



1793

# De iterata integratione formularum integralium, dum aliquis exponens pro variabili assumitur

Leonhard Euler

Follow this and additional works at: <https://scholarlycommons.pacific.edu/euler-works>

 Part of the [Mathematics Commons](#)

Record Created:

2018-09-25

## Recommended Citation

Euler, Leonhard, "De iterata integratione formularum integralium, dum aliquis exponens pro variabili assumitur" (1793). *Euler Archive - All Works*. 653.

<https://scholarlycommons.pacific.edu/euler-works/653>

This Article is brought to you for free and open access by the Euler Archive at Scholarly Commons. It has been accepted for inclusion in Euler Archive - All Works by an authorized administrator of Scholarly Commons. For more information, please contact [mgibney@pacific.edu](mailto:mgibney@pacific.edu).

DE  
ITERATA INTEGRATIONE  
FORMVLARVM INTEGRALIVM  
DVM ALIQVIS EXPONENS PRO VARIABILI  
ASSVMITVR.

Auctore

L. EVLERO.

---

Conuent. exhib. die 19 Aug. 1776.

---

Problema I.

**C**um sit  $\int x^{\theta-1} dx \left[ \begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = \frac{1}{\theta}$ , hanc formulam denuo integrare, sumto exponente  $\theta$  variabili.

Solutio.

§. I. Quoniam hic de integratione agitur, ut ea determinetur, integrale ita capi assumamus, ut evanescat certo casu, posito scilicet  $\theta = \alpha$ . Multiplicetur ergo utrinque per elementum  $d\theta$ , et integratione juxta hanc legem instituta pro parte dextra habebimus  $\int \frac{d\theta}{\theta} = l\theta - l\alpha = l\frac{\theta}{\alpha}$ . At pro parte sinistra notum est, hanc integrationem a signo summatorio  $\int$  penitus non turbari, et quia jam sola littera  $\theta$  pro variabili habetur,  $\frac{dx}{x}$  vero ut constans spectatur, ob  $x^{\theta-1} dx = \frac{dx}{x} x^{\theta}$ , habebimus

$$\int x^\theta \partial \theta = \frac{x^\theta}{l x} - \frac{x^\alpha}{l x};$$

quo valore substituto membrum sinistrum erit

$$\int \frac{\partial x}{x} \cdot \frac{x^\theta - x^\alpha}{l x},$$

quamobrem ista integratio iterata nos perducit ad hanc aequationem:

$$\int \frac{x^{\theta-1} - x^{\alpha-1} \partial x}{l x} \left[ \begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = l \frac{\theta}{\alpha}.$$

### Corollarium 1.

§. 2. Si eodem modo formula integralis

$$\int x^{n+\theta-1} \partial x \left[ \begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = \frac{1}{n+\theta}$$

denuo integretur, sumto  $\theta$  variabili, reperietur haec aequatio integrata:

$$\int (x^{n+\theta-1} - x^{n+\alpha-1}) \frac{\partial x}{l x} \left[ \begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = \int \frac{n+\theta}{n+\alpha}.$$

At si  $\theta$  negative capiatur, tum etiam  $\alpha$  negative accipi debet, unde aequatio denuo integrata haec prodibit:

$$\int (x^{n-\theta-1} - x^{n-\alpha-1}) \frac{\partial x}{l x} \left[ \begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = \int \frac{n-\theta}{n-\alpha}.$$

### Corollarium 2.

§. 3. Hic igitur notentur istae integrationes, quas in parte sinistra institui oportet, et quibus pro aliis formulis in posterum erit utendum, ubi semper assumamus, integralia ita capi debere, ut evanescant posito  $\theta = \alpha$ . Primo scilicet erit

$$\int x^\theta \partial \theta = \frac{x^\theta - x^\alpha}{l x}.$$

Praeterea vero simili modo

$$\int x^{n+\theta} \partial \theta = \frac{x^{n+\theta} - x^{n+\alpha}}{l x};$$

atque hinc porro intelligitur fore

$$\int x^{n+\lambda\theta} \partial \theta = \frac{x^{n+\lambda\theta} - x^{n+\lambda\alpha}}{\lambda l x},$$

vnde patet, si  $\lambda$  capiatur negative, fore

$$\int x^{n-\lambda\theta} \partial \theta = \frac{x^{n-\lambda\theta} - x^{n-\lambda\alpha}}{-\lambda l x}.$$

