1793

Methodus facilis investigandi radium osculi ex principio maximorum et minimorum petita

Leonhard Euler

METHODVS FACILIS INUESTIGANDI RADIVM OSCVLI EX PRINCIPIO MAXIMORVM ET MINIMORVM PETITA.

Auctore

L. EVLERO.

Conuent. exhib. die 11 *Sept.* 1776.

Problema.

Proposita curua quacunque eius radium osculi inuenire.

Solutio.

§. 1. Sit A Y curua proposita, aequatioue quacunque inter binas coordinatas $AX = x$ et $XY = y$ expressa, ita vt y spectari possit tanquam certa functio ipsius x, vnde fiat $\partial y = p\,\partial x$; et quia p denuo certam functionem ipsius x designat, sit porro $\partial p = q\,\partial x$, quibus positis inuestigari proponitur radius osculi huius curuae in puncto Y, siue quaeri debet punctum O, ex quo tanquam centro si describatur circulus per Y transiens, hic circulus non solum curuam in Y tangat, sed etiam communem habiturus sit curuaturam, quo casu is dicitur curuam osculari, eiusque radius sub nomine radii osculi designari solet. Tab. I. fig. 3.

§. 2. Quodsi ad hanc curuam in Y ducatur normalis YN, eius quodlibet punctum O hac gaudet proprietate, vt

eius

eius diſtantia O Y inuariata maneat, etiamſi punctum Y per interuallum infinite paruum promoueatur. Verum ſi punctum O fuerit centrum circuli oſculantis, quantitas interualli O Y non ſolum non variabitur, dum per differentialia prima procedimus, ſed etiam nullam variationem patietur, etiamſi per differentialia ſecunda procedamus; quamobrem ex hoc ipſo principio licebit iſtud centrum circuli oſculantis O determinare.

§. 3. Hunc in finem ex puncto hoc quaeſito O ad axem demittatur perpendiculum O P ac vocentur interualla $AP = f$ et $PO = g$, eritque $XP = f - x$; et ducta axi parallela O Q fiet interuallum $QY = y - g$, atque hinc colligitur $OY^2 = (f - x)^2 + (y - g)^2$, cuius ergo ante omnia differentiale primum debet annihilari, vnde ob $\partial y = p\,\partial x$ fiet

$$-2\,\partial x\,(f - x) + 2\,p\,\partial x\,(y - g) = 0, \text{ ſiue}$$

$$-f + x + p\,(y - g) = 0;$$

deinde vero etiam huius expreſſionis differentiale denuo ad nihilum reuocari debebit, vnde ob $\partial p = q\,\partial x$ orietur iſta aequatio:

$$\partial x + y\,\partial p + p\,\partial y - g\,\partial p = 0, \text{ ſiue}$$

$$1 + q\,(y - g) + pp = 0,$$

ex qua colligimus

$$g = y + \frac{1 + pp}{q}.$$

§. 4. At vero ex priore aequatione colligitur $f = x + p\,(y - g)$, vbi ſi loco g valor modo inuentus ſubſtituatur, prodibit $f = x - p\,\frac{(1 + pp)}{q}$; ſicque per ſola elementa ad curuam pertinentia, ſcilicet x, y, p et q, centrum circuli oſculantes O ita determinatur, vt ſit

$$AP = x - \frac{p\,(1 + pp)}{q} \text{ et } PO = y + \frac{1 + pp}{q}$$

quod ergo punctum nullam plane ambiguitatem inuoluit.

§. 5.

§. 5. Inuento autem puncto O longitudo radii osculi nulla amplius laborat difficultate. Cum enim sit
$OQ = -\frac{p(1+pp)}{q}$ et $QY = -\frac{(1+pp)}{q}$ erit $OY^2 = \frac{(1+pp)^3}{qq}$
quod cum sit quadratum radii osculi, erit ipse radius osculi $= \pm \frac{(1+pp)^{\frac{3}{2}}}{q}$, quae est expressio notissima radii osculi. Cum enim sit $q = \frac{\partial p}{\partial x}$, erit radius osculi $= \pm \partial x \frac{(1+pp)^{\frac{3}{2}}}{\partial p}$, vbi ambiguitas signi nihil turbat, quia locus puncti O iam ante est definitus.

