari

quari debet. Ex imaginaria igitur haec oritur aequatio:

$$2 \operatorname{cof.} n \Phi \operatorname{fin.} n \omega + \operatorname{fin.} 2 n \omega = 0,$$

vnde per fin. $n \omega$ diuidendo prodit

$$2 \operatorname{cof.} n \Phi + \operatorname{cof.} n \omega = 0.$$

At vero ex parte reali deducitur

$$1 + 2 \operatorname{cof.} n \Phi \operatorname{cof.} n \omega + \operatorname{cof.} 2 n \omega = 0$$

vnde quia

$$1 + \operatorname{cof.} 2 n \omega = 2 \operatorname{cof.} n \omega^2, \text{ erit}$$

$$\operatorname{cof.} n \Phi + \operatorname{cof.} n \omega = 0$$

prorfus vt ante. Vnde patet, angulum ω ita accipi debere, vt fiat $\operatorname{cof.} n \omega = - \operatorname{cof.} n \Phi$, cui conditioni infinitis modis fatisfieri poteft, fumendo vel $n \omega = \pi \pm n \Phi$ vel $n \omega = 3 \pi \pm n \Phi$, vel $n \omega = 5 \pi \pm n \Phi$, atque adeo in genere $n \omega = (2 i + 1) \pi \pm n \Phi$. Atque hinc adeo n valores diuerfi pro ω obtinebuntur; totidem vero nobis eft opus ad denominatorem implendum. Forma igitur generalis anguli ω erit $\omega = \frac{(2 i + 1) \pi}{n} \pm \Phi$; et quicunque huiusmodi valor ipfi ω tribuatur, denominatoris factor erit $1 - 2 v \operatorname{cof.} \omega + v v$, quo euanefcente fimul ipfe denominator euanefcet, fietque fcilicet $v^{2 n} = - 2 v^n \operatorname{cof.} n \Phi - 1$.

§. 22. Inuentis iam omnibus factoribus denominatoris, ipfa formula propofita in totidem partes refolui poterit, quarum denominatores fint ifti ipfi factores trinomiales $1 - 2 v \operatorname{cof.} \omega + v v$; quam ob rem pro quolibet tali factore fractionem ei refpondentem, hoc eft eius numeratorem, inueftigari oportebit, qui cum ex numeratore ipfius formae propofitae deduci debeat, ponamus breuitatis gratia numeratorem formulae integralis propofitae $R \partial v + S \partial \Phi$, ita vt fit

$$R = v^{m-1} [\operatorname{cof.} m \Phi + v^n \operatorname{cof.} (m - n) \Phi] \text{ et}$$

$$S = v^m [\operatorname{fin.} m \Phi + v^n \operatorname{fin.} (m - n) \Phi].$$

Iam

Iam primo euoluamus fractionem $\frac{R}{1 + 2 v^n \operatorname{cof.} n \Phi + v^{2n}}$, quam inuoluere fingamus hanc fractionem fimplicem: $\frac{r}{1 - 2 v \operatorname{cof.} \omega + v v}$, pro cuius numeratore r conftat, eius valorem deriuari debere ex fractione $\frac{R(1 - 2 v \operatorname{cof.} \omega + v v)}{1 + 2 v^n \operatorname{cof.} n \Phi + v^{2n}}$, pofito $1 - 2 v \operatorname{cof.} \omega + v v = 0$, vbi operationem ita inftitui oportet, vt pro r quantitas integra obtineatur.

§. 23. Quoniam autem cafu $1 - 2 v \operatorname{cof.} \omega + v v = 0$ tam numerator quam denominator euanefcit, notum eft hoc cafu iftam fractionem $\frac{1 - 2 v \operatorname{cof.} \omega + v v}{1 + 2 v^n \operatorname{cof.} n \Phi + v^{2n}}$ aequari huic:

$$\frac{v - \operatorname{cof.} \omega}{n v^{2n-1} + n v^{n-1} \operatorname{cof.} n \Phi} = \frac{v v - v \operatorname{cof.} \omega}{n v^n (v^n + \operatorname{cof.} n \Phi)},$$

cuius denominator, ob $v^{2n} = -2 v^n \operatorname{cof.} n \Phi - 1$, dabit $-n v^n \operatorname{cof.} n \Phi - n$; numerator vero, ob $v v = 2 v \operatorname{cof.} \omega - 1$, erit $v \operatorname{cof.} \omega - 1$, ideoque fractio $= \frac{-v \operatorname{cof.} \omega + 1}{n (v^n \operatorname{cof.} n \Phi - 1)}$. Ex denominatore autem nihilo aequato fit $v^n = \operatorname{cof.} n \Phi + \sqrt{-1} \operatorname{fin.} n \Phi$, qui valor in hoc denominatore fubftitutus dat

$$\frac{1 - v \operatorname{cof.} \omega}{n \operatorname{cof.} n \Phi^2 + n \sqrt{-1} \operatorname{fin.} n \Phi \operatorname{cof.} n \Phi - n} = \frac{v \operatorname{cof.} \omega - 1}{n \operatorname{fin.} n \Phi (\operatorname{fin.} n \Phi - \sqrt{-1} \operatorname{cof.} n \Phi)}.$$

Numerator vero, pofito $v = \operatorname{cof.} \omega + \sqrt{-1} \operatorname{fin.} \omega$, abibit in $-\operatorname{fin.} \omega (\operatorname{fin.} \omega - \sqrt{-1} \operatorname{cof.} \omega)$, ficque tota haec fractio erit $-\frac{\operatorname{fin.} \omega (\operatorname{fin.} \omega - \sqrt{-1} \operatorname{cof.} \omega)}{n \operatorname{fin.} n \Phi (\operatorname{fin.} n \Phi - \sqrt{-1} \operatorname{cof.} n \Phi)}$. Nunc haec fractio fupra et infra ducatur in $\operatorname{fin.} n \Phi + \sqrt{-1} \operatorname{cof.} n \Phi$, prodibitque

$$-\frac{\operatorname{fin.} \omega [\operatorname{cof.} (\omega - n \Phi) + \sqrt{-1} \operatorname{fin.} (\omega - n \Phi)]}{n \operatorname{fin.} n \Phi}.$$

Verum imaginaria, quae hic adhuc fuperfunt, noftrum negotium prorſus turbant. Interim tamen hoc incommodum tolli poterit, ſi quantitas v in numeratorem introducatur. Ponamus igitur numeratorem eſſe $Av+B$, ita vt A et B ſint quantitates reales; vnde cum ſit

$$A v + B = A \operatorname{cof.} \omega + B + \sqrt{-1}\, A \operatorname{fin.} \omega,$$

partes reales et imaginariae ſeorſim aequentur, ſicque eſſe debet

$$A \operatorname{fin.} \omega = -\operatorname{fin.} (\omega - n\Phi) \operatorname{fin.} \omega,$$

vnde ſit $A = -\operatorname{fin.} (\omega - n\Phi)$, quo valore ſubſtituto partes reales dabunt

$$-\operatorname{cof.} \omega \operatorname{fin.} (\omega - n\Phi) + B = -\operatorname{fin.} \omega \operatorname{cof.} (\omega - n\Phi),$$

vnde ſit $B = -\operatorname{fin.} n\Phi$, ſicque fractio noſtra erit

$$\frac{-v \operatorname{fin.} (\omega - n\Phi) - \operatorname{fin.} n\Phi}{n \operatorname{fin.} n\Phi}.$$

§. 24. Nunc igitur tantum ſupereſt vt iſta formula multiplicetur per R, eius ſcilicet valorem, quem accipiet poſito

$$v v - 2 v \operatorname{cof.} \omega + 1 = 0.$$

Erat autem

$$R = v^{m-1} [\operatorname{cof.} m\Phi + v^n \operatorname{cof.} (m-n)\Phi],$$

quod poſito $v = \operatorname{cof.} \omega + \sqrt{-1} \operatorname{fin.} \omega$ abit in

$$\operatorname{cof.} m\Phi \operatorname{cof.} (m-1)\omega + \operatorname{cof.} (m-n)\Phi \operatorname{cof.} (m+n-1)\omega$$
$$+ \sqrt{-1}\, [\operatorname{cof.} m\Phi \operatorname{fin.} (m-1)\omega + \operatorname{cof.} (m-n)\Phi \operatorname{fin.} (m+n-1)\omega],$$

cuius loco, vt imaginaria extirpemus, ſcribamus

$$C v + D, \text{ ſiue } C \operatorname{cof.} \omega + D + \sqrt{-1}\, C \operatorname{fin.} \omega,$$

vnde erit

$$C = \frac{\operatorname{cof.} m\Phi \operatorname{fin.} (m-1)\omega + \operatorname{cof.} (m-n)\Phi \operatorname{fin.} (m+n-1)\omega}{\operatorname{fin.} \omega}$$

et hinc

$$D = \frac{\operatorname{cof.} m\Phi \operatorname{fin.} (2-m)\omega + \operatorname{cof.} (m-n)\Phi \operatorname{fin.} (2-m-n)\omega}{\operatorname{fin.} \omega}.$$

§. 25.

§. 25. Inuentis nunc valoribus litterarum A, B, C, D, erit numerator noster quaesitus $r = (A v + B)(C v + D)$. Quia autem hic adhuc inest quadratum $v v$, eius loco scribendum restat $2 v \operatorname{cos.} \omega - 1$, sicque erit valor iustus

$$r = 2 A C v \operatorname{cos.} \omega - A C + (A D + B C) v + B D.$$

Consequenter pars integralis huic factori respondens pro variabili v erit $\int \frac{r \partial v}{1 - 2 v \operatorname{cos.} \omega + v v}$. Simili modo pro altera variabili Φ fractio partialis ex fractione $\frac{S}{1 - 2 v^n \operatorname{cos.} n \Phi + v^{2n}}$ deriuari debet, quae si statuatur $\frac{s}{1 - 2 v \operatorname{cos.} \omega + v v}$, atque quantitas S redigatur ad formam $E v + F$, simili modo reperietur $s = (A v + B)(E v + F)$, vbi autem insuper loco $v v$ scribi debet $2 v \operatorname{cos.} \omega - 1$, quo facto pro variabili Φ habebitur formula $\int \frac{s \partial \Phi}{1 - 2 v \operatorname{cos.} \omega + v v}$.

§. 26. Quod si iam haec colligamus, pars integralis ex quolibet denominatoris factore $1 - 2 v \operatorname{cos.} \omega + v v$ oriunda erit $\int \frac{r \partial v + s \partial \Phi}{1 - 2 v \operatorname{cos.} \omega + v v}$; vbi imprimis notandum est, hic criterium notissimum circa integrabilitatem formularum duas variabiles inuoluentium certe locum esse habiturum. Sufficiet autem plerumque alterutram tantum variabilem considerasse.

§. 27. Hic quidem ad valorem litterae P inueniendum sufficere posset formula $\int \frac{r \partial v}{1 - 2 v \operatorname{cos.} \omega + v v}$, in qua sola v vt variabilis tractetur, cuius integrale, vti constat, per logarithmos et arcus circulares exhiberi potest. Interim tamen hoc idem integrale etiam erui debet ex altera formula $\int \frac{s \partial \Phi}{1 - 2 v \operatorname{cos.} \omega + v v}$, in qua solus angulus Φ cum angulo ω ab eo pendente variabilis assumitur, quae integratio eo magis est notatu digna, quod plura multipla anguli Φ in ea occurrunt, neque adhuc methodus tales formulas tractandi satis est exculta. At vero haec nimis funt generalia, quam vt ea, quae in iis funt contenta, clare perfpicere queamus; vnde haud parum lucis nobis accendetur, fi quosdam cafus fimpliciffimos contemplabimur.

Applicatio
ad formulam differentialem $\frac{\partial z}{1+z}$, vbi eft $m = 1$ et $n = 1$.

§. 28. Cum huius formulae integrale fit $l(1+z)$, pofito $z = x + y\sqrt{-1}$, feu potius, vti in genere fecimus,

$$z = v(\text{cof.}\,\Phi + \sqrt{-1}\,\text{fin.}\,\Phi),$$

integrale

$$l(1 + v\,\text{cof.}\,\Phi + v\sqrt{-1}\,\text{fin.}\,\Phi)$$

euoluitur in formam $P + Q\sqrt{-1}$, exiftente

$$P = l\sqrt{(1 + 2v\,\text{cof.}\,\Phi + vv)} \text{ et}$$
$$Q = \text{A tang.}\,\frac{v\,\text{fin.}\,\Phi}{1 + v\,\text{cof.}\,\Phi}.$$

§. 29. Nunc igitur eosdem valores per integrationem eruere conemur. Pofitis autem $m = n = 1$, formulae generales pro P et Q exhibitae fequentes induent formas:

$$P = \int \frac{\partial v(\text{cof.}\,\Phi + v) - v\,\partial\Phi\,\text{fin.}\,\Phi}{1 + 2v\,\text{cof.}\,\Phi + vv} \text{ et}$$
$$Q = \int \frac{\partial v\,\text{fin.}\,\Phi + v\,\partial\Phi\,\text{cof.}\,\Phi + vv\,\partial\Phi}{1 + 2v\,\text{cof.}\,\Phi + vv},$$

vbi formula prior manifefto habet integrale

$$\tfrac{1}{2}l(1 + 2v\,\text{cof.}\,\Phi + vv),$$

pofterior vero integrale habet A tang. $\frac{v\,\text{fin.}\,\Phi}{1 + v\,\text{cof.}\,\Phi}$, quemadmodum differentiatio manifefto declarat, ita vt hic non opus fuerit alterum angulum ω in calculum introducere.