### Problema 2.

*Cum sit, uti jam saepius est ostensum,*

$$\int \frac{x^{\theta-1} \partial x}{1+x^\nu} \left[ \begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array} \right] = \frac{\pi}{\nu \operatorname{fin.} \frac{\theta\pi}{\nu}},$$

*hanc aequationem de novo integrare, sumto exponente  $\theta$  pro variabili.*

### Solutio.

§. 4. Perpetuo hic, ut hactenus, integralia ita accipi statuamus, ut evanescant posito  $\theta = \alpha$ ; quo observato pro parte dextra habebimus  $\int \frac{\pi \partial \theta}{\nu \operatorname{fin.} \frac{\theta\pi}{\nu}}$ , quae formula posito  $\frac{\theta\pi}{\nu} = \Phi$  abit in hanc:  $\int \frac{\partial \Phi}{\operatorname{fin.} \Phi}$ , cujus integrale novimus esse  $l \operatorname{tang.} \frac{1}{2} \Phi$ ; quamobrem adjecta debita constante pro hac parte habebimus

$$\int \frac{\pi \partial \theta}{\nu \operatorname{fin.} \frac{\theta\pi}{\nu}} = l \operatorname{tang.} \frac{\theta\pi}{2\nu} - l \operatorname{tang.} \frac{\alpha\pi}{2\nu} = l \frac{\operatorname{tang.} \frac{\theta\pi}{2\nu}}{\operatorname{tang.} \frac{\alpha\pi}{2\nu}}.$$

Pro

Pro parte autem sinistra, vbi solus factor  $x^{\theta-1}$  est variabilis, erit

$$\int x^{\theta-1} d\theta = \frac{x^{\theta-1} - x^{\alpha-1}}{l x}.$$

Hoc igitur valore introducto formula nostra integralis denuo integrata erit

$$\int \frac{d x (x^{\theta-1} - x^{\alpha-1})}{(1 + x^{\nu}) l x} \left[ \begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array} \right] = \int \frac{\text{tang. } \frac{\theta \pi}{2 \nu}}{\text{tang. } \frac{\alpha \pi}{2 \nu}}.$$

### Corollarium.

§. 5. Quodsi ergo sumamus  $\alpha = \frac{1}{2} \nu$ , quoniam  $\text{tang. } \frac{\pi}{4} = 1$ , hoc casu, ponendo potius  $\nu = 2 \alpha$ , habebimus hanc aequationem integram satis memorabilem:

$$\int \frac{d x (x^{\theta-1} - x^{\alpha-1})}{(1 + x^{2 \alpha}) l x} \left[ \begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array} \right] = l \text{ tang. } \frac{\theta \pi}{4 \alpha}.$$

### Problema 3.

*Cum sit, vti jam satis constat:*

$$\int \frac{(x^{\theta-1} + x^{\nu-\theta-1}) d x}{1 + x^{\nu}} \left[ \begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = \frac{\pi}{\nu \text{ fin. } \frac{\theta \pi}{\nu}},$$

*hanc aequationem denuo integrare per exponentem variabilem  $\theta$ , ita vt integralia evanescant posito  $\theta = \alpha$ .*

### Solutio.

§. 6. Multiplicando igitur per  $d\theta$  et integrando, pro parte dextra, prorsus vt in praecedente problemate, habebimus

$$\int \frac{\text{tang. } \frac{\theta \pi}{2 \nu}}{\text{tang. } \frac{\alpha \pi}{2 \nu}}.$$