§. 6. Hinc iam facile formulae vulgares pro radio osculi dari solitae deduci possunt. Ac primo quidem cum sit $\partial . \frac{p}{\sqrt{1+pp}} = \frac{\partial p}{(1+pp)^{\frac{3}{2}}}$, si ponamus $\frac{p}{\sqrt{1+pp}} = t$, erit $\frac{\partial p}{(1+pp)^{\frac{3}{2}}} = \partial t$, ex quo valore erit radius osculi $= \frac{\partial x}{\partial t}$.

§. 7. Deinde etiam radius osculi per sola differentialia tam primi quam secundi gradus exprimi solet. Cum enim sit $p = \frac{\partial y}{\partial x}$, erit $1+pp = \frac{\partial x^2 + \partial y^2}{\partial x^2}$, ideoque $(1+pp)^{\frac{3}{2}} = \frac{(\partial x^2 + \partial y^2)^{\frac{3}{2}}}{\partial x^3}$; tum vero nullo differentiali pro constanti sumpto erit $\partial p = \frac{\partial x \partial\partial y - \partial y \partial\partial x}{\partial x^2}$, quibus substitutis erit radius osculi $= \frac{(\partial x^2 + \partial y^2)^{\frac{3}{2}}}{\partial x \partial\partial y - \partial y \partial\partial x}$.

§. 8. Sin autem elementum ∂x pro constanti accipiatur, fiet radius osculi $= \frac{(\partial x^2 + \partial y^2)^{\frac{3}{2}}}{\partial x \partial\partial y}$; at si alterum elementum ∂y constans assumatur, fiet radius osculi $= -\frac{(\partial x^2 + \partial y^2)^{\frac{3}{2}}}{\partial y \partial\partial x}$.

§. 9. Quodsi porro elementum curuae in computum trahatur, idque vocetur $= \partial s$, vt sit $\partial s^2 = \partial x^2 + \partial y^2$, erit radius osculi $= \frac{\partial s^3}{\partial x \partial\partial y - \partial y \partial\partial x}$, vbi nullum differentiale pro constanti est assumtum.

§. 10. Sin autem iſtud elementum curuae ∂s conſtans accipere velimus, erit $\partial s \partial \partial s = 0$, ideoque $\partial x \partial \partial x + \partial y \partial \partial y = 0$, ex qua aequatione fit primo $\partial \partial y = - \frac{\partial x \partial \partial x}{\partial y}$, ideoque denominator ille $\partial x \partial \partial y - \partial y \partial \partial x$ fiet $= - \frac{\partial \partial x (\partial x^2 + \partial y^2)}{\partial y} = - \frac{\partial s^2 \partial \partial x}{\partial y}$, ſicque hoc caſu radius oſculi erit $= - \frac{\partial y \partial s}{\partial \partial x}$.

§. 11. Simili modo cum ſit $\partial \partial x = - \frac{\partial y \partial \partial y}{\partial x}$, erit denominator

$$\partial x \partial \partial y - \partial y \partial \partial x = \frac{\partial \partial y (\partial x^2 + \partial y^2)}{\partial x} = \frac{\partial s^2 \partial \partial y}{\partial x},$$

vnde radius oſculi colligitur $= \frac{\partial x \partial s}{\partial \partial y}$. Hoc modo ſumto elemento ∂s conſtante duae habebuntur formulae pro radio oſculi, quae ſunt $- \frac{\partial s \partial y}{\partial \partial x}$ et $\frac{\partial s \partial x}{\partial \partial y}$.

OBSER-