Appli-

Applicatio
ad formulam differentialem $\frac{\partial z}{1+zz}$, vbi $m=1$ et $n=2$.

§. 30. Hunc casum iam supra euoluimus, vbi vidimus, posito $z=x+y\sqrt{-1}$, integrale esse

$$\text{A tang.}\, x+y\sqrt{-1}=-\tfrac{1}{2}\text{A tang.}\frac{2x}{xx+yy-1}+\frac{\sqrt{-1}}{4}\, l\, \frac{xx+(y+1)^2}{xx+(y-1)^2}.$$

Hinc ergo si ponamus $x=v\cos.\Phi$ et $y=v\sin.\Phi$, erit pro integrali $P+Q\sqrt{-1}$

$$P=-\tfrac{1}{2}\text{A tang.}\frac{2v\cos.\Phi}{vv-1}=\tfrac{1}{2}\text{A tang.}\frac{2v\cos.\Phi}{1-vv}\text{ et}$$

$$Q=\tfrac{1}{4}\, l\, \frac{1+2v\sin.\Phi+vv}{1-2v\sin.\Phi+vv}.$$

Hos igitur valores videamus quemadmodum per integrationem eliciamus.

§. 31. Cum igitur hic sit $m=1$ et $n=2$, formulae generales praebebunt

$$P=\int\frac{\partial v(1+vv)\cos.\Phi-v\partial\Phi(1-vv)\sin.\Phi}{1+2vv\cos.2\Phi+v^4},$$

$$Q=\int\frac{\partial v(1-vv)\sin.\Phi+v\partial\Phi(1+vv)\cos.\Phi}{1+2vv\cos.2\Phi+v^4},$$

vbi notetur denominatoris binos factores, ob

$$\cos.2\omega=-\cos.2\Phi=\cos.(\pi\pm2\Phi),$$

hincque vel $\omega=90^\circ+\Phi$, vel $\omega=90^\circ-\Phi$, esse

$$1+2v\sin.\Phi+vv \text{ et } 1-2v\sin.\Phi+vv.$$

Hinc ad resolutionem expediendam consideremus in genere fractionem $\frac{S}{1+2vv\cos.2\Phi+v^4}$, quam resolui ponamus in has partes:

$$\frac{F}{1+2v\sin.\Phi+vv}+\frac{G}{1-2v\sin.\Phi+vv},$$

vbi nouimus hos numeratores ita definiri debere, vt sit

F =

$$F = \frac{S}{1 - 2v \sin.\Phi + vv}, \text{ posito } 1 + 2v \sin.\Phi + vv = 0, \text{ et}$$
$$G = \frac{S}{1 + 2v \sin.\Phi + vv}, \text{ posito } 1 - 2v \sin.\Phi + vv = 0.$$

§. 32. Quoniam nunc tam pro P quam Q binas habemus partes, alteram per ∂v, alteram vero per $\partial \Phi$ datam, sit primo $S = (1 + vv) \cos.\Phi$, vnde fit

$$F = \frac{(1 + vv)\cos.\Phi}{1 - 2v \sin.\Phi + vv}, \text{ posito } 1 + vv = -2v \sin.\Phi,$$

vnde statim fit

$$F = \frac{-2v \sin.\Phi \cos.\Phi}{-4v \sin.\Phi} = \tfrac{1}{2} \cos.\Phi;$$

similique modo erit

$$G = \frac{(1 + vv)\cos.\Phi}{1 + 2v \sin.\Phi + vv}, \text{ posito } 1 + vv = +2v \sin.\Phi,$$

sicque erit $G = +\frac{1}{2}\cos.\Phi$: quamobrem pro P pars integralis elementum ∂v continens erit

$$P = \tfrac{1}{2}\int \frac{\partial v \cos.\Phi}{1 + 2v \sin.\Phi + vv} + \tfrac{1}{2}\int \frac{\partial v \cos.\Phi}{1 - 2v \sin.\Phi + vv}.$$

§. 33. Pro parte autem vbi Φ est variabile, habebimus $S = -v(1 - vv)\sin.\Phi$, vnde fiet

$$F = \frac{-v(1 - vv)\sin.\Phi}{1 - 2v \sin.\Phi + vv}, \text{ posito } 1 + vv = -2v \sin.\Phi,$$

ideoque $vv = -2v \sin.\Phi - 1$, vnde fit $F = \frac{1}{2}(1 + v \sin.\Phi)$. Simili modo erit $G = \frac{-v(1 - vv)\sin.\Phi}{1 + 2v \sin.\Phi + vv}$, posito scilicet $1 + vv = 2v \sin.\Phi$, quo facto fit $G = -\frac{1}{2}(1 - v \sin.\Phi)$. Hinc igitur valor completus quantitatis P ex vtraque variabilitate erit

$$P = \tfrac{1}{2}\int \frac{\partial v \cos.\Phi + (1 + v \sin.\Phi)\partial\Phi}{1 + 2v \sin.\Phi + vv} + \tfrac{1}{2}\int \frac{\partial v \cos.\Phi - (1 - v \sin.\Phi)\partial\Phi}{1 - 2v \sin.\Phi + vv}.$$

§. 34. Pari modo pro quantitate Q primo habemus $S = (1 - vv)\sin.\Phi$, ideoque fiet

$$F = \frac{(1 - vv)\sin.\Phi}{1 - 2v \sin.\Phi + vv}, \text{ posito } 1 + vv = -2v \sin.\Phi,$$

vnde

vnde fit $F = \frac{1 + v\, \text{fin.}\, \Phi}{2v}$. Hic autem v ex denominatore extrudere oportet, quem in finem multiplicetur fupra et infra per $v + 2\, \text{fin.}\, \Phi$, vt denominator fiat $2(vv + 2v\, \text{fin.}\, \Phi) = -2$; numerator autem tunc erit

$$vv\, \text{fin.}\, \Phi + v + 2v\, \text{fin.}\, \Phi^2 + 2\, \text{fin.}\, \Phi,$$

ideoque $F = -\frac{1}{2}(v + \text{fin.}\, \Phi)$. Simili modo erit

$$G = \frac{(1 - vv)\, \text{fin.}\, \Phi}{1 + 2v\, \text{fin.}\, \Phi + vv}, \text{ pofito fcilicet } 1 + vv = 2v\, \text{fin.}\, \Phi,$$

quó facto fit $G = \frac{1 - vv}{4v}$, et ob $1 = 2v\, \text{fin.}\, \Phi - vv$, erit $G = \frac{1}{2}(\text{fin.}\, \Phi - v)$. Sicque pars prior pro Q variabilem v continens erit

$$Q = \frac{1}{2}\int \frac{\partial v(v + \text{fin.}\, \Phi)}{1 + 2v\, \text{fin.}\, \Phi + vv} + \frac{1}{2}\int \frac{\partial v(\text{fin.}\, \Phi - v)}{1 - 2v\, \text{fin.}\, \Phi + vv}.$$

Pro altera vero parte variabilem Φ habente erit $S = v(1 + vv)\text{cof.}\Phi$, hincque colligitur

$$F = \frac{v(1 + vv)\, \text{cof.}\, \Phi}{1 - 2v\, \text{fin.}\, \Phi + vv}, \text{ pofito } 1 + 2v\, \text{fin.}\, \Phi + vv = 0,$$

fiue $1 + vv = -2v\, \text{fin.}\, \Phi$, vnde fit $F = \frac{1}{2} v\, \text{cof.}\, \Phi$; tum vero erit

$$G = \frac{v(1 + vv)\, \text{cof.}\, \Phi}{1 + 2v\, \text{fin.}\, \Phi + vv}, \text{ pofito } 1 + vv = +2v\, \text{fin.}\, \Phi,$$

ideoque $G = \frac{1}{2} v\, \text{cof.}\, \Phi$, ficque valor completus ipfius Q erit

$$Q = \frac{1}{2}\int \frac{\partial v(v + \text{fin.}\, \Phi) + v\,\partial\Phi\, \text{cof.}\, \Phi}{1 + 2v\, \text{fin.}\, \Phi + vv} + \frac{1}{2}\int \frac{\partial v(\text{fin.}\, \Phi - v) + v\,\partial\Phi\, \text{cof.}\, \Phi}{1 - 2v\, \text{fin.}\, \Phi + vv}.$$

§. 35. Incipiamus ab euolutione pofterioris valoris Q, vtpote facillima, quoniam in vtraque formula numerator manifefto eft dimidium differentiale denominatoris, vnde ftatim obtinetur $Q = \frac{1}{4} l \frac{1 + 2v\, \text{fin.}\, \Phi + vv}{1 - 2v\, \text{fin.}\, \Phi + vv}$, qui valor prorfus congruit cum fupra dato. Pro littera P autem notetur effe

$$\int \frac{f\,\partial v}{1 - 2v\, \text{cof.}\, \omega + vv} = \frac{f}{\text{fin.}\, \omega} \text{A tang.} \frac{v\, \text{fin.}\, \omega}{1 + v\, \text{cof.}\, \omega},$$

vnde cum noftro cafu pro parte priore fit $f = \text{cof.}\, \Phi$, $\text{cof.}\, \omega = -\text{fin.}\, \Phi$ et $\text{fin.}\, \omega = \text{cof.}\, \Phi$, erit

$\int$

$$\int \frac{\partial v \cos.\Phi}{1+2v\sin.\Phi+vv} = \text{A tang.}\,\frac{v\cos.\Phi}{1+v\sin.\Phi},$$

si quidem angulus Φ vt constans tractetur. At vero ex eius variabilitate non prodit altera pars, quae est $\int \frac{\partial\Phi(1+v\sin.\Phi)}{1+2v\sin.\Phi+vv}$, sed eius loco differentiatio praebet $\frac{-v\,\partial\Phi\sin.\Phi-vv\,\partial\Phi}{1+2v\sin.\Phi+vv}$. In hunc ergo dissensum accuratius inquiri conueniet.

§. 36. Primo quidem nullum est dubium quin differentiatio formulae A tang. $\frac{v\cos.\Phi}{1+v\sin.\Phi}$ praebeat partem priorem; sed idem contingeret, si constans quaecunque adiiceretur, quare cum in hac integratione angulus Φ pro constante sit habitus, ista constans vtique adhuc ipsum angulum Φ continere potest. Hancobrem in genere statuamus integrale quaesitum esse

$$\text{A tang.}\,\frac{v\cos.\Phi}{1+v\sin.\Phi}+\int \varPhi\,\partial\Phi,$$

existente $\varPhi$ functione ipsius Φ, et iam huius formulae differentiale, posito v constante, erit

$$\frac{-v\,\partial\Phi\sin.\Phi-vv\,\partial\Phi}{1+2v\sin.\Phi+vv}+\varPhi\,\partial\Phi = \partial\Phi\left\{\frac{\varPhi+2v\varPhi\sin.\Phi+\varPhi vv-v\sin.\Phi-vv}{1+2v\sin.\Phi+vv}\right\};$$

vbi si sumatur $\varPhi = 1$, ipsum nostrum differentiale prodit

$$\frac{\partial\Phi(1+v\sin.\Phi)}{1+2v\sin.\Phi+vv},$$

ita vt ista pars sit

$$\text{A tang.}\,\frac{v\cos.\Phi}{1+v\sin.\Phi}+\Phi = \text{A tang.}\,\frac{v\cos.\Phi}{1+v\sin.\Phi}+\text{A tang.}\,\frac{\sin.\Phi}{\cos.\Phi},$$

qui duo arcus contracti praebent A tang. $\frac{\sin.\Phi+v}{\cos.\Phi}$, haecque formula differentiata ipsum producit integrale datum.

§. 37. Pro altera autem parte ipsius P, quae est

$$\int \frac{\partial v\cos.\Phi-(1-v\sin.\Phi)\,\partial\Phi}{1-2v\sin.\Phi+vv},$$

cum haec forma a priori tantum in hoc discrepet, quod angulus Φ sit negatiue sumtus, idem discrimen in integrali introductum dabit A tang. $\frac{v-\sin.\Phi}{\cos.\Phi}$. Sicque completus valor quantitatis P erit

P =

$$P = \tfrac{1}{2}\, A\ \mathrm{tang.}\, \frac{v + \mathrm{fin.}\,\Phi}{\mathrm{cof.}\,\Phi} + \tfrac{1}{2}\, A\ \mathrm{tang.}\, \frac{v - \mathrm{fin.}\,\Phi}{\mathrm{cof.}\,\Phi}$$

qui duo arcus in vnum contracti dabunt

$$P = \tfrac{1}{2}\, A\ \mathrm{tang.}\, \frac{2v\, \mathrm{cof.}\,\Phi}{1 - vv},$$

qui valor pariter perfecte congruit cum fupra dato.

Applicatio
ad cafum quo $m = 1$ et $n = 3$, feu formulam differentialem $\frac{\partial z}{1 + z^3}$.