Pro parte autem sinistra, quia formula  $\frac{\partial x}{1+x^v}$  est constans, et exponens  $\theta$  in duobus terminis occurrit, pro priore termino habebimus

$$\int x^{\theta-1} \partial \theta = \frac{x^{\theta-1} - x^{\alpha-1}}{l x},$$

pro altero vero termino ex §. 3. habebimus

$$\int x^{v-\theta-1} \partial \theta = \frac{x^{v-\alpha-1} - x^{v-\theta-1}}{l x},$$

quibus valoribus substitutis orietur ista noua integratio :

$$\int \frac{\partial x}{l x} \cdot \frac{x^{\theta-1} - x^{\alpha-1} + x^{v-\alpha-1} - x^{v-\theta-1}}{1+x^v} \left[ \begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = \int \frac{\text{tang. } \frac{\theta \pi}{2v}}{\text{tang. } \frac{\alpha \pi}{2v}}.$$

### Corollarium 1.

§. 7. Ista aequatio aliquanto succinctius ita repraesentari potest:

$$\int \frac{\partial x}{x l x} \frac{(x^{\theta} - x^{\alpha} + x^{v-\alpha} - x^{v-\theta})}{1+x^v} \left[ \begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = \int \frac{\text{tang. } \frac{\theta \pi}{2v}}{\text{tang. } \frac{\alpha \pi}{2v}}$$

vbi cum sit  $x^{v-\alpha} - x^{v-\theta} = x^{v-\alpha-\theta} (x^{\theta} - x^{\alpha})$ , ista aequatio ita commodius per factores repraesentari poterit:

$$\int \frac{\partial x}{x l x} \frac{(x^{\theta} - x^{\alpha})(1+x^{v-\alpha-\theta})}{1+x^v} \left[ \begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = \int \frac{\text{tang. } \frac{\theta \pi}{2v}}{\text{tang. } \frac{\alpha \pi}{2v}}.$$

### Corollarium 2.

§. 8. Quodsi hic capiamus  $\theta = v - \alpha$ , vt fiat  $x^{v-\alpha-\theta} = 1$ , pro parte dextra erit  $\text{tang. } \frac{(v-\alpha)\pi}{2v} = \text{cotang. } \frac{\alpha \pi}{2v}$ , vnde totum hoc membrum erit  $2 \int \text{cot. } \frac{\alpha \pi}{2v}$ ; quare cum pro parte sinistra

factor

factor  $1 + x^{\nu-\alpha-\theta}$  evadat  $= 2$ , vtrinqve per 2 dividendo habebimus

$$\int \frac{\partial x}{x \ln x} \cdot \frac{x^{\nu-\alpha} - x^{\alpha}}{1 + x^{\nu}} \left[ \begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = l \cot. \frac{\alpha \pi}{2 \nu}.$$

### Corollarium 3.

§. 9. Quodsi sumamus  $\nu = 2\alpha$ , vt fiat  $\text{tang. } \frac{\alpha \pi}{2\nu} = 1$ , pro parte sinistra factor  $1 + x^{\nu-\alpha-\theta}$  abit in  $1 + x^{\alpha-\theta}$ , dum prior factor  $x^{\theta} - x^{\alpha}$  ita repraesentari potest:  $x^{\theta}(1 - x^{\alpha-\theta})$ ; unde amborum productum erit  $x^{\theta}(1 - x^{\nu\alpha-2\theta})$ , quamobrem integratio nostra ita se habebit:

$$\int \frac{x^{\theta-1} \partial x}{\ln x} \cdot \frac{1 - x^{2\alpha-2\theta}}{1 + x^{2\alpha}} \left[ \begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = l \text{ tang. } \frac{\theta \pi}{4\alpha}.$$

### Scholion.

§. 10. Istaе integrationes eo majorem attentionem mereantur, quod in iis tres exponentes  $\alpha$ ,  $\theta$ ,  $\nu$  indefiniti occurrunt, quos singulos pro lubitu vtrunqve determinare licet, ita vt istae formulae multo latius pateant, quam eae quas non ita pridem ex iisdem fundamentis derivavi.

### Problema 4.

*Cum sit, vti jam abunde est demonstratum,*

$$\int \frac{x^{\theta-1} - x^{\nu-\theta-1}}{1 - x^{\nu}} \partial x \left[ \begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = \frac{\pi}{\nu \text{ tang. } \frac{\theta \pi}{\nu}},$$

*hanc formulam denuo integrare, sumto exponente  $\theta$  variabili, ita vt integralia evanescant posito  $\theta = \alpha$ .*

### Solutio.

§. 11. Quodsi ergo hic per  $\partial \theta$  multiplicemus, pro parte dextra habebimus  $\frac{\pi \partial \theta}{\nu \operatorname{tang.} \frac{\theta \pi}{\nu}}$ , quae formula, posito  $\frac{\pi \theta}{\nu} = \Phi$ , abit in  $\frac{\partial \Phi}{\operatorname{tang.} \Phi} = \frac{\partial \Phi \operatorname{cof.} \Phi}{\operatorname{fin.} \Phi}$ , cujus integrale manifesto est  $\int \operatorname{fin.} \Phi$ ; quamobrem constanti debita adjecta, pro parte dextra habebimus

$$\int \operatorname{fin.} \frac{\theta \pi}{\nu} - \int \operatorname{fin.} \frac{\alpha \pi}{\nu} = \int \frac{\operatorname{fin.} \frac{\theta \pi}{\nu}}{\operatorname{fin.} \frac{\alpha \pi}{\nu}}.$$

Pro parte autem sinistra, quae ita repraesentetur:

$$\int \frac{\partial x}{x} \cdot \frac{x^\theta - x^{\nu-\theta}}{1 - x^\nu},$$

habebimus

$$\int x^\theta \partial \theta = \frac{x^\theta - x^\alpha}{l x} \text{ et}$$

$$\int x^{\nu-\theta} \partial \theta = \frac{x^{\nu-\alpha} - x^{\nu-\theta}}{l x},$$

quibus valoribus substitutis orietur sequens aequatio integrata:

$$\int \frac{\partial x}{x l x} \cdot \frac{(x^\theta - x^\alpha - x^{\nu-\alpha} + x^{\nu-\theta})}{1 - x^\nu} \left[ \begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = \int \frac{\operatorname{fin.} \frac{\theta \pi}{\nu}}{\operatorname{fin.} \frac{\alpha \pi}{\nu}}$$

vbi iterum tres exponentes indefiniti occurrunt,  $\alpha$ ,  $\theta$ ,  $\nu$ .

### Corollarium I.

§. 12. Cum sit, vti jam ante obseruauimus,

$$x^{\nu-\alpha} - x^{\nu-\theta} = x^{\nu-\alpha-\theta} (x^\theta - x^\alpha),$$

formula nostra commodius ita per factores exprimi poterit:

$$\int \frac{\partial x}{x l x} \cdot \frac{(x^\theta - x^\alpha)(1 - x^{\nu-\alpha-\theta})}{1 - x^\nu} \left[ \begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = \int \frac{\operatorname{fin.} \frac{\theta \pi}{\nu}}{\operatorname{fin.} \frac{\alpha \pi}{\nu}}.$$

vbi



vbi si sumeremus  $\nu = \alpha + \theta$ , membrum finiftrum evanefceret, dextrum autem manifefto quoque evanefceret.

### Corollarium 2.

§. 13. Quodfi autem hic fumamus  $\nu = 2\alpha$ , pro dextra foret fin.  $\frac{\alpha\pi}{\nu} = 1$ , vnde hoc cafu formula noftra integralis erit

$$\int \frac{\partial x}{x l x} \cdot \frac{(x^\theta - x^\alpha)(1 - x^{\alpha-\theta})}{1 - x^{2\alpha}} \left[ \begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = l \text{ fin. } \frac{\theta \pi}{2\alpha},$$

quae forma evidenter in hanc contrahitur:

$$\int \frac{x^{\theta-1} \partial x}{l x} \cdot \frac{(1 - x^{\alpha-\theta})^2}{1 - x^{2\alpha}} \left[ \begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = l \text{ fin. } \frac{\theta \pi}{2\alpha}.$$

### Scholion.

§. 14. Has igitur egregias integrationes deduximus ex formulis integralibus jam pridem erutis, quatenus in iis exponentes indefiniti occurrunt; quod fi ergo aliae hujusmodi formulae integrales infuper innotefcerent, eas fimili modo tractare liceret; verum haecenus nullae tales formulae funt inventae quae ad hunc fcopum accommodari poffunt, quam ob caufam integrationes hic exhibitae fumma attentione Geometrarum dignae funt exiftimandae.

### Additamentum.

§. 15. