



1793

Observationes generales circa series, quarum termini secundum sinus vel cosinus angulorum multiplorum progrediuntur

Leonhard Euler

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OBSERVATIONES GENERALES

CIRCA SERIES, QVARVM TERMINI SECVNDVM
SINVS VEL COSINVS ANGVLORVM MVLTIPLO-
RVM PROGREDIVNTVR.

Auctore

L. EVLERO.

Conuent. exhib. die 6 Mart. 1776.

§. 1.

Quodsi huius seriei $A + Bx + Cxx + Dx^3 + \text{etc.}$ summa-
tio fuerit cognita, ita vt, quicumque valor litterae x tribuatur,
eius summa assignari queat, tum etiam semper tam summa
huius seriei:

$$A + B \cos. \phi + C \cos. 2 \phi + D \cos. 3 \phi + \text{etc.}$$

quam huius:

$$B \sin. \phi + C \sin. 2 \phi + D \sin. 3 \phi + E \sin. 4 \phi + \text{etc.}$$

exhiberi poterit. Cum enim summa primae seriei exprimat
per certam quandam functionem ipsius x , quam hoc caracte-
re $\Delta : x$ designemus, ita vt fit

$$\Delta : x = A + Bx + Cxx + Dx^3 + \text{etc.}$$

si loco x scribamus tam

$$\cos. \phi + \sqrt{-1} \sin. \phi, \text{ quam } \cos. \phi - \sqrt{-1} \sin. \phi$$

sum-

summa ferierum inde resultantium erit

$2A + 2B \operatorname{cof.} \Phi + 2C \operatorname{cof.} 2\Phi + 2D \operatorname{cof.} 3\Phi + 2E \operatorname{cof.} 4\Phi + \text{etc.}$
 cuius ergo summa erit

$\Delta : (\operatorname{cof.} \Phi + \sqrt{-1} \operatorname{fin.} \Phi) + \Delta : (\operatorname{cof.} \Phi - \sqrt{-1} \operatorname{fin.} \Phi);$
 fin autem posteriorem a priore subtrahamus, prodibit ista series:
 $2B\sqrt{-1} \operatorname{fin.} \Phi + 2C\sqrt{-1} \operatorname{fin.} 2\Phi + 2D\sqrt{-1} \operatorname{fin.} 3\Phi$
 $+ 2E\sqrt{-1} \operatorname{fin.} 4\Phi \text{ etc.}$

cuius ergo summa erit

$$\Delta : (\operatorname{cof.} \Phi + \sqrt{-1} \operatorname{fin.} \Phi) - \Delta : (\operatorname{cof.} \Phi - \sqrt{-1} \operatorname{fin.} \Phi).$$

§. 2. Quo expressiones has commodiores reddamus, statuamus breuitatis gratia

$\operatorname{cof.} \Phi + \sqrt{-1} \operatorname{fin.} \Phi = p; \operatorname{cof.} \Phi - \sqrt{-1} \operatorname{fin.} \Phi = q$
 eritque, uti in vulgus notum est, $p q = 1$ ideoque $q = \frac{1}{p}$; tum vero erit

$$\operatorname{cof.} \Phi = \frac{p+q}{2}; \operatorname{cof.} 2\Phi = \frac{p^2+q^2}{2}; \operatorname{cof.} 3\Phi = \frac{p^3+q^3}{2};$$

$$\operatorname{cof.} 4\Phi = \frac{p^4+q^4}{2}; \text{ etc.}$$

Praeterea vero pro finibus habebitur

$$\operatorname{fin.} \Phi = \frac{p-q}{2\sqrt{-1}}; \operatorname{fin.} 2\Phi = \frac{p^2-q^2}{2\sqrt{-1}}; \operatorname{fin.} 3\Phi = \frac{p^3-q^3}{2\sqrt{-1}}; \text{ etc.}$$

quibus constitutis nanciscimur has duas summationes:

$$A \operatorname{cof.} 0\Phi + B \operatorname{cof.} \Phi + C \operatorname{cof.} 2\Phi + D \operatorname{cof.} 3\Phi + E \operatorname{cof.} 4\Phi + \text{etc.}$$

$$= \frac{\Delta : p + \Delta : q}{2}; \text{ et}$$

$$A \operatorname{fin.} 0\Phi + B \operatorname{fin.} \Phi + C \operatorname{fin.} 2\Phi + D \operatorname{fin.} 3\Phi + \text{etc.} = \frac{\Delta : p - \Delta : q}{2\sqrt{-1}}$$

§. 3. Sumamus nunc pro serie principali potestatem quamcunque Binomii euolutam, quae est

$$(1+x)^n = 1 + \frac{n}{1} x + \frac{n(n-1)}{1 \cdot 2} x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^3 + \text{etc.}$$

ita

ita vt hoc casu fit $\Delta : x = (1 + x)^n$; tum vero, vt hanc expressionem contrahamus, designemus singulos coefficientes, vt iam aliquoties fecimus, his characteribus: $\binom{n}{0}$, $\binom{n}{1}$, $\binom{n}{2}$, $\binom{n}{3}$, $\binom{n}{4}$, ita vt fit

$$\begin{aligned} \binom{n}{0} &= 1, \\ \binom{n}{1} &= n, \\ \binom{n}{2} &= \frac{n}{1} \cdot \frac{n-1}{2}, \\ \binom{n}{3} &= \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}, \\ &\text{etc.} \end{aligned}$$

vbi obseruasse iuuabit esse in genere $\binom{n}{i} = \binom{n}{n-i}$, ideoque $\binom{n}{n} = \binom{n}{0} = 1$. Praeterea vero euidens est, quoties fuerit i vel numerus negatiuus, vel positiuus, maior quam n , tum semper esse $\binom{n}{i} = 0$, siquidem n fuerit numerus integer. His ergo obseruatis habebimus hanc summationem principalem:

$$(1 + x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \binom{n}{4}x^4 + \text{etc.}$$

vnde ergo per praecepta modo tradita deriuabimus binas sequentes summationes:

$$\begin{aligned} \binom{n}{0} \text{cos. } 0\Phi + \binom{n}{1} \text{cos. } \Phi + \binom{n}{2} \text{cos. } 2\Phi + \binom{n}{3} \text{cos. } 3\Phi + \text{etc.} \\ = \frac{(1+p)^n + (1+q)^n}{2} \text{ et} \end{aligned}$$

$$\begin{aligned} \binom{n}{0} \text{sin. } 0\Phi + \binom{n}{1} \text{sin. } \Phi + \binom{n}{2} \text{sin. } 2\Phi + \binom{n}{3} \text{sin. } 3\Phi + \text{etc.} \\ = \frac{(1+p)^n - (1+q)^n}{2\sqrt{-1}}. \end{aligned}$$

Quouis autem casu, quamquam formulae pro p et q assumptae sunt imaginariae, tamen semper istas formulas ad valores reales reuocare licebit, quemadmodum in sequentibus binis problematibus sumus ostensuri.

Problema I.

Proposita hac serie cosinum:

$$1 + \frac{n}{1} \text{cos. } \Phi + \frac{n}{1} \cdot \frac{n-1}{2} \text{cos. } 2\Phi + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \text{cos. } 3\Phi + \text{etc.} = s,$$

Noua Acta Acad. Imp. Sc. T. VII.

M

ita

ita ut per characteres stabilitos fit

$s = \binom{n}{0} \cos. 0 \Phi + \binom{n}{1} \cos. \Phi + \binom{n}{2} \cos. 2 \Phi + \binom{n}{3} \cos. 3 \Phi + \text{etc.}$
 eius summam realiter exprimere.

Solutio.

§. 4. Cum igitur fit, ut modo vidimus

$$s = \frac{(1+p)^n + (1+q)^n}{2},$$

existente $p = \cos. \Phi + \sqrt{-1} \sin. \Phi$ et $q = \cos. \Phi - \sqrt{-1} \sin. \Phi$, totum negotium huc redit, ut ista expressio pro s exhibita ab imaginariis liberetur; evidens enim est, si formulae $(1+p)^n$ et $(1+q)^n$ actu evoluantur, tum imaginaria se sponte esse destructura, quandoquidem hinc ipsa series summanda exoritur; quam ob rem in aliam resolutionem nobis erit inquirendum, ut sine evolutione adhibita imaginaria e medio tollantur, id quod sequenti modo fieri poterit.

§. 5. Cum fit $pq = 1$, formula $1+p$ ita exprimi poterit, ut sit $1+p = (\sqrt{p} + \sqrt{q}) \sqrt{p}$, similique modo erit $1+q = (\sqrt{p} + \sqrt{q}) \sqrt{q}$, hisque valoribus introductis prodibit nostra summa

$$s = \frac{1}{2} (\sqrt{p} + \sqrt{q})^n (p^{\frac{n}{2}} + q^{\frac{n}{2}}).$$

Cum iam in genere sit $p^\alpha + q^\alpha = 2 \cos. \alpha \Phi$, erit

$$p^{\frac{1}{2}} + q^{\frac{1}{2}} = 2 \cos. \frac{1}{2} \Phi \text{ et } p^{\frac{n}{2}} + q^{\frac{n}{2}} = 2 \cos. \frac{1}{2} n \Phi,$$

quibus valoribus substitutis summa quaesita iam realiter sequenti modo exprimetur: $s = 2^n \cos. \frac{1}{2} \Phi^n \cos. \frac{1}{2} n \Phi$.

§. 6. Hoc igitur pacto summationem maxime memorabilem sumus adepti, quae ita se habet, ut semper fit

==== (91) ====

$$1 + \frac{n}{1} \operatorname{cof.} \Phi + \frac{n}{1} \cdot \frac{n-1}{2} \operatorname{cof.} 2\Phi + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \operatorname{cof.} 3\Phi + \text{etc.}$$

$$= 2^n \operatorname{cof.} \frac{1}{2} \Phi^n \operatorname{cof.} \frac{1}{2} n \Phi,$$

quae semper veritati est consentanea, quicumque numeri pro n substituantur, siue integri siue fracti, siue etiam negatiui. Operae ergo pretium erit ex quouis genere casus simpliciores ob oculos exponere.

Euolutio casuum,

quibus exponens n est numerus integer positiuus.

§. 7. Consideremus casus sequentes :

1°. Sit $n=0$, et ipsa series in unitatem coalescit, summa autem erit $= 1$.

2°. Sit $n=1$, et series abibit in $1 + \operatorname{cof.} \Phi$; summa autem inuenta praebet $2 \operatorname{cof.} \frac{1}{2} \Phi^2$. Constat autem esse $2 \operatorname{cof.} \frac{1}{2} \Phi^2 = 1 + \operatorname{cof.} \Phi$.

3°. Sit $n=2$, et series abit in $1 + 2 \operatorname{cof.} \Phi + \operatorname{cof.} 2\Phi$; summa autem oritur $= 4 \operatorname{cof.} \frac{1}{2} \Phi^2 \operatorname{cof.} \Phi$. Modo autem vidimus esse $2 \operatorname{cof.} \frac{1}{2} \Phi^2 = 1 + \operatorname{cof.} \Phi$, quae forma per $2 \operatorname{cof.} \Phi$ multiplicata producit

$$2 \operatorname{cof.} \Phi \times 2 \operatorname{cof.} \frac{1}{2} \Phi^2 = 1 + 2 \operatorname{cof.} \Phi + \operatorname{cof.} 2\Phi.$$

4°. Sit nunc $n=3$, et series nascitur haec: $1 + 3 \operatorname{cof.} \Phi + 3 \operatorname{cof.} 2\Phi + \operatorname{cof.} 3\Phi$, cuius summa est $= 8 \operatorname{cof.} \frac{1}{2} \Phi^3 \operatorname{cof.} \frac{3}{2} \Phi$, quae formula per reductiones satis cognitae ipsam seriem producit.

5°. Sit nunc $n=4$, et series abit in

$$1 + 4 \operatorname{cof.} \Phi + 6 \operatorname{cof.} 2\Phi + 4 \operatorname{cof.} 3\Phi + \operatorname{cof.} 4\Phi$$

cuius ergo summa erit $2^4 \operatorname{cof.} \frac{1}{2} \Phi^4 \operatorname{cof.} 2\Phi$, cuius veritas etiam non difficulter ostenditur. Sicque semper veritatem per reductiones cognitae ostendere licebit.

Euolutio casuum,

quibus pro n numerus integer negatius accipitur.

§. 8. Statuamus 1°. $n = -1$, atque hinc nascetur sequens series infinita:

$1 - \text{cof. } \Phi + \text{cof. } 2\Phi - \text{cof. } 3\Phi + \text{cof. } 4\Phi - \text{cof. } 5\Phi \text{ etc.}$
 in infinitum, cuius ergo summa per nostram seriem generalem erit $\frac{\text{cof. } \frac{1}{2}\Phi}{2 \text{ cof. } \frac{1}{2}\Phi} = \frac{1}{2}$, quod quidem iam dudum a Geometris est

obseruatum. Quodsi enim haec series, cuius summa tantisper ponatur $= s$, ducatur in $2 \text{ cof. } \frac{1}{2}\Phi$, reperietur per reductiones notissimas

$$2s \text{ cof. } \frac{1}{2}\Phi = \left\{ \begin{array}{l} 2 \text{ cof. } \frac{1}{2}\Phi - \text{cof. } \frac{3}{2}\Phi + \text{cof. } \frac{5}{2}\Phi - \text{cof. } \frac{7}{2}\Phi + \text{cof. } \frac{9}{2}\Phi \\ -\text{cof. } \frac{1}{2}\Phi + \text{cof. } \frac{3}{2}\Phi - \text{cof. } \frac{5}{2}\Phi + \text{cof. } \frac{7}{2}\Phi - \text{cof. } \frac{9}{2}\Phi \end{array} \right\} \text{ etc.}$$

quod manifesto redit ad $2s \text{ cof. } \frac{1}{2}\Phi = \text{cof. } \frac{1}{2}\Phi$ ideoque $s = \frac{1}{2}$.

§. 9. Statuamus nunc $n = -2$, et series orietur sequens:

$$1 - 2 \text{ cof. } \Phi + 3 \text{ cof. } 2\Phi - 4 \text{ cof. } 3\Phi + 5 \text{ cof. } 4\Phi - 6 \text{ cof. } 5\Phi \text{ etc.}$$

cuius ergo summa erit $= \frac{\text{cof. } \Phi}{4 \text{ cof. } \frac{1}{2}\Phi^2}$, cuius veritas etiamnunc sequenti modo ostendi potest. Posita seriei summa $= s$, erit

$$2s \text{ cof. } \frac{1}{2}\Phi = \left\{ \begin{array}{l} 2 \text{ cof. } \frac{1}{2}\Phi - 2 \text{ cof. } \frac{3}{2}\Phi + 3 \text{ cof. } \frac{5}{2}\Phi - 4 \text{ cof. } \frac{7}{2}\Phi \\ -2 \text{ cof. } \frac{1}{2}\Phi + 3 \text{ cof. } \frac{3}{2}\Phi - 4 \text{ cof. } \frac{5}{2}\Phi + 5 \text{ cof. } \frac{7}{2}\Phi \end{array} \right\} \text{ etc.}$$

qui valor coalescit in sequentem seriem:

$$2s \text{ cof. } \frac{1}{2}\Phi = \text{cof. } \frac{3}{2}\Phi - \text{cof. } \frac{5}{2}\Phi + \text{cof. } \frac{7}{2}\Phi - \text{cof. } \frac{9}{2}\Phi \text{ etc.}$$

Multiplicetur denuo per $2 \text{ cof. } \frac{1}{2}\Phi$, ac prodibit

$$4s \text{ cof. } \frac{1}{2}\Phi^2 = \left\{ \begin{array}{l} \text{cof. } \Phi + \text{cof. } 2\Phi - \text{cof. } 3\Phi + \text{cof. } 4\Phi + \text{cof. } 5\Phi \\ -\text{cof. } 2\Phi + \text{cof. } 3\Phi - \text{cof. } 4\Phi - \text{cof. } 5\Phi \end{array} \right\} \text{ etc.} = \text{cof. } \Phi$$

ideo-

ideoque $s = \frac{\text{cof. } \Phi}{4 \text{ cof. } \frac{1}{2} \Phi^2}$, vti inuenimus, siue erit etiam

$$s = \frac{\text{cof. } \Phi}{2(1 + \text{cof. } \Phi)}$$

§. 10. Sit nunc $n = -3$, orieturque haec series infinita:

$$1 - 3 \text{ cof. } \Phi + 6 \text{ cof. } 2\Phi - 10 \text{ cof. } 3\Phi + 15 \text{ cof. } 4\Phi - 21 \text{ cof. } 5\Phi \text{ etc.}$$

cuius ergo summa erit $= \frac{\text{cof. } \frac{3}{2} \Phi}{8 \text{ cof. } \frac{1}{2} \Phi^3}$. Haec autem expressio

porro reducitur ad hanc

$$s = \frac{1}{2} - \frac{3}{8 \text{ cof. } \frac{1}{2} \Phi^2} = \frac{1}{2} - \frac{3}{4(1 + \text{cof. } \Phi)}$$

ita vt quoque fit $s = \frac{-1 + 2 \text{ cof. } \Phi}{4(1 + \text{cof. } \Phi)}$.

§. 11. Simili modo sequentes adipiscemur summationes:

$$1 - 4 \text{ cof. } \Phi + 10 \text{ cof. } 2\Phi - 20 \text{ cof. } 3\Phi \text{ etc.} = \frac{\text{cof. } 2\Phi}{16 \text{ cof. } \frac{1}{2} \Phi^4}$$

$$1 - 5 \text{ cof. } \Phi + 15 \text{ cof. } 2\Phi - 35 \text{ cof. } 3\Phi + 70 \text{ cof. } 4\Phi \text{ etc.} = \frac{\text{cof. } \frac{5}{2} \Phi}{32 \text{ cof. } \frac{1}{2} \Phi^5}$$

$$1 - 6 \text{ cof. } \Phi + 21 \text{ cof. } 2\Phi - 56 \text{ cof. } 3\Phi + 126 \text{ cof. } 4\Phi \text{ etc.} = \frac{\text{cof. } 3\Phi}{64 \text{ cof. } \frac{1}{2} \Phi^6}$$

$$1 - 7 \text{ cof. } \Phi + 28 \text{ cof. } 2\Phi - 84 \text{ cof. } 3\Phi + 210 \text{ cof. } 4\Phi \text{ etc.} = \frac{\text{cof. } \frac{7}{2} \Phi}{128 \text{ cof. } \frac{1}{2} \Phi^7}$$

Euolutio casus

quo $n = \frac{1}{2}$.

§. 12. Hinc ergo formabitur sequens series infinita:

$$1 + \frac{1}{2} \text{ cof. } \Phi - \frac{1 \cdot 1}{2 \cdot 4} \text{ cof. } 2\Phi + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} \text{ cof. } 3\Phi - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} \text{ cof. } 4\Phi \text{ etc.}$$

M 3

cuius

cuius ergo summa erit $\equiv \cos. \frac{1}{2} \Phi \sqrt{2} \cos. \frac{1}{2} \Phi$, cuius veritatem
 haud facile erit aliunde comprobare; certis autem casibus ma-
 nifesto in oculos incurrit. Veluti si fuerit $\Phi = 0$, habebitur

$$1 + \frac{1}{2} - \frac{1 \cdot 1}{2 \cdot 4} + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} \text{ etc. } \equiv \sqrt{2};$$

series enim manifesto oritur ex evolutione $(1 + 1)^{\frac{1}{2}} \equiv \sqrt{2}$.
 Faciamus nunc $\Phi = 180^\circ$, ut sit $\frac{1}{2} \Phi = 90^\circ$, et series erit

$$1 - \frac{1}{2} - \frac{1 \cdot 1}{2 \cdot 4} - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} \text{ etc. } \equiv 0$$

quod etiam inde evidens est, quia haec series nascitur ex for-
 ma $(1 - 1)^{\frac{1}{2}}$. Sit etiam $\Phi = 90^\circ$ et series inde nata erit

$$1 + \frac{1 \cdot 1}{2 \cdot 4} - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} + \frac{1 \cdot 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12} \text{ etc. } \equiv \cos. 22^\circ, 30' \sqrt[4]{2}.$$

Est vero

$$\cos. 22^\circ, 30' = \sqrt{\frac{1 + \cos. 45^\circ}{2}} = \sqrt{\frac{1 + \sqrt{2}}{2}}$$

unde summa concluditur $\equiv \sqrt{\frac{1 + \sqrt{2}}{2}}$, ficque habetur haec sum-
 matio maxime memoratu digna:

$$1 + \frac{1 \cdot 1}{2 \cdot 4} - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} + \frac{1 \cdot 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12} \text{ etc. } \equiv \sqrt{\frac{1 + \sqrt{2}}{2}}.$$

Sumamus etiam $\Phi = 60^\circ$, et orietur ista series:

$$1 + \frac{1}{2} \cdot \frac{1}{2} + \frac{1 \cdot 1}{2 \cdot 4} \cdot \frac{1}{2} - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} + \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{1}{2} + \frac{1 \cdot 1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} \cdot \frac{1}{2} + \frac{1 \cdot 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12} \cdot \frac{1}{2} \text{ etc.}$$

cuius seriei ergo summa erit $\cos. 15^\circ \sqrt[4]{3}$. Cum igitur sit

$$\cos. 15^\circ = \sqrt{\frac{1 + \cos. 30^\circ}{2}} = \sqrt{\frac{2 + \sqrt{3}}{4}},$$

seriei summa erit $\frac{1}{2} \sqrt[4]{(3 + 2\sqrt{3})}$.

Evolutio casus

quo $n = -\frac{1}{2}$.

§. 13. Hinc ergo sequens formabitur series infinita:

$$1 - \frac{1}{2} \cos. \Phi + \frac{1 \cdot 3}{2 \cdot 4} \cos. 2\Phi - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cos. 3\Phi + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cos. 4\Phi \text{ etc.}$$

cuius

cuius ergo summa erit $= \frac{\text{cof. } \frac{1}{4} \Phi}{\sqrt{2} \text{ cof. } \frac{1}{2} \Phi}$. Hinc si fuerit $\Phi = 0$,
 oritur haec summatio :

$$1 - \frac{1}{2} + \frac{1 \cdot 3}{2 \cdot 4} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \text{ etc. } = \frac{1}{\sqrt{2}}.$$

Nascitur enim haec series ex forma $(1 + x)^{-\frac{1}{2}}$. Sit nunc
 $\Phi = 180^\circ$, et series resultans erit

$$1 + \frac{1}{2} + \frac{1 \cdot 3}{2 \cdot 4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} + \text{etc.} = \infty.$$

Oritur enim haec series ex evolutione $(1 - x)^{-\frac{1}{2}}$. Sumamus
 etiam $\Phi = 90^\circ$ et series erit

$$1 - \frac{1 \cdot 3}{2 \cdot 4} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} - \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12} \text{ etc. } = \frac{\text{cof. } 22^\circ, 30'}{\sqrt[4]{2}}.$$

Ante autem vidimus esse $\text{cof. } 22^\circ, 30' = \sqrt{\frac{1 + \sqrt{2}}{2\sqrt{2}}}$, vnde ista
 summa erit $= \frac{1}{2} \sqrt{1 + \sqrt{2}}$.

§. 14. In genere etiam pro exponentibus quibuscun-
 que n operae pretium erit angulo Φ certos tribuere valores,
 ac primo quidem sumpto $\Phi = 0$ habebimus

$$1 + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \binom{n}{4} + \text{etc.} = 2^n;$$

haec scilicet series est ipsa formula $(1 + x)^n$ euoluta. Suma-
 mus nunc $\Phi = 180^\circ$ et orietur haec series:

$$1 - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \binom{n}{4} - \text{etc.} = 0,$$

scilicet haec series est $= (1 - x)^n$. Sit etiam $\Phi = 90^\circ$ et se-
 ries hinc nata erit

$$1 - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \binom{n}{8} - \binom{n}{10} \text{ etc.}$$

cuius ergo summa erit

$$2^n \text{ cof. } 45^n \text{ cof. } n 45^\circ = 2^{\frac{1}{2}n} \text{ cof. } n 45^\circ.$$

§. 15. Haec postrema series eo maiori attentione digna videtur, quod eius veritas non parum est abscondita; vnde haud abs re erit casus aliquot speciales contemplari, ac primo quidem pro numeris integris positivis:

- 1°. Si $n=0$ erit $1=1$,
 - 2°. Si $n=1$ erit $1=\cos.45^\circ\sqrt{2}=1$,
 - 3°. Si $n=2$ erit $1-1=2\cos.90^\circ=0$,
 - 4°. Si $n=3$ erit $1-3=2^{\frac{3}{2}}\cos.3.45^\circ=-2$,
 - 5°. Si $n=4$ erit $1-6+1=2^2\cos.4.45^\circ=-4$,
 - 6°. Si $n=5$ erit $1-10+5=2^{\frac{5}{2}}\cos.5.45^\circ=-4$,
 - 7°. Si $n=6$ erit $1-15+15-1=2^3\cos.6.45^\circ=0$,
 - 8°. Si $n=7$ erit $1-21+35-7=2^{\frac{7}{2}}\cos.7.45^\circ=2^3$,
 - 9°. Si $n=8$ erit $1-28+70-28+1=2^4\cos.8.45^\circ=2^4$,
- etc. etc.

§. 16. Maiorem attentionem merentur casus, quibus pro n numeri negativi accipiuntur, quippe quibus series infinitae proueniunt.

- 1°. Si $n=-1$ erit $1-1+1-1+1-1+1-1+1-1$ etc.

$$= \frac{\cos.45^\circ}{\sqrt{2}} = \frac{1}{2},$$
- 2°. Si $n=-2$ erit $1-3+5-7+9-11+13-15+17$ etc.

$$= 2\cos.2.45^\circ=0,$$
- 3°. Si $n=-3$ erit $1-6+15-28+45-66+91$ etc.

$$= \frac{\cos.3.45^\circ}{\sqrt{8}} = -\frac{1}{4},$$
- 4°. Si $n=-4$ erit $1-10+35-84+165-286+455$ etc.

$$= \frac{\cos.4.45^\circ}{4} = -\frac{1}{2},$$

5°.

5°. Si $n = -5$ erit $1 - 15 + 70 - 210 + 495 - 1001$ etc.

$$= \frac{\text{cof. } 5. 45^\circ}{2^{\frac{5}{2}}} = -\frac{1}{8}$$

6°. Si $n = -6$ erit $1 - 21 + 126 - 462 + 1287 - 3003$ etc.

$$= \frac{\text{cof. } 6. 45^\circ}{8} = 0,$$

etc.

etc.

In genere autem pro his casibus erit-

$$1 - \frac{\lambda(\lambda+1)}{1 \cdot 2} + \frac{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{\lambda(\lambda+1) \cdot \dots \cdot (\lambda+5)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \frac{\lambda(\lambda+1) \cdot \dots \cdot (\lambda+7)}{1 \cdot 2 \cdot \dots \cdot 8} - \frac{\lambda(\lambda+1) \cdot \dots \cdot (\lambda+9)}{1 \cdot 2 \cdot \dots \cdot 10} \text{ etc.}$$

cuius ergo seriei summa erit

$$= \frac{\text{cof. } \lambda. 45^\circ}{2^{\frac{1}{2} \lambda}}$$

Problema 2.

Proposita hac serie finuum:

$$\frac{n}{1} \text{ fin. } \Phi + \frac{n}{1} \cdot \frac{n-1}{2} \text{ fin. } 2 \Phi + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \text{ fin. } 3 \Phi + \text{etc.} = s,$$

ita ut per characteres supra adhibitos fit

$$s = \binom{n}{0} \text{ fin. } 0 \Phi + \binom{n}{1} \text{ fin. } \Phi + \binom{n}{2} \text{ fin. } 2 \Phi + \binom{n}{3} \text{ fin. } 3 \Phi + \text{etc.}$$

valorem huius summae s realiter exprimere.

Solutio.

§. 17. Quodsi hic iterum introducamus litteras

$$p = \text{cof. } \Phi + \sqrt{-1} \text{ fin. } \Phi \text{ et}$$

$$q = \text{cof. } \Phi - \sqrt{-1} \text{ fin. } \Phi,$$

quoniam est $p^n - q^n = 2 \sqrt{-1} \text{ fin. } n \Phi$, series proposita in duas sequentes discerpetur

$$2s\sqrt{-1} = \left\{ \begin{array}{l} + \binom{n}{1} p + \binom{n}{2} p p + \binom{n}{3} p^3 + \binom{n}{4} p^4 \\ - \binom{n}{1} q - \binom{n}{2} q q - \binom{n}{3} q^3 - \binom{n}{4} q^4 \end{array} \right\} \text{ etc.}$$

Vnde manifesto erit

$$2s\sqrt{-1} = (1+p)^n - (1+q)^n.$$

§. 18. Hic iam iterum obseruasse iuuabit esse

$$1+p = (\sqrt{p} + \sqrt{q})\sqrt{p} \text{ et}$$

$$1+q = (\sqrt{p} + \sqrt{q})\sqrt{q}$$

quibus valoribus adhibitis erit

$$2s\sqrt{-1} = (\sqrt{p} + \sqrt{q})^n (p^{\frac{1}{2}n} - q^{\frac{1}{2}n}).$$

Quoniam igitur est

$$p^{\frac{1}{2}n} - q^{\frac{1}{2}n} = 2\sqrt{-1} \sin. \frac{1}{2}n\Phi \text{ et}$$

$$\sqrt{p} + \sqrt{q} = 2 \cos. \frac{1}{2}\Phi,$$

hinc per $2\sqrt{-1}$ diuidendo prodibit summa quaesita realiter expressa $s = 2^n \cos. \frac{1}{2}\Phi^n \sin. \frac{1}{2}n\Phi$.