§. 38. Quod fi hic ponatur $z = \mathrm{cof.}\,\Phi + \sqrt{-1}\, \mathrm{fin.}\,\Phi$ et integrale inde refultans ftatuatur $\int \frac{\partial z}{1 + z^3} = P + Q\sqrt{-1}$, ex formulis generalibus fupra datis erit

$$P = \int \frac{\partial v(\mathrm{cof.}\,\Phi + v^3\, \mathrm{cof.}\, 2\Phi) - v\,\partial\Phi(\mathrm{fin.}\,\Phi - v^3\, \mathrm{fin.}\, 2\Phi)}{1 + 2v^3\, \mathrm{cof.}\, 3\Phi + v^6} \text{ et}$$

$$Q = \int \frac{\partial v(\mathrm{fin.}\,\Phi - v^3\, \mathrm{fin.}\, 2\Phi) + v\,\partial\Phi(\mathrm{cof.}\,\Phi + v^3\, \mathrm{cof.}\, 2\Phi)}{1 + 2v^3\, \mathrm{cof.}\, 3\Phi + v^6}.$$

§. 39. Hic igitur denominator tres habebit factores trinomiales, quorum forma fi ponatur $1 - 2v\, \mathrm{cof.}\,\omega + vv$, debet effe $\mathrm{cof.}\, 3\omega = -\mathrm{cof.}\, 3\Phi$. Aequabitur ergo 3ω vel $\pi + 3\Phi$, vel $\pi - 3\Phi$, vel $3\pi - 3\Phi$, vnde ergo oriuntur hi tres valores ipfius ω:

$$\omega = 60^\circ + \Phi,\ \omega = 60^\circ - \Phi,\ \omega = 180^\circ - \Phi.$$

Nunc igitur in genere confideremus hanc fractionem: $\frac{S}{1 + 2v^3\, \mathrm{cof.}\, 3\Phi + v^6}$, cuius vna fractio partialis fit $\frac{F}{1 - 2v\, \mathrm{cof.}\,\omega + vv}$; atque, vt fupra animaduertimus, valorem ipfius F deriuari oportet ex forma $\frac{S(1 - 2v\, \mathrm{cof.}\,\omega + vv)}{1 + 2v^3\, \mathrm{cof.}\, 3\Phi + v^6}$, fi ftatuatur $1 - 2v\, \mathrm{cof.}\,\omega + vv = 0$; tum autem ifta fractio reducetur ad hanc formam: $\frac{S(vv - v\, \mathrm{cof.}\,\omega)}{3v^6 + 3v^3\, \mathrm{cof.}\, 3\Phi}$. Cum autem fit

$$v^6 = -2v^3\, \mathrm{cof.}\, 3\Phi - 1,$$

denominator erit

$$-3(v^3\, \mathrm{cof.}\, 3\Phi + 1),$$

numerator vero S (v coſ. $\Phi - 1$), ſicque fractio reſoluenda erit $\frac{S(1 - v \cos. \omega)}{3(v \cos. 3\Phi + 1)} = F$, poſtquam ſcilicet ex denominatore quantitas v fuerit eliſa.

§. 40. Quoniam igitur per hypotheſin habemus

$v = \cos. \omega + \sqrt{-1} \sin. \omega$, erit

$v^3 = \cos. 3\omega + \sqrt{-1} \sin. 3\omega$;

vbi notetur eſſe coſ $3\omega = - \cos. 3\Phi$; tum vero erit ſin. $3\omega = \pm \sin. 3\Phi$. Scilicet pro primo valore, quo $\omega = 60^\circ + \Phi$, ſiue $3\omega = 180^\circ + 3\Phi$, erit ſin. $3\omega = - \sin. 3\Phi$; pro ſecundo valore, quo $3\omega = 180^\circ - 3\Phi$, erit ſin. $3\omega = + \sin. 3\Phi$; pro tertio caſu, quo $3\omega = 3\pi - 3\Phi$, erit etiam ſin. $3\omega = + \sin. 3\Phi$. Hoc autem valore poſito denominator noſter erit

$$3(-\cos. 3\Phi^2 \pm \sqrt{-1} \sin. 3\Phi \cos. 3\Phi + 1),$$

vbi ſignum ſuperius valet pro valore tertio et ſecundo anguli ω, inferius autem pro primo. Hic denominator etiam hoc modo concinnius exprimi poteſt:

$$3 \sin. 3\Phi (\sin. 3\Phi \pm \sqrt{-1} \cos. 3\Phi).$$

§. 41. Nunc igitur tam numeratorem quam denominatorem ducamus in ſin. $3\Phi \mp \sqrt{-1} \cos. 3\Phi$, eritque

$$F = \frac{S(1 - v \cos. \omega)(\sin. 3\Phi \mp \sqrt{-1} \cos. 3\Phi)}{3 \sin. 3\Phi}.$$

At ſi etiam loco v ſcribamus coſ. $\omega + \sqrt{-1}$ ſin. ω, fiet

$$F = \frac{S \sin. \omega (\sin. \omega - \sqrt{-1} \cos. \omega)(\sin. 3\Phi \mp \sqrt{-1} \cos. 3\Phi)}{3 \sin. 3\Phi}.$$

Hinc ſi bini factores imaginarii numeratoris in ſe inuicem ducantur, reperietur

$$F = \frac{S \sin. \omega [\mp \cos. (\omega \pm 3\Phi) \mp \sqrt{-1} \sin. (\omega \pm 3\Phi)]}{3 \sin. 3\Phi}$$

vbi imaginaria non amplius curamus, quoniam, vti ſupra vidimus, introducendo litteram v, ea rurſus tollere licet.

§. 41.

§. 41. Nunc autem pro S quatuor habemus valores ad binas litteras P et Q definiendas. Primo enim pro P et elemento ∂v erit $S = \text{cof.}\,\Phi + v^3\,\text{cof.}\,2\,\Phi$, vbi loco v^3 fcribamus valorem jam ante ufurpatum $-\text{cof.}\,3\,\Phi \pm \sqrt{-1}\,\text{fin.}\,3\,\Phi$; vnde fiet

$$S = \text{cof.}\,\Phi - \text{cof.}\,3\,\Phi\,\text{cof.}\,2\,\Phi \pm \sqrt{-1}\,\text{fin.}\,3\,\Phi\,\text{cof.}\,2\,\Phi$$
$$= \text{fin.}\,3\,\Phi\,(\text{fin.}\,2\,\Phi \pm \sqrt{-1}\,\text{cof.}\,2\,\Phi),$$

ficque erit valor nofter

$$F = \tfrac{1}{3}\,\text{fin.}\,\omega\,(\text{fin.}\,2\,\Phi \pm \sqrt{-1}\,\text{cof.}\,2\,\Phi)\,[\mp\text{cof.}\,(\omega \pm 3\,\Phi)$$
$$\mp \sqrt{-1}\,\text{fin.}\,(\omega \pm 3\,\Phi)],$$

qui valor pro fignis fuperioribus erit

$$F = -\tfrac{1}{3}\,\text{fin.}\,\omega\,[\text{fin.}\,(\omega + \Phi) + \sqrt{-1}\,\text{cof.}\,(\omega + \Phi)],$$

at pro fignis inferioribus prodit

$$F = +\tfrac{1}{3}\,\text{fin.}\,\omega\,[\text{fin.}\,(\omega - \Phi) - \sqrt{-1}\,\text{cof.}\,(\Phi - \Phi)].$$

§. 42. Nunc autem neceffe eft imaginaria hinc fecludi, ad quod efficiendum ftatuamus

$$\text{fin.}\,(\omega + \Phi) + \sqrt{-1}\,\text{cof.}\,(\omega + \Phi) = A\,v + B$$
$$= A\,\text{cof.}\,\omega + B + \sqrt{-1}\,A\,\text{fin.}\,\omega,$$

vnde manifefto deducitur

$$A = \frac{\text{cof.}\,(\omega + \Phi)}{\text{fin.}\,\omega} \text{ et } B = -\frac{\text{cof.}\,(2\,\omega + \Phi)}{\text{fin.}\,\omega},$$

ficque habebimus pro priore cafu

$$F = -\tfrac{1}{3}\,v\,\text{cof.}\,(\omega + \Phi) + \tfrac{1}{3}\,\text{cof.}\,(2\,\omega + \Phi);$$

pro pofteriore vero

$$A = -\frac{\text{cof.}\,(\omega - \Phi)}{\text{fin.}\,\omega} \text{ et } B = \frac{\text{cof.}\,\Phi}{\text{fin.}\,\omega}, \text{ ideoque}$$
$$F = -\tfrac{1}{3}\,v\,\text{cof.}\,(\omega - \Phi) + \tfrac{1}{3}\,\text{cof.}\,\Phi.$$

Verum non opus eft vlterius progredi, quoniam evolutio horum cafuum fpecialium nobis jam viam fternit ad formam generaneralem euoluendam, quam ergo in fequente problemate profequemur.

Problema generale.

Si ponatur $z = v(\text{cof.}\,\Phi + \sqrt{-1}\,\text{fin.}\,\Phi)$, *inueftigare integrale hujus formulae:* $\int \frac{z^{m-1}\,\partial z}{1+z^n}$.

Solutio.

§. 43. Cum ob valorem ipfius z imaginarium integrale quaefitum pariter effe debeat imaginarium, id fub forma $P + Q\sqrt{-1}$ complectamur, ita vt P et Q fint quantitates reales, hanc ob rem erit facta fubftitutione indicata

$$\int \frac{z^{m-1}\,\partial z}{1+z^n} = P + Q\sqrt{-1}.$$

§. 44. Cum porro fit $z = v(\text{cof.}\,\Phi + \sqrt{-1}\,\text{fin.}\,\Phi)$, erit

$z^n = v^n(\text{cof.}\,n\Phi + \sqrt{-1}\,\text{fin.}\,n\Phi)$ et

$\frac{\partial z}{z} = \frac{\partial v}{v} + \partial\Phi\sqrt{-1}$.

Hinc igitur formula propofita abibit in hanc:

$$\frac{v^m(\text{cof.}\,m\Phi + \sqrt{-1}\,\text{fin.}\,m\Phi)(\frac{\partial v}{v} + \partial\Phi\sqrt{1})}{1 + v^n(\text{cof.}\,n\Phi + \sqrt{-1}\,\text{fin.}\,n\Phi)},$$

vbi pro fequente ratiocinio notetur, denominatorem evanefcere, fi ponatur

$$v^n = -\frac{1}{\text{cof.}\,n\Phi + \sqrt{-1}\,\text{fin.}\,n\Phi} = -\text{cof.}\,n\Phi + \sqrt{-1}\,\text{fin.}\,n\Phi.$$

Nunc vero ut denominator ab imaginariis liberetur, fupra et infra multiplicetur per $1 + v^n(\text{cof.}\,n\Phi - \sqrt{-1}\,\text{fin.}\,n\Phi)$ et formula differentialis, quam per ∂V defignemus, erit

$$\partial V = \frac{v^m(\text{cof.}\,m\Phi + \sqrt{-1}\,\text{fin.}\,m\Phi)(\frac{\partial v}{v} + \partial\Phi\sqrt{-1})[(1 + v^n(\text{cof.}\,n\Phi - \sqrt{-1}\,\text{fin.}\,n\Phi)]}{1 + 2v^n\,\text{cof.}\,n\Phi + v^{2n}}.$$

Nu-

Numerator autem reduci poteſt ad hanc formam:

$$v^m\left(\frac{\partial v}{v}+\partial\phi\sqrt{-1}\right)[\text{coſ.}\,m\phi+\sqrt{-1}\,\text{ſin.}\,m\phi+v^n(\text{coſ.}(m-n)\phi+\sqrt{-1}\,\text{ſin.}(m-n)\phi)]$$

cujus partes reales et imaginariae ita a ſe invicem ſegregabuntur, ut ſit pars realis

$$v^{m-1}\partial v[\text{coſ.}\,m\phi+v^n\text{coſ.}(m-n)\phi]-v^m\partial\phi(\text{ſin.}\,m\phi+v^n\text{ſin.}(m-n)\phi)]$$

pars vero imaginaria per $\sqrt{-1}$ diviſa

$$v^{m-1}\partial v[\text{ſin.}\,m\phi+v^n\text{ſin.}(m-n)\phi+v^m\partial\phi(\text{coſ.}\,m\phi+v^n\text{coſ.}(m-n)\phi)]$$

§. 45. Ponamus nunc brevitatis gratia

$$R=\text{coſ.}\,m\,\phi+v^n\,\text{coſ.}\,(m-n)\,\phi \text{ et}$$

$$S=\text{ſin.}\,m\,\phi+v^n\,\text{ſin.}\,(m-n)\,\phi$$

et ambae quantitates quaeſitae P et Q per ſequentes formulas integrales exprimentur:

$$P=\int\frac{R\,v^{m-1}\,\partial v-S\,v^m\,\partial\phi}{1+2\,v^n\,\text{coſ.}\,n\,\phi+v^{2n}} \text{ et}$$

$$Q=\int\frac{S\,v^{m-1}\,\partial v+R\,v^m\,\partial\phi}{1+2\,v^n\,\text{coſ.}\,n\,\phi+v^{2n}}.$$

Totum negotium ergo huc redit, ut primo denominatoris factores trinomiales inveſtigentur, tum vero ex ſingulis fractiones partiales eruantur.

§. 46. Ponamus igitur denominatoris factorem quemcunque eſſe $1-2\,v\,\text{coſ.}\,\omega+v\,v$, atque neceſſe erit ut poſito $1-2\,v\,\text{coſ.}\,\omega+v\,v=0$ etiam denominator evaneſcat, id quod ante jam animadvertimus fieri caſu

$$v^n=-\text{coſ.}\,n\,\phi+\sqrt{-1}\,\text{ſin.}\,n\,\phi.$$

At vero cum ſit $v=\text{coſ.}\,\omega+\sqrt{-1}\,\text{ſin.}\,\omega$, erit hinc

$$v^n=\text{coſ.}\,n\,\omega+\sqrt{-1}\,\text{ſin.}\,n\,\omega,$$

vnde manifeſtum eſt eſſe debere $\text{coſ.}\,n\,\omega=-\text{coſ.}\,n\,\phi$ et ſin. $n\omega = + $ ſin. $n\Phi$. Hinc patet angulorum $n\omega$ et $n\Phi$ ſummam aequari debere angulo $i\pi$, denotante i numerum imparem quemcunque, ita vt $n\omega = i\pi - n\Phi$, ideoque $\omega = \frac{i\pi}{n} - \Phi$. Evidens autem eſt hoc modo pro ω tot diverſos valores reperiri, quot exponens n habet vnitates. Singuli enim iſti valores prodibunt, ſi loco i ſumantur numeri impares 1, 3, 5, 7 etc. vſque ad $2n - 1$; quamobrem ſinguli iſti factores totidem producunt fractiones partiales, idque pro ſingulis partibus, quibus litterae P et Q exprimuntur.

§. 47. Ad hanc reſolutionem inſtituendam conſideremus in genere fractionem $\frac{N}{1 + 2v^n \text{cof.}\, n\Phi + v^{2n}}$, vnde pro factore $1 - 2v\,\text{cof.}\,\omega + vv$ oriatur fractio partialis $= \frac{F}{1 - 2v\,\text{cof.}\,\omega + vv}$, reliquae vero partes omnes deſignentur per Ω, ita vt ſit

$$\frac{N}{1 + 2v^n \text{cof.}\, n\Phi + v^{2n}} = \frac{F}{1 - 2v\,\text{cof.}\,\omega + vv} + \Omega,$$

vnde colligimus

$$F = \frac{N(1 - 2v\,\text{cof.}\,\omega + vv)}{1 + 2v^n \text{cof.}\, n\Phi + v^{2n}} - \Omega(1 - 2v\,\text{cof.}\,\omega + vv).$$

Ex quo intelligitur, valorem ipſius F ex ſola parte priore elici poſſe, ſi ſtatuatur $1 - 2v\,\text{cof.}\,\omega + vv = 0$. At vero tum prioris partis tam numerator quam denominator evaneſcet, vnde ſecundum regulam notiſſimam differentialia ſubſtitui debent, quo facto fiet

$$F = \frac{N(2v - 2\,\text{cof.}\,\omega)}{2nv^{2n-1} + 2nv^{n-1}\text{cof.}\, n\Phi} = \frac{N(v - \text{cof.}\,\omega)}{nv^{n-1}(v^n + \text{cof.}\, n\Phi)};$$

§. 48. Cum autem casu, quo ista euanescentia numeratoris et denominatoris evenit, fit

$$v = \text{cos.}\,\omega + \sqrt{-1}\,\text{sin.}\,\omega \text{ et}$$

$$v^n = -\text{cos.}\,n\,\Phi + \sqrt{-1}\,\text{sin.}\,n\,\Phi,$$

his valoribus substitutis fiet

$$F = \frac{N\,\text{sin.}\,\omega}{n\,v^{n-1}\,\text{sin.}\,n\,\Phi} = \frac{N\,v\,\text{sin.}\,\omega}{n\,v^n\,\text{sin.}\,n\,\Phi}.$$

Nunc igitur tantum opus est, vt loco N diversae partes, quae supra in numeratoribus formularum P et Q occurrerunt, substituantur, hincque ope aequationis $v\,v - 2\,v\,\text{cos.}\,\omega + 1 = 0$ singulae expressiones infra secundam potestatem ipsius v deprimantur.

§. 49. Hunc in finem in usum vocetur sequens lemma: *Si fuerit* $v\,v - 2\,v\,\text{cos.}\,\omega + 1 = 0$, *semper erit*

$$v^\lambda\,\text{sin.}\,\omega = v\,\text{sin.}\,\lambda\,\omega - \text{sin.}\,(\lambda - 1)\,\omega,$$

cujus veritas haud difficulter demonstratur. Tantum autem opus est vt pro littera N gemini valores evolvantur, qui sunt $N = R\,v^{m-1}$ et $N = S\,v^{m-1}$, quibus deinceps adjungi debebit siue ∂v, siue $v\,\partial\Phi$. Sit igitur primo

$$N = R\,v^{m-1} = v^{m-1}\,\text{cos.}\,m\,\Phi + v^{m+n-1}\,\text{cos.}\,(m-n)\,\Phi$$

eritque

$$F = \frac{v^{m-n}\,\text{sin}\,\omega\,\text{cos.}\,m\,\Phi + v^m\,\text{sin.}\,\omega\,\text{cos.}\,(m-n)\,\Phi}{n\,\text{sin.}\,n\,\Phi}$$

vbi secundum lemma habebimus

$$v^{m-n}\,\text{sin.}\,\omega = v\,\text{sin.}\,(m-n)\,\omega - \text{sin.}\,(m-n-1)\,\omega \text{ et}$$

$$v^m\,\text{sin.}\,\omega = v\,\text{sin.}\,m\,\omega - \text{sin.}\,(m-1)\,\omega$$

vnde ergo conficitur

$$F = \frac{1}{n\,\text{sin.}\,n\,\Phi}\left[\begin{array}{l} +\,v\,\text{sin.}\,(m-n)\,\omega\,\text{cos.}\,m\,\Phi - \text{sin.}\,(m-n-1)\,\omega\,\text{cos.}\,m\,\Phi \\ +\,v\,\text{sin.}\,m\,\omega\,\text{cos.}\,(m-n)\,\Phi - \text{sin.}\,(m-1)\,\omega\,\text{cos.}\,(m-n)\,\Phi \end{array}\right].$$

§. 50.

§. 50. In hac expressione littera v ducitur in formulam

$$\text{fin.}\,(m-n)\,\omega\,\text{cof.}\,m\,\Phi + \text{cof.}\,(m-n)\,\Phi\,\text{fin.}\,m\,\omega,$$

pro cujus resolutione notetur esse

$$\text{cof.}\,n\,\omega = -\,\text{cof.}\,n\,\Phi \text{ et } \text{fin.}\,n\,\omega = +\,\text{fin.}\,n\,\Phi,$$

hincque fiet

$$\text{fin.}\,(m-n)\,\omega = -\,\text{fin.}\,m\,\omega\,\text{cof.}\,n\,\Phi - \text{cof.}\,m\,\omega\,\text{fin.}\,n\,\Phi,$$

tum vero

$$\text{cof.}\,(m-n)\,\Phi = \text{cof.}\,m\,\Phi\,\text{cof.}\,n\,\Phi + \text{fin.}\,m\,\Phi\,\text{fin.}\,n\,\Phi,$$

quibus valoribus substitutis quantitas litteram v afficiens erit $-\,\text{fin.}\,n\,\Phi\,\text{cof.}\,m\,(\omega+\Phi)$. Hinc autem pars reliqua oritur, si mutato signo loco $m\,\omega$ scribatur $(m-1)\,\omega$, sicque integer valor quaesitus erit

$$F = -\tfrac{1}{n}\,[v\,\text{cof.}\,m\,(\omega+\Phi) - \text{cof.}\,(m-1)\,\omega + m\,\Phi].$$

§. 51. Supra autem vidimus esse $\omega + \Phi = \frac{i\pi}{n}$, ideoque $m\,(\omega+\Phi) = \frac{m\,i\,\pi}{n}$, cujus loco scribamus ζ, quo facto pro casu praesente, quo $N = R\,v^{m-1}$, fractionis quaesitae numerator erit

$$F = -\tfrac{1}{n}\,[v\,\text{cof.}\,\zeta - \text{cof.}\,(\zeta-\omega)],$$

quem igitur duplici modo adhiberi convenit; namque pro littera P is multiplicari debet per ∂v, pro littera Q vero per $v\,\partial\Phi$.

§. 52. Simili modo pro casu

$$N = S\,v^{m-1} = v^{m-1}\,\text{fin.}\,m\,\Phi + v^{m+n-1}\,\text{fin.}\,(m-n)\,\Phi,$$

oritur

$$F = \frac{v^{m-n}\,\text{fin.}\,\omega\,\text{fin.}\,m\,\Phi + v^{m}\,\text{fin.}\,\omega\,\text{fin.}\,[m-n)\,\Phi}{n\,\text{fin.}\,n\,\Phi}.$$

Jam loco potestatum ipsius v scribamus valores supra assignatos, ac prodibit

F =

$$F = \frac{1}{\pi \sin. n\phi} \left[\frac{v \sin. m\phi \sin. (m-n)\omega - \sin. m\phi \sin. (m-n-1)\omega}{v \sin. (m-n)\phi \sin. m\omega - \sin. (m-n)\phi \sin. (m-1)\omega}\right].$$

Cum jam fit

$$\sin. (m-n)\omega = - \sin. m\omega \cos. n\omega - \cos. m\omega \sin. n\omega \text{ et}$$

$$\sin. (m-n)\phi = \sin. m\phi \cos. n\phi - \cos. m\phi \sin. n\phi$$

littera v affecta eſt hac quantitate:

$$- \sin. n\phi \sin. (\phi + \omega) m = - \sin. n\phi \sin. \zeta$$

vnde integer valor erit

$$F = - \tfrac{1}{n} [v \sin. \zeta - \sin. (\zeta - \omega)]$$

qui valor pro P duci debet in $- v \partial \phi$, pro Q autem in $+ \partial v$.

§. 53. His igitur valoribus inventis ſinguli anguli ω, quorum numerus eſt $= n$, dabunt totidem partes pro quantitatibus quaeſitis P et Q, ſcilicet valor $\omega = \frac{i\pi}{n} - \phi$, exiſtente $\frac{m i \pi}{n} = \zeta$, dabit

$$P = - \frac{1}{n} \int \frac{\partial v [v \cos. \zeta - \cos. (\zeta - \omega)] - v \partial \phi [v \sin. \zeta - \sin. (\zeta - \omega)]}{1 - 2 v \cos \omega + v v} \text{ et}$$

$$Q = - \frac{1}{n} \int \frac{\partial v [v \sin. \zeta - \sin. (\zeta - \omega)] + v \partial \phi [v \cos. \zeta - \cos. (\zeta - \omega)]}{1 - 2 v \cos. \omega + v v}.$$

Vbi quidem $\partial \phi$ adhuc multiplicatur per $v v$, cujus loco ſcribi poſſet $2 v \cos. \omega - 1$; verum omiſſa hac ſubſtitutione nullus error committitur.

§. 54. Videamus nunc, quomodo ipſa harum formularum integratio inſtitui queat. Ac primo quidem angulum ϕ pro conſtante habeamus, vt ſit

$$P = - \frac{1}{n} \int \frac{\partial v [v \cos. \zeta - \cos. (\zeta - \omega)]}{1 - 2 v \cos. \omega + v v} \text{ et}$$

$$Q = - \frac{1}{n} \int \frac{\partial v [v \sin. \zeta - \sin. (\zeta - \omega)]}{1 - 2 v \cos. \omega + v v}.$$

Ponatur igitur

$$M = \int \frac{\partial v [v \cos. \zeta - \cos. (\zeta - \omega)]}{1 - 2 v \cos. \omega + v v}, \text{ eritque}$$

$$M - \cos. \zeta \, l [\sqrt{(1 - 2 v \cos. \omega + v v)}]$$

 $= \int$

$$= \int \frac{\partial v\,[v \operatorname{cof.} \zeta - \operatorname{cof.} (\zeta - \omega)]}{1 - 2 v \operatorname{cof.} \omega + v v} - \int \frac{(v \operatorname{cof.} \zeta - \operatorname{cof.} \zeta \operatorname{cof.} \omega)\, \partial v}{1 - 2 v \operatorname{cof.} \omega + v v}$$

$$= \int \frac{\partial v\,[\operatorname{cof.} \zeta \operatorname{cof.} \omega - \operatorname{cof.} (\zeta - \omega)]}{1 - 2 v \operatorname{cof.} \omega + v v} = - \int \frac{\partial v \operatorname{fin.} \zeta \operatorname{fin.} \omega}{1 - 2 v \operatorname{cof.} \omega + v v},$$

ficque integrale erit

$$- \operatorname{fin.} \zeta \text{ A tang. } \frac{v \operatorname{fin.} \omega}{1 - v \operatorname{cof.} \omega}, \text{ ideoque}$$

$$M = \operatorname{cof.} \zeta\, l \sqrt{(1 - 2 v \operatorname{cof.} \omega + v v)} - \operatorname{fin.} \zeta \text{ A tang. } \frac{v \operatorname{fin.} \omega}{1 - v \operatorname{cof.} \omega};$$

confequenter habebimus

$$P = - \frac{\operatorname{cof.} \zeta}{n}\, l \sqrt{(1 - 2 v \operatorname{cof.} \omega + v v)} + \frac{\operatorname{fin.} \zeta}{n} \text{ A tang. } \frac{v \operatorname{fin.} \omega}{1 - v \operatorname{cof.} \omega}.$$

§. 55. Hoc valore ex fola variabilitate ipfius v orto, videamus quomodo cum angulo variabili Φ confiftat. Hunc in finem differentiemus hanc ipfam formulam inuentam, ftatuendo folum angulum ω variabilem, fiquidem $\partial \omega = - \partial \Phi$, ob angulum ζ conftantem, eritque differentiale

$$- \frac{1}{n} \left[\frac{- v \partial \Phi \operatorname{fin.} \omega \operatorname{cof.} \zeta + v \partial \Phi (\operatorname{cof.} \omega - v) \operatorname{fin.} \zeta}{1 - 2 v \operatorname{cof.} \omega + v v} \right] = \frac{1}{n} \left[\frac{v \operatorname{fin.} \zeta - \operatorname{fin.} (\zeta - \omega)\, v \partial \Phi}{1 - 2 v \operatorname{cof.} \omega + v v} \right],$$

quod prorfus conuenit cum forma propofita, ita vt iuftus valor pro P fit

$$P = - \frac{\operatorname{cof.} \zeta}{n}\, l \sqrt{(1 - 2 v \operatorname{cof.} \omega + v v)} + \frac{\operatorname{fin.} \zeta}{n} \text{ A tang. } \frac{v \operatorname{fin.} \omega}{1 - v \operatorname{cof.} \omega}.$$

§. 56. Eodem modo procedamus pro valore Q, fitque

$$M = \int \frac{\partial v\,[v \operatorname{fin.} \zeta - \operatorname{fin.} (\zeta - \omega)]}{1 - 2 v \operatorname{cof.} \omega + v v}, \text{ eritque}$$

$$M - \operatorname{fin.} \zeta\, l \sqrt{(1 - 2 v \operatorname{cof.} \omega + v v)} =$$

$$\int \frac{\partial v\,[v \operatorname{fin.} \zeta - \operatorname{fin.} (\zeta - \omega)]}{1 - 2 v \operatorname{cof.} \omega + v v} - \int \frac{(v \operatorname{fin.} \zeta - \operatorname{cof.} \omega \operatorname{fin.} \zeta)\, \partial v}{1 - 2 v \operatorname{cof.} \omega + v v} =$$

$$\int \frac{\partial v \operatorname{cof.} \zeta \operatorname{fin.} \omega}{1 - 2 v \operatorname{cof.} \omega + v v} = \operatorname{cof.} \zeta \text{ A tang. } \frac{v \operatorname{fin.} \omega}{1 - v \operatorname{cof.} \omega},$$

vnde manifefto colligitur

$$Q = - \frac{\operatorname{fin.} \zeta}{n}\, l \sqrt{(1 - 2 v \operatorname{cof.} \omega + v v)} - \frac{\operatorname{cof.} \zeta}{n} \text{ A tang. } \frac{v \operatorname{fin.} \omega}{1 - v \operatorname{cof.} \omega}$$

quae expreffio variabilitati ipfius Φ etiam eft confentanea.

§. 57.

§. 57. Nunc igitur casum formulae $\int \frac{\partial z}{(1+z^3)}$, quem iam bis frustra sumus aggressi, facile expedire licebit. Cum enim hic sit $m = 1$ et $n = 3$, pro littera i tres sumi debebunt valores 1, 3 et 5, vnde pro nostris formulis integralibus sequentes valores emergunt:

i	1	3	5
ω	$60^\circ - \Phi$	$180^\circ - \Phi$	$300^\circ - \Phi$
$\sin.\omega$	$\sin.(60^\circ - \Phi)$	$\sin.\Phi$	$-\sin.(60^\circ + \Phi)$
$\cos.\omega$	$\cos.(60^\circ - \Phi)$	$-\cos.\Phi$	$\cos.(60^\circ + \Phi)$
ζ	60°	180°	300°
$\sin.\zeta$	$\frac{\sqrt{3}}{2}$	0	$-\frac{\sqrt{3}}{2}$
$\cos.\zeta$	$\frac{1}{2}$	-1	$\frac{1}{2}$

§. 58. Ex his iam ternis valoribus tam pro P quam Q ternas partes adipiscemur, quae erunt:

Pro P

Pars I. $-\frac{1}{6} l \sqrt{[1 - 2v\cos.(60^\circ - \Phi) + vv]} + \frac{1}{2\sqrt{3}} \text{A tang.} \frac{v \sin.(60^\circ - \Phi)}{1 - v\cos.(60^\circ - \Phi)}$,

Pars II. $+\frac{1}{3} l \sqrt{(1 + 2v\cos.\Phi + vv)} + 0$

Pars III. $-\frac{1}{6} l \sqrt{[1 - 2v\cos.(60^\circ + \Phi) + vv]} + \frac{1}{2\sqrt{3}} \text{A tang.} \frac{v \sin.(60^\circ + \Phi)}{1 - v\cos.(60^\circ + \Phi)}$.

Vbi notasse iuuabit partem primam et tertiam ita coniunctim exprimi posse:

$$-\tfrac{1}{12} l [1 - 2v\cos.\Phi + 2vv(\tfrac{1}{2} + \cos.2\Phi) - 2v^3\cos.\Phi + v^4]$$
$$+ \frac{1}{2\sqrt{3}} \text{A tang.} \frac{2v\cos.\Phi\sqrt{3} - vv\sqrt{3}}{2 - 2v\cos.\Phi - vv}.$$

Pro Q

Pars I. $-\frac{1}{2\sqrt{3}} l \sqrt{[1 - 2v\cos.(60^\circ - \Phi) + vv]} - \frac{1}{6} \text{A tang.} \frac{v \sin.(60^\circ - \Phi)}{1 - v\cos.(60^\circ - \Phi)}$,

Pars II. $-0 \qquad + \frac{1}{3} \text{A tang.} \frac{v \sin.\Phi}{1 + v\cos.\Phi}$,

Pars III. $+\frac{1}{2\sqrt{3}} l \sqrt{[1 - 2v\cos.(60^\circ + \Phi) + vv]} + \frac{1}{6} \text{A tang.} \frac{v \sin.(60^\circ + \Phi)}{1 - v\cos.(60^\circ + \Phi)}$.

Hic iterum partes prima et tertia contrahi poſſent, ſed praeſtabit formulis primo inuentis vti. Hinc iam iſtam tractationem ſequenti Theoremate concludemus.

Theorema.

§. 59. Poſito $z = v(\cos.\Phi + \sqrt{-1}\sin.\Phi)$, ſi ſtatuatur $\int \frac{\partial z}{1+z^3} = P + Q\sqrt{-1}$, hae quantitates P et Q ita exprimentur:

$$P = \begin{cases} -\frac{1}{6} l\sqrt{[1-2v\cos.(60^\circ-\Phi)+vv]} + \frac{1}{2\sqrt{3}}\,\mathrm{A\,tang.}\frac{v\sin.(60^\circ-\Phi)}{1-v\cos.(60^\circ-\Phi)} \\ +\frac{1}{3} l\sqrt{[1+2v\cos.\Phi+vv]} \\ -\frac{1}{6} l\sqrt{[1-2v\cos.(60^\circ+\Phi)+vv]} + \frac{1}{2\sqrt{3}}\,\mathrm{A\,tang.}\frac{v\sin.(60^\circ+\Phi)}{1-v\cos.(60^\circ+\Phi)} \end{cases}$$

$$Q = \begin{cases} -\frac{1}{2\sqrt{3}} l\sqrt{[1-2v\cos.(60^\circ-\Phi)+vv]} - \frac{1}{6}\,\mathrm{A\,tang.}\frac{v\sin.(60^\circ-\Phi)}{1-v\cos.(60^\circ-\Phi)} \\ \qquad\qquad +\frac{1}{3}\,\mathrm{A\,tang.}\frac{v\sin.\Phi}{1+v\cos.\Phi} \\ +\frac{1}{2\sqrt{3}} l\sqrt{[1-2v\cos.(60^\circ+\Phi)+vv]} + \frac{1}{6}\,\mathrm{A\,tang.}\frac{v\sin.(60^\circ+\Phi)}{1-v\cos.(60^\circ+\Phi)}. \end{cases}$$

Corollarium.

§. 60. Si ergo ſumamus angulum $\Phi = 0$, vt fiat $z = v$, pro formula integrali $\int \frac{\partial v}{1+v^3} = P + Q\sqrt{-1}$ erit:

$$P = \begin{cases} -\frac{1}{6} l\sqrt{(1-v+vv)} + \frac{1}{2\sqrt{3}}\,\mathrm{A\,tang.}\frac{v\sqrt{3}}{2-v} \\ +\frac{1}{3} l\sqrt{(1+v)} \\ -\frac{1}{6} l\sqrt{(1-v+vv)} + \frac{1}{2\sqrt{3}}\,\mathrm{A\,tang.}\frac{v\sqrt{3}}{2-v} \end{cases}$$

$$Q = \begin{cases} -\frac{1}{2\sqrt{3}} l\sqrt{(1-v+vv)} - \frac{1}{6}\,\mathrm{A\,tang.}\frac{v\sqrt{3}}{2-v} \\ +\frac{1}{2\sqrt{3}} l\sqrt{(1-v+vv)} + \frac{1}{6}\,\mathrm{A\,tang.}\frac{v\sqrt{3}}{2-v}. \end{cases}$$

Sicque erit $Q = 0$, vti natura rei poſtulat. Nam quia ipſa formula integranda eſt realis, etiam integrale partem imaginariam con-

continere nequit. Ceterum ipsum hoc integrale satis est notum.

Corollarium 2.

§. 61. Consideremus etiam casum quo $\Phi = 90°$, ideoque $z = v\sqrt{-1}$, et formula integranda erit

$$\int \frac{\partial v \sqrt{-1}}{1 - v^3\sqrt{-1}} = P + Q\sqrt{-1};$$

quantitates vero P et Q ita exprimentur:

$$P = \begin{cases} -\frac{1}{6} l \sqrt{(1 - v\sqrt{3} + vv)} - \frac{1}{2\sqrt{3}} \text{A tang.} \frac{v}{2 - v\sqrt{3}} \\ +\frac{1}{3} l \sqrt{(1 + vv)} \\ -\frac{1}{6} l \sqrt{(1 + v\sqrt{3} + vv)} + \frac{1}{2\sqrt{3}} \text{A tang.} \frac{v}{2 + v\sqrt{3}} \end{cases}$$

$$Q = \begin{cases} -\frac{1}{2\sqrt{3}} l \sqrt{(1 - v\sqrt{3} + vv)} + \frac{1}{6} \text{A tang.} \frac{v}{2 - v\sqrt{3}} \\ \qquad + \frac{1}{3} \text{A tang.}\, v \\ +\frac{1}{2\sqrt{3}} l \sqrt{(1 + v\sqrt{3} + vv)} + \frac{1}{6} \text{A tang.} \frac{v}{2 + v\sqrt{3}}. \end{cases}$$

Corollarium 3.

§. 62. Praeterea vero etiam casus memoratu dignus occurrit, quo $\Phi = 60°$, ideoque $z = \frac{v}{2} + \frac{v\sqrt{-3}}{2}$ et $z^3 = -v^3$, ita vt formula integranda sit $\frac{\partial v(\frac{1}{2} + \frac{1}{2}\sqrt{-3})}{1 - v^3}$; tum igitur erit:

$$P = \begin{cases} -\frac{1}{6} l (1 - v) \\ +\frac{1}{3} l \sqrt{(1 + v + vv)} \\ -\frac{1}{6} l \sqrt{(1 + v + vv)} + \frac{1}{2\sqrt{3}} \text{A tang.} \frac{v\sqrt{3}}{2+v} \end{cases}$$

$$Q = \begin{cases} -\frac{1}{2\sqrt{3}} l (1 - v) \\ +\frac{1}{3} \text{A tang.} \frac{v\sqrt{3}}{2+v} \\ +\frac{1}{2\sqrt{3}} l \sqrt{(1 + v + vv)} + \frac{1}{6} \text{A tang.} \frac{v\sqrt{3}}{2+v}, \end{cases}$$

vbi manifesto $P : Q = 1 : \sqrt{3}$, prorsus vti natura rei postulat.

SVPPLEMENTVM

ad differtationem praecedentem, circa integrationem formulae $\int \frac{z^{m-1}\partial z}{1-z^n}$, cafu quo ponitur

$z = v(\text{cof.}\,\Phi + \sqrt{-1}\,\text{fin.}\,\Phi)$.

§. 1. Refolutio formulae $\int \frac{z^{m-1}\partial z}{1+z^n}$, quam fupra in problemate, pro cafu quo $z = v(\text{cof.}\,\Phi + \sqrt{-1}\,\text{fin.}\,\Phi)$ dedimus, eximia et notatu digniffima artificia complectitur, quae animo firmius imprimere haud inutile erit. Cum igitur formula, quam hic tractandam fufcipimus, non minore attentione fit digna quam ea quam fupra tractauimus, eius integrale per eandem methodum exhibere conftitui; vbi fimul occafionem inueniemus nouum compendium in calculo adhibendi.

Problema.

Si ponatur $z = v(\text{cof.}\,\Phi + \sqrt{-1}\,\text{fin.}\,\Phi)$, *inueftigare integrale huius formulae:* $\int \frac{z^{m-1}\partial z}{1-z^n}$.

Solutio.

§. 2. Cum ob valorem ipfius z imaginarium integrale quaefitum etiam effe debeat imaginarium, id fub forma $P + Q\sqrt{-1}$ complectamur, ita vt P et Q fint quantitates reales. Hanc ob rem erit facta fubftitutione

$$\int \frac{z^{m-1}\partial z}{1-z^n} = P + Q\sqrt{-1}.$$

§. 3. Cum porro fit $z = v(\text{cof.}\,\Phi + \sqrt{-1}\,\text{fin.}\,\Phi)$, hincque $\frac{\partial z}{z} = \frac{\partial v}{v} + \partial\Phi\sqrt{-1}$, erit numerator

z

$$z^{m-1}\partial z = v^m(\text{cof.}\, m\Phi + \sqrt{-1}\,\text{fin.}\, m\Phi)\left(\frac{\partial v}{v} + \partial\Phi\sqrt{-1}\right),$$

denominator vero erit

$$1 - v^n(\text{cof.}\, n\Phi + \sqrt{-1}\,\text{fin.}\, n\Phi),$$

qui ergo euanefcit ponendo

$$v^n = \frac{1}{\text{cof.}\, n\Phi + \sqrt{-1}\,\text{fin.}\, n\Phi} = \text{cof.}\, n\Phi - \sqrt{-1}\,\text{fin.}\, n\Phi.$$

§. 4. Iam vt imaginaria ex denominatore tollantur, fupra et infra multiplicemus per

$$1 - v^n(\text{cof.}\, n\Phi - \sqrt{-1}\,\text{fin.}\,\Phi),$$

ficque fractio noftra euoluenda erit

$$\partial V = \frac{z^{m-1}\partial z(1 - v^n\,\text{cof.}\, n\Phi + v^n\sqrt{-1}\,\text{fin.}\, n\Phi)}{1 - 2v^n\,\text{cof.}\, n\Phi + v^{2n}}.$$

Quod fi iam hic loco $z^{m-1}\partial z$ valor modo affignatus fubftituatur et partes reales ab imaginariis fegregentur, ob

$$(\text{cof.}\, m\Phi + \sqrt{-1}\,\text{fin.}\, m\Phi)(\text{cof.}\, n\Phi - \sqrt{-1}\,\text{fin.}\, n\Phi)$$
$$= \text{cof.}\,(m-n)\Phi + \sqrt{-1}\,\text{fin.}\,(m-n)\Phi$$

prodibit pars realis ita expreffa:

$$v^{m-1}\partial v\,[\text{cof.}\, m\Phi - v^n\,\text{cof.}\,(m-n)\Phi]$$
$$- v^m\partial\Phi\,[\text{fin.}\, m\Phi - v^n\,\text{fin.}\,(m-n)\Phi],$$

pars vero imaginaria per $\sqrt{-1}$ diuifa:

$$v^{m-1}\partial v\,[\text{fin.}\, m\Phi - v^n\,\text{fin.}\,(m-n)\Phi]$$
$$+ v^m\partial\Phi\,(\text{cof.}\, m\Phi - v^n\,\text{cof.}\,(m-n)\Phi].$$

§. 5. Quod fi iam breuitatis gratia ftatuamus

$$R = v^{m-1}[\text{cof.}\, m\Phi - v^n\,\text{cof.}\,(m-n)\Phi] \text{ et}$$
$$S = v^{m-1}[\text{fin.}\, m\Phi - v^n\,\text{cof.}\,(m-n)\Phi],$$

ambae litterae quaefitae P et Q per fequentes formulas integrales exprimentur:

P =

$$P = \int \frac{R\,\partial v - S\,v\,\partial\phi}{1 - 2\,v^n \cos. n\,\phi + v^{2n}} \text{ et}$$

$$Q = \int \frac{S\,\partial v + R\,v\,\partial\phi}{1 - 2\,v^n \cos. n\,\phi + v^{2n}}.$$

Has igitur duas formulas integrare oportebit, quod fiet, dum denominatoris ſingulos factores trinomiales inueſtigabimus et ex ſingulis fractiones partiales inde oriundas definiemus.

§. 6. Conſideremus igitur in genere hanc fractionem: $\frac{N}{1 - 2\,v^n \cos. n\,\phi + v^{2n}}$, et fingamus denominatoris factorem eſſe $1 - 2\,v \cos. \omega + v\,v$, vbi angulus ω ita debet eſſe comparatus, vt poſito

$$1 - 2\,v \cos. \omega + v\,v = 0, \text{ ſiue}$$

$$v = \cos. \omega + \sqrt{-1} \sin. \omega,$$

ſimul quoque denominator euaneſcat, id quod fit, vti vidimus, quando $v^n = \cos. n\,\phi - \sqrt{-1} \sin. n\,\phi$. At vero ex factore ſuppoſito fit $v^n = \cos. n\,\omega + \sqrt{-1} \sin. n\,\omega$, vnde ſtatui debebit $\cos. n\,\omega = \cos. n\,\phi$ et $\sin. n\,\omega = -\sin. n\,\phi$, id quod euenit in genere quando $n\,\omega + n\,\phi = i\,\pi$, denotante i omnes numeros pares, ſicque erit $n\,\omega = i\,\pi - n\,\phi$, ideoque $\omega = \frac{i\pi}{n} - \phi$, vnde n diuerſi valores pro angulo ω deducuntur, dum ſcilicet loco i ſcribuntur ſucceſſiue numeri 0, 2, 4, 6, etc. vsque ad $2\,n$, excluſo poſtremo.

§. 7. Ponamus nunc fractionem partialem ex iſto factore oriundam eſſe $\frac{F}{1 - 2\,v \cos. \omega + v\,v}$, atque ex ſuperioribus patet ſtatui debere

$$F = \frac{N(1 - 2\,v \cos. \omega + v\,v)}{1 - 2\,v^n \cos. n\,\phi + v^{2n}},$$

vnde

vnde ſcilicet ope aequationis $vv - 2v \cos. \omega + 1 = 0$ pro F huiusmodi forma $Av + B$ elici debet. Quoniam vero hoc caſu tam numerator quam denominator euaneſcit, differentialibus in ſubſidium vocatis fiet

$$F = \frac{N(v - \cos. \omega)}{n v^{n-1}(v^n - \cos. n\Phi)}.$$

§. 8. Cum nunc caſu quo $vv - 2v \cos. \omega + 1 = 0$ fit

$$v - \cos. \omega = \sqrt{-1} \sin. \omega \text{ et}$$

$$v^n - \cos. n\Phi = -\sqrt{-1} \sin. n\Phi, \text{ erit}$$

$$F = -\frac{N v \sin. \omega}{n v^n \sin. n\Phi},$$

qui valor prorſus conuenit cum eo qui ſupra eſt repertus. Hic igitur tantum opus eſt, vt loco N ſiue R ſiue S ſubſtituatur, indeque forma praeſcripta pro iſto numeratore F deriuetur, in vſum vocando lemma ſupra allatum

Euolutio fractionis

$$\frac{R v \sin. \omega}{n v^n \sin. n\Phi} \text{ ſiue } \frac{v^m \sin. \omega [\cos. m\Phi - v^n \cos. (m-n)\Phi]}{n v^n \sin. n\Phi}.$$

§. 9. Hinc ergo erit

$$F = -\frac{v^{m-n} \sin. \omega \cos. m\Phi + v^m \sin. \omega \cos. (m-n)\Phi}{n \sin. n\Phi}.$$

Per lemma autem memoratum habebitur

$$\sin. \omega\, v^{m-n} = v \sin. (m-n)\omega - \sin. (m-n-1)\omega.$$

Cum igitur ſit $n\omega = i\pi - n\Phi$, erit

$$\sin. (m-n)\omega = \sin. (m\omega + n\Phi) \text{ et}$$

$$\sin. (m-n-1)\omega = \sin. [(m-1)\omega + n\Phi].$$

Deinde vero eſt

$$\text{ſin.}\,\omega.\,v^m = v\,\text{ſin.}\,m\,\omega - \text{ſin.}\,(m-1)\,\omega;$$

quibus valoribus ſubſtitutis erit

$$F = -\frac{1}{n\,\text{ſin.}\,n\,\Phi}\left[\begin{array}{l} v\,\text{coſ.}\,m\,\Phi\,\text{ſin.}\,(m\,\omega + n\,\Phi) - \text{coſ.}\,m\,\Phi\,\text{ſin.}\,[(m-1)\,\omega + n\,\Phi] \\ - v\,\text{ſin.}\,m\,\omega\,\text{coſ.}\,(m-n)\,\Phi] + \text{ſin.}\,(m-1)\,\omega\,\text{coſ.}\,(m-n)\,\Phi \end{array}\right].$$

Facta iam euolutione formularum

$$\text{ſin.}\,(m\,\omega + n\,\Phi) = \text{ſin.}\,m\,\omega\,\text{coſ.}\,n\,\Phi + \text{coſ.}\,m\,\omega\,\text{ſin.}\,n\,\Phi \text{ et}$$

$$\text{coſ.}\,(m-n)\,\Phi = \text{coſ.}\,m\,\Phi\,\text{coſ.}\,n\,\Phi + \text{ſin.}\,m\,\Phi\,\text{ſin.}\,n\,\Phi,$$

littera v hic multiplicatur per hanc formam:

$$\text{ſin.}\,n\,\Phi\,\text{coſ.}\,m\,\Phi\,\text{coſ.}\,m\,\omega - \text{ſin.}\,n\,\Phi\,\text{ſin.}\,m\,\Phi\,\text{ſin.}\,m\,\omega$$

$$= \text{ſin.}\,n\,\Phi\,\text{coſ.}\,(m\,\Phi + m\,\omega),$$

reliqui vero termini, quia ab his tantum in eo differunt vt loco $m\,\omega$ ſcribi debeat $(m-1)\,\omega$, erunt:

$$-\text{ſin.}\,n\,\Phi\,\text{coſ.}\,[m\,(\omega+\Phi) - \omega]$$

ſicque pro numeratore quem quaerimus erit

$$F = -\frac{1}{n}\,v\,\text{coſ.}\,m\,(\omega+\Phi) + \frac{1}{n}\,\text{coſ.}\,[m\,(\omega+\Phi) - \omega].$$

Euolutio fractionis

$$\frac{\mathfrak{S}\,v\,\text{ſin.}\,\omega}{n\,v^n\,\text{ſin.}\,n\,\Phi} = \frac{v^m\,\text{ſin.}\,\omega\,[\text{ſin.}\,m\,\Phi - v^n\,\text{ſin.}\,(m-n)\,\Phi]}{n\,v^n\,\text{ſin.}\,n\,\Phi}.$$

§. 10. Hoc caſu erit

$$F = -\frac{v^{m-n}\,\text{ſin.}\,\omega\,\text{ſin.}\,m\,\Phi + v^m\,\text{ſin.}\,\omega\,\text{ſin.}\,(m-n)\,\Phi}{n\,\text{ſin.}\,n\,\Phi}.$$

Hic igitur eodem lemmate in ſubſidium vocato erit

$$F = -\frac{1}{n\,\text{ſin.}\,n\,\Phi}\left[\begin{array}{l} v\,\text{ſin.}\,m\,\Phi\,\text{ſin.}\,(m\,\omega + n\,\Phi) - \text{ſin.}\,m\,\Phi\,\text{ſin.}\,[(m-1)\,\omega + n\,\Phi] \\ - v\,\text{ſin.}\,(m-n)\,\Phi\,\text{ſin.}\,m\,\omega + \text{ſin.}\,(m-n)\,\Phi\,\text{ſin.}\,(m-1)\,\omega \end{array}\right];$$

vbi per ſimilem euolutionem quantitas, qua v multiplicatur, inuenitur $= \text{ſin.}\,n\,\Phi\,\text{ſin.}\,[m\,(\omega \pm \Phi)]$; reliqua vero pars erit

$$-\sin. n\,\Phi \sin. [m(\omega+\Phi)-\omega],$$

hinc igitur pro littera S valor quaesitus numeratoris erit

$$F = -\tfrac{1}{n} v \sin. m(\omega+\Phi) + \tfrac{1}{n}\sin. [m(\omega+\Phi)-\omega].$$

§. 11. Cum igitur sit $\omega+\Phi=\frac{i\pi}{n}$, ponamus breuitatis gratia angulum $m(\omega+\Phi)=\frac{m i \pi}{n}=\zeta$, atque pro littera R erit

$$F = -\tfrac{1}{n}[v \cos. \zeta - \cos. (\zeta-\omega)]$$

at vero pro S erit

$$F = -\tfrac{1}{n}[v \sin. \zeta - \sin. (\zeta-\omega)],$$

quibus valoribus inuentis pro denominatoris factore $1-2v\cos.\omega+vv$ partes, ex quibus litterae P et Q componuntur, per sequentes formulas integrales exprimentur:

$$P = -\tfrac{1}{n}\int \frac{[v \cos. \zeta - \cos. (\zeta-\omega)]\,\partial v - v\,\partial\Phi\,[v \sin. \zeta - \sin. (\zeta-\omega)]}{1-2v\cos.\omega+vv}$$

$$Q = -\tfrac{1}{n}\int \frac{[v \sin. \zeta - \sin. (\zeta-\omega)]\,\partial v + v\,\partial\Phi\,[v \cos. \zeta - \cos. (\zeta-\omega)]}{1-2v\cos.\omega+vv}.$$

§. 12. Quoniam hae formulae prorsus conueniunt cum iis, quas supra sumus nacti, et ne signa quidem sunt immutata, peculiari integratione non indigemus, sed pro quantitatibus P et Q sequentes habebimus valores integratos:

$$P = -\tfrac{\cos. \zeta}{n}\, l\sqrt{(1-2v\cos.\omega+vv)} + \tfrac{\sin. \zeta}{n}\, \text{A tang.} \frac{v\sin.\omega}{1-v\cos.\omega} \quad \text{et}$$

$$Q = -\tfrac{\sin. \zeta}{n}\, l\sqrt{(1-2v\cos.\omega+vv)} - \tfrac{\cos. \zeta}{n}\, \text{A tang.} \frac{v\sin.\omega}{1-v\cos.\omega}.$$

Tales scilicet formulae ex singulis factoribus denominatoris formae $1-2v\cos.\omega+vv$ deriuari et in vnam summam colligi debent, vt veri valores pro P et Q obtineantur, vbi tantum recordari oportet esse $\omega=\frac{i\pi}{n}-\Phi$ et $\zeta=\frac{m i \pi}{n}$; pro i autem hic numeros pares accipi oportet.

Exemplum 1.

§. 13. *Sit* $m = 1$ *et* $n = 1$, *ita vt quaeri debeat* $\int \frac{\partial z}{1-z} = P + Q\sqrt{-1}$, *posito scilicet* $z = v(\cos.\Phi + \sqrt{-1}\sin.\Phi)$.

Quia hic $n = 1$, vnicus valor pro ω locum habet, resultans ex $i = 0$, eritque ergo $\omega = -\Phi$ et $\zeta = 0$, vnde statim colligimus

$$P = -l\sqrt{(1 - 2v\cos.\Phi + vv)} \text{ et } Q = -A\,\text{tang.}\frac{v\sin.\Phi}{1 - v\cos.\Phi}.$$

Exemplum 2.

§. 14. *Sit* $m = 1$ *et* $n = 2$, *ideoque formula integranda* $\int \frac{\partial z}{1-zz} = P + Q\sqrt{-1}$, *posito* $z = v(\cos.\Phi + \sqrt{-1}\sin.\Phi)$.

Quia hic est $n = 2$, pro ω duos habebimus valores ex $i = 0$ et $i = 2$ oriundos, vnde

Si $i = 0$, erit $\omega = -\Phi$ et $\zeta = 0$

Si $i = 2$, erit $\omega = \pi - \Phi$ et $\zeta = \pi$.

Hinc igitur statim colligemus

$$P = \begin{cases} -\frac{1}{2}l\sqrt{(1 - 2v\cos.\Phi + vv)} + 0 \\ +\frac{1}{2}l\sqrt{(1 + 2v\cos.\Phi + vv)} + 0. \end{cases}$$

$$Q = \begin{cases} 0 + \frac{1}{2}A\,\text{tang.}\frac{v\sin.\Phi}{1 - v\cos.\Phi} \\ 0 + \frac{1}{2}A\,\text{tang.}\frac{v\sin.\Phi}{1 + v\cos.\Phi}. \end{cases}$$

Exemplum 3.

§. 15. *Sit nunc* $m = 2$ *et* $n = 2$, *ideoque formula integranda* $\int \frac{z\partial z}{1-zz} = P + Q\sqrt{-1}$, *posito scilicet* $z = v(\cos.\Phi + \sqrt{-1}\sin.\Phi)$.

Hic ergo primo sumi debet $i = 0$, tum vero $i = 2$, vnde

Si $i = 0$, erit $\omega = -\Phi$ et $\zeta = 0$

Si $i = 2$, erit $\omega = \pi - \Phi$ et $\zeta = 2\pi$

vnde valores pro P et Q eruuntur sequentes

P =

$$P = \begin{cases} -\tfrac{1}{2}\, l \sqrt{(1 - 2v \cos.\Phi + vv)} - 0 \\ -\tfrac{1}{2}\, l \sqrt{(1 + 2v \cos.\Phi + vv)} - 0. \end{cases}$$

$$Q = \begin{cases} 0 + \tfrac{1}{2}\, A\ \text{tang.}\ \frac{v \sin.\Phi}{1 - v \cos.\Phi} \\ 0 - \tfrac{1}{2}\, A\ \text{tang.}\ \frac{v \sin.\Phi}{1 - v \cos.\Phi}. \end{cases}$$

Exemplum 4.

§. 16. *Sit* $m = 1$ *et* $n = 3$, *ideoque formula integranda* $\int \frac{\partial z}{1 - z^3} = P + Q\sqrt{-1}$, *posito* $z = v(\cos.\Phi + \sqrt{-1}\sin.\Phi)$.

Hic igitur ternos valores pro angulo ω habebimus, quos sequenti modo repraesentemus:

i	0	2	4
ω	$-\Phi$	$120^\circ - \Phi$	$240^\circ - \Phi$
$\sin.\omega$	$-\sin.\Phi$	$+\sin.(60^\circ + \Phi)$	$-\sin.(60^\circ - \Phi)$
$\cos.\omega$	$+\cos.\Phi$	$-\cos.(60^\circ + \Phi)$	$+\cos.(60^\circ - \Phi)$
ζ	0	120°	240
$\sin.\zeta$	0	$+\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$
$\cos.\zeta$	$+1$	$-\frac{1}{2}$	$-\frac{1}{2}$

Hinc ergo inueniemus

$$P = \begin{cases} -\tfrac{1}{3} l \sqrt{(1 - 2v\cos.\Phi + vv)} + 0 \\ +\tfrac{1}{6} l \sqrt{[1 + 2v\cos.(60^\circ + \Phi) + vv]} + \frac{1}{2\sqrt{3}}\, A\ \text{tang.}\ \frac{v \sin.(60^\circ + \Phi)}{1 + v\cos.(60^\circ + \Phi)} \\ +\tfrac{1}{6} l \sqrt{[1 + 2v\cos.(60^\circ - \Phi) + vv]} + \frac{1}{2\sqrt{3}}\, A\ \text{tang.}\ \frac{v \sin.(60^\circ - \Phi)}{1 + v\cos.(60^\circ - \Phi)}. \end{cases}$$

$$Q = \begin{cases} 0 \qquad + \tfrac{1}{3}\, A\ \text{tang.}\ \frac{v \sin.\Phi}{1 - v\cos.\Phi} \\ -\frac{1}{2\sqrt{3}}\, l \sqrt{[1 + 2v\cos.(60^\circ + \Phi) + vv]} + \tfrac{1}{6}\, A\ \text{tang.}\ \frac{v \sin.(60^\circ + \Phi)}{1 + v\cos.(60^\circ + \Phi)} \\ +\frac{1}{2\sqrt{3}}\, l \sqrt{[1 + 2v\cos.(60^\circ - \Phi) + vv]} - \tfrac{1}{6}\, A\ \text{tang.}\ \frac{v \sin.(60^\circ - \Phi)}{1 + v\cos.(60^\circ - \Phi)}. \end{cases}$$

Exemplum 5.

§. 17. *Sumatur nunc $m = 2$, manente $n = 3$, vt formula integranda fit* $\int \frac{z\,\partial z}{1 - z^3} = P + Q\sqrt{-1}$, *posito* $z = v(\text{cos.}\Phi + \sqrt{-1}\,\text{sin.}\Phi)$.

Hic notetur, valores ipsius ω prorsus eosdem manere vt ante, sicque etiam logarithmi et arcus circulares iidem manebunt; valores autem pro ζ erunt sequentes:

Si $i = 0$, erit $\zeta = 0$, sin. $\zeta = 0$ et cos. $\zeta = +1$.
Si $i = 2$, erit $\zeta = \frac{4}{3}\pi$, sin. $\zeta = -\frac{\sqrt{3}}{2}$ et cos. $\zeta = -\frac{1}{2}$.
Si $i = 4$, erit $\zeta = \frac{8}{3}\pi$, sin. $\zeta = +\frac{\sqrt{3}}{2}$ et cos. $\zeta = -\frac{1}{2}$.

Hinc igitur fiet

$$P = \begin{cases} -\frac{1}{3} l\sqrt{(1 - 2v\,\text{cof.}\Phi + vv)} + 0 \\ +\frac{1}{6} l\sqrt{[1 - 2v\,\text{cof.}(60^\circ + \Phi) + vv]} - \frac{1}{2\sqrt{3}}\,\text{A tang.}\,\frac{v\,\text{fin.}(60^\circ + \Phi)}{1 + v\,\text{cof.}(60^\circ + \Phi)} \\ +\frac{1}{6} l\sqrt{[1 + 2v\,\text{cof.}(60^\circ - \Phi) + vv]} - \frac{1}{2\sqrt{3}}\,\text{A tang.}\,\frac{v\,\text{fin.}(60^\circ - \Phi)}{1 + v\,\text{cof.}(60^\circ - \Phi)}. \end{cases}$$

$$Q = \begin{cases} 0 + \frac{1}{3}\,\text{A tang.}\,\frac{v\,\text{fin.}\Phi}{1 - v\,\text{cof.}\Phi} \\ + \frac{1}{2\sqrt{3}} l\sqrt{[1 + 2v\,\text{cof.}(60^\circ + \Phi) + vv]} + \frac{1}{6}\,\text{A tang.}\,\frac{v\,\text{fin.}(60^\circ + \Phi)}{1 + v\,\text{cof.}(60^\circ + \Phi)} \\ - \frac{1}{2\sqrt{3}} l\sqrt{[1 + 2v\,\text{cof.}(60^\circ - \Phi) + vv]} + \frac{1}{6}\,\text{A tang.}\,\frac{v\,\text{fin.}(60^\circ - \Phi)}{1 + v\,\text{cof.}(60^\circ - \Phi)}. \end{cases}$$

Exemplum 6.

§. 18. *Sit nunc $m = 1$ et $n = 4$, vt formula integranda fiat* $\int \frac{\partial z}{1 - z^4} = P + Q\sqrt{-1}$, *posito* $z = v\,(\text{cos.}\,\Phi + \sqrt{-1}\,\text{sin.}\,\Phi)$.

Quia hic $n = 4$, pro angulis ω et ζ quaternos valores adipiscimur, scilicet

i	o	2	4	6
ω	$-\Phi$	$\frac{1}{2}\pi-\Phi$	$\pi-\Phi$	$\frac{3}{2}\pi-\Phi$
fin. ω	$-$ fin. Φ	$+$ cof. Φ	$+$ fin. Φ	$-$ cof. Φ
cof. ω	$+$ cof. Φ	$+$ fin. Φ	$-$ cof. Φ	$-$ fin. Φ
ζ	o	90°	180°	270°
fin. ζ	o	$+1$	o	-1
cof. ζ	$+1$	o	-1	o

Hinc jam litterae P et Q fequenti modo exprimentur:

$$P=\begin{cases}-\frac{1}{4}l\sqrt{(1-2v\,\text{cof.}\,\Phi+vv)}+0\\ 0 \qquad\qquad\qquad\qquad +\frac{1}{4}\,A\,\text{tang.}\,\frac{v\,\text{cof.}\,\Phi}{1-v\,\text{fin.}\,\Phi}\\ +\frac{1}{4}l\sqrt{(1+2v\,\text{cof.}\,\Phi+vv)}+0\\ 0 \qquad\qquad\qquad\qquad +\frac{1}{4}\,A\,\text{tang.}\,\frac{v\,\text{cof.}\,\Phi}{1+v\,\text{fin.}\,\Phi}.\end{cases}$$

$$Q=\begin{cases}0 \qquad\qquad\qquad\qquad +\frac{1}{4}\,A\,\text{tang.}\,\frac{v\,\text{fin.}\,\Phi}{1-v\,\text{cof.}\,\Phi}\\ -\frac{1}{4}l\sqrt{(1-2v\,\text{fin.}\,\Phi+vv)}+0\\ 0 \qquad\qquad\qquad\qquad +\frac{1}{4}\,A\,\text{tang.}\,\frac{v\,\text{fin.}\,\Phi}{1+v\,\text{cof.}\,\Phi}\\ +\frac{1}{4}l\sqrt{(1+2v\,\text{fin.}\,\Phi+vv)}+0.\end{cases}$$

§. 19. Super hoc exemplo notaffe juuabit effe

$$\int\frac{\partial z}{1-z^4}=\tfrac{1}{2}\int\frac{\partial z}{1-zz}+\tfrac{1}{2}\int\frac{\partial z}{1+zz}.$$

Modo autem vidimus pro formula $\int\frac{\partial z}{1-zz}$ effe

$$P=-\tfrac{1}{2}l\sqrt{(1-2v\,\text{cof.}\,\Phi+vv)}+\tfrac{1}{2}l\sqrt{(1+2v\,\text{cof.}\,\Phi+vv)} \text{ et}$$
$$Q=+\tfrac{1}{2}\,A\,\text{tang.}\,\frac{v\,\text{fin.}\,\Phi}{1-v\,\text{cof.}\,\Phi}\qquad+\tfrac{1}{2}\,A\,\text{tang.}\,\frac{v\,\text{fin.}\,\Phi}{1+v\,\text{cof.}\,\Phi}.$$

Pro altera vero formula $\int\frac{\partial z}{1+zz}$ in fuperiore differtatione §. 30. et feqq. inuenimus

$$P=\tfrac{1}{2}\,A\,\text{tang.}\,\frac{2v\,\text{cof.}\,\Phi}{1-vv} \text{ et } Q=\tfrac{1}{4}\,l\,\frac{1+2v\,\text{fin.}\,\Phi+vv}{1-2v\,\text{fin.}\,\Phi+vv}$$

quos autem valores ob arcum circuli hic contractum potius ex formulis problematis generalis §. 54. et feqq. deriuemus. Erit

Erit enim, posito ibi $m = 1$, $n = 2$, pro forma integrali $\int \frac{\partial z}{1 + zz}$ valor

$$P = \tfrac{1}{2}\,\text{A tang.}\, \frac{v\, cos.\,\Phi}{1 - v\, sin.\,\Phi} + \tfrac{1}{2}\,\text{A tang.}\, \frac{v\, cos.\,\Phi}{1 + v\, sin.\,\Phi}$$

$$Q = -\tfrac{1}{2} l \sqrt{(1 + 2v\, \text{sin.}\,\Phi + vv)} - \tfrac{1}{2} l \sqrt{(1 - 2v\, \text{sin.}\,\Phi + vv)}.$$

Additis ergo binis P et Q per binarium diuisis prodit pro forma integrali $\int \frac{\partial z}{1 - z^4}$ valor

$$P = \begin{cases} +\tfrac{1}{4} l \sqrt{(1 + 2v\, \text{cos.}\,\Phi + vv)} + \tfrac{1}{4}\,\text{A tang.}\, \frac{v\, cos.\,\Phi}{1 - v\, sin.\,\Phi} \\ -\tfrac{1}{4} l \sqrt{(1 - 2v\, \text{cos.}\,\Phi + vv)} + \tfrac{1}{4}\,\text{A tang.}\, \frac{v\, cos.\,\Phi}{1 + v\, sin.\,\Phi} \end{cases}.$$

$$Q = \begin{cases} +\tfrac{1}{4} l \sqrt{(1 + 2v\, \text{sin.}\,\Phi + vv)} + \tfrac{1}{4}\,\text{A tang.}\, \frac{v\, sin.\,\Phi}{1 - v\, cos.\,\Phi} \\ -\tfrac{1}{4} l \sqrt{(1 - 2v\, \text{sin.}\,\Phi + vv)} + \tfrac{1}{4}\,\text{A tang.}\, \frac{v\, sin.\,\Phi}{1 + v\, cos.\,\Phi} \end{cases}$$

prorsus vti supra inuenimus.

§. 20. Quanquam haec solutio satis est commoda et sine multis ambagibus ad optatum finem perducit, tamen aliam hic subjungam, quae quidem multo simplicior et breuior, ita tamen est comparata, vt ejus bonitas nequidem perspici queat, atque eatenus tantum admitti possit, quatenus ad veritatem jam aliunde cognitam perducit. In eo autem ista solutio a praecedente solutione recedit, quod primo denominatorem $1 - z^n$ ab imaginariis liberare non est opus; deinde etiam numerator ita tractari potest, vt quantitas v inde penitus elidatur, neque permixtio quantitatum realium et imaginariarum vllam moram facessat.

Alia solutio Problematis.

§. 21. Cum posito $z = v\,(\text{cos.}\,\Phi + \sqrt{-1}\,\text{sin.}\,\Phi)$ esse debeat

$$\int \frac{z^{m-1}\,\partial z}{1 - z^n} = P + Q\sqrt{-1},$$

statim

ſtatim conſidero denominatoris factorem $1 - 2v \cos.\omega + vv$; quo ergo poſito $= 0$ etiam ipſe denominator euaneſcere debet; inde autem fit $v = \cos.\omega + \sqrt{-1}\sin.\omega$, et cum ſit

$$z = v(\cos.\Phi + \sqrt{-1}\sin.\Phi), \text{ erit}$$

$$z^n = v^n(\cos.n\Phi + \sqrt{-1}\sin.n\Phi).$$

Quare cum ſit $v^n = \cos.n\omega + \sqrt{-1}\sin.n\omega$, hinc fiet

$$z^n = \cos.(n\omega + n\Phi) + \sqrt{-1}\sin.(n\omega + n\Phi),$$

quae expreſſio cum vnitati debeat eſſe aequalis, erit $\cos.(n\omega + n\Phi) = 1$, vnde fit $n\omega + n\Phi = i\pi$, denotante i numerum parem quemqunque, ſicque altera pars $\sqrt{-1}\sin.(n\omega + n\Phi)$ ſponte euaneſcit. Cum igitur hinc ſit $n\omega = i\pi - n\Phi$, erit $\omega = \frac{i\pi}{n} - \Phi$, vnde n diuerſi valores pro ω eliciuntur.

§. 22. Statuamus nunc fractionem partialem ex hoc factore oriundam eſſe $= \frac{F}{1 - 2v\cos.\omega + vv}$, atque vt ſupra vidimus, ſtatui debet

$$F = z^{m-1}\partial z \cdot \frac{1 - 2v\cos.\omega + vv}{1 - z^n},$$

vnde ope aquationis $vv - 2v\cos.\omega + 1 = 0$ iſte valor F penitus a litteris z et v debet liberari. Quoniam autem hinc fractionis illius tam numerator quam denominator euaneſcit, ſumtis differentialibus, ob $\partial.z^n = nz^n\frac{\partial z}{z} = nz^n\frac{\partial v}{v}$, quandoquidem in hac reductione anguli ω et Φ vt conſtantes ſpectari poſſunt, illa fractio induet hanc formam: $\frac{2(v - \cos.\omega)v}{nz^n}$. Quoniam igitur $v - \cos.\omega = \sqrt{-1}\sin.\omega$ et $z^n = 1$, erit iſta fractio $= -\frac{2v\sqrt{-1}\sin.\omega}{n}$, ſicque habebimus

$$F = -\frac{2v}{n}z^{m-1}\partial z\sqrt{-1}\sin.\omega.$$

§. 23. Cum nunc, sumto etiam angulo ϕ variabili, sit $\frac{\partial z}{z} = \frac{\partial v}{v} + \partial \phi \sqrt{-1}$, ideoque

$$\frac{2 v \sqrt{-1}}{n} \cdot \frac{\partial z}{z} = \frac{2}{n} \partial v \sqrt{-1} - 2 v \frac{\partial \phi}{n},$$

habebimus

$$F = -\frac{2}{n} z^m \partial v \sqrt{-1} \text{ fin. } \omega + \frac{2}{n} v z^m \partial \phi \text{ fin. } \omega, \text{ siue}$$

$$F = \frac{2}{n} z^m \text{ fin. } \omega (v \partial \phi - \partial v \sqrt{-1}).$$

Nunc vero, vti ante euoluimus potestatem z^n, hic simili modo euoluamus potestatem z^m, eritque

$$z^m = \text{cof. } (m \omega + m \phi) + \sqrt{-1} \text{ fin. } (m \omega + m \phi),$$

quo valore introducto fiet

$$F = \frac{2}{n} \text{ fin. } \omega (v \partial \phi - \partial v \sqrt{-1}) [\text{cof. } (m\omega + m\phi) + \sqrt{-1} \text{ fin. } (m\omega + m\phi)].$$

Cum denique sit $\omega = \frac{i\pi}{n} - \phi$, erit $m\omega + m\phi = \frac{m i \pi}{n}$, quem ergo angulum si vocemus $= \zeta$, valor litterae F quaesitus erit

$$F = \frac{2}{n} \text{ fin. } \omega (v \partial \phi - \partial v \sqrt{-1}) (\text{cof. } \zeta + \sqrt{-1} \text{ fin. } \zeta),$$

quem partiamur in has partes:

$$F = + \frac{2}{n} \partial v \text{ fin. } \omega (\text{fin. } \zeta - \sqrt{-1} \text{ cof. } \zeta)$$
$$+ \frac{2}{n} v \partial \phi \text{ fin. } \omega (\text{cof. } \zeta + \sqrt{-1} \text{ fin. } \zeta).$$

§. 24. Quia haec expressio ex partibus realibus et imaginariis constat, videri posset partes reales sumi debere pro valore litterae P, imaginarias pro $Q \sqrt{-1}$; verum hinc in crassissimum errorem illaberemur, quemadmodum ex collatione cum superiore solutione manifestum est. Interim tamen obseruaui, ex hac ipsa formula veros valores pro P et Q elici posse. Scilicet pro valore ipsius P inueniendo haec tota formula ex realibus et imaginariis permixta in valorem realem transformetur; tum enim eius semissis pro littera P valebit. Simili modo pro littera Q eandem expressionem totam in formam simplicipliciter imaginariam transfundi oportet, cuius pariter femiffis pro valore litterae Q adhiberi debebit; fcilicet cum valor ipfius F coëfficientem habeat 2, ex altera femiffi littera P, ex altera vero littera Q formari debet.

§. 25. Hinc ergo omiffo factore formulam pro F inventam primo ad litteram P accomodemus, qui valor cum debeat effe realis, ftatuatur $= A v + B$, et loco v valorem cof. $\omega + \sqrt{-1}$ fin. ω fubftituendo habebimus hanc aequationem:

$$\left.\begin{array}{l} + \frac{1}{n} \partial v \,\text{fin.}\, \omega (\text{fin.}\, \zeta - \sqrt{-1} \,\text{cof.}\, \zeta) \\ + \frac{1}{n} v \partial \Phi \,\text{fin.}\, \omega (\text{cof.}\, \zeta + \sqrt{-1} \,\text{fin.}\, \zeta) \end{array}\right\} = A \,\text{cof.}\, \omega + B + A \sqrt{-1} \,\text{fin.}\, \omega.$$

Hinc iam partibus realibus et imaginariis feorfim aequatis primo ex imaginariis elicitur:

$$A \,\text{fin.}\, \omega = \frac{1}{n} \,\text{fin.}\, \omega (- \partial v \,\text{cof.}\, \zeta + v \partial \Phi \,\text{fin.}\, \zeta),$$

vnde fit

$$A = - \frac{1}{n} (\partial v \,\text{cof.}\, \zeta - v \partial \Phi \,\text{fin.}\, \zeta).$$

Hic iam valor in aequalitate partium realium fubftitutus dabit

$$\frac{1}{n} \text{fin.}\, \omega (\partial v \,\text{fin.}\, \zeta + v \partial \Phi \,\text{cof.}\, \zeta) = - \frac{\text{cof.}\, \omega}{n} (\partial v \,\text{cof.}\, \zeta - v \partial \Phi \,\text{fin.}\, \zeta) + B$$

vnde colligitur

$$B = \frac{1}{n} \partial v \,\text{cof.}\, (\zeta - \omega) - \frac{1}{n} v \partial \Phi \,\text{fin.}\, (\zeta - \omega).$$

Hinc ergo pro littera P erit

$$F = - \frac{1}{n} \partial v [v \,\text{cof.}\, \zeta - \text{cof.}\, (\zeta - \omega)] + \frac{1}{n} v \partial \Phi [v \,\text{fin.}\, \zeta - \text{fin.}\, (\zeta - \omega)],$$

ficque ex factore denominatoris $1 - 2 v \,\text{cof.}\, \omega + v v$ habebimus

$$P = - \frac{1}{n} \int \frac{\partial v [v \,\text{cof.}\, \zeta - \text{cof.}\, (\zeta - \omega)] - v \partial \Phi [v \,\text{fin.}\, \zeta - \text{fin.}\, (\zeta - \omega)]}{1 - 2 v \,\text{cof.}\, \omega + v v}.$$

§. 26. Pro littera Q altera femiffis litterae F aequetur huic quantitati fimpliciter imaginariae: $(C v + D) \sqrt{-1}$, vnde exorietur ifta aequatio:

$$\left.\begin{array}{l} +\frac{1}{n}\partial v\sin.\omega(\sin.\zeta-\sqrt{-1}\cos.\zeta) \\ +\frac{1}{n}v\partial\Phi\sin.\omega(\cos.\zeta+\sqrt{-1}\sin.\zeta) \end{array}\right\} = C\cos.\omega\sqrt{-1}+D\sqrt{-1}-C\sin.\omega.$$

Hinc ex partibus realibus concluditur

$$C = -\frac{1}{n}(\partial v\sin.\zeta + v\partial\Phi\cos.\zeta),$$

quo valore substituto ex partibus imaginariis haec emerget aequatio:

$$-\frac{1}{n}\sin.\omega(\partial v\cos.\zeta - v\partial\Phi\sin.\zeta) = -\frac{\cos.\omega}{n}(\partial v\sin.\zeta + v\partial\Phi\cos.\zeta) + D,$$

vnde eruitur

$$D = \frac{1}{n}\partial v\sin.(\zeta-\omega) + \frac{1}{n}v\partial\Phi\cos.(\zeta-\omega).$$

Hinc ergo pro littera Q habemus:

$$F = -\frac{1}{n}\partial v[v\sin.\zeta - \sin.(\zeta-\omega)]$$
$$-\frac{1}{n}v\partial\Phi[v\cos.\zeta - \cos.(\zeta-\omega)],$$

vnde valor ipsius Q ex factore $1-2v\cos.\omega+vv$ oriundus erit:

$$Q = -\frac{1}{n}\int\frac{\partial v[v\sin.\zeta - \sin.(\zeta-\omega)] + v\partial\Phi[v\cos.\zeta - \cos.(\zeta-\omega)]}{1-2v\cos.\omega+vv}.$$

§. 27. Quoniam haec solutio tam egregie cum praecedente conuenit, id profecto casui fortuito tribui nequit; quam ob rem mihi quidem haec solutio prorsus singularis haud parum in recessu habere videtur, vnde eam Geometris perscrutandam proponere non dubito, vt eius soliditatem ex firmis principiis deriuare conentur.

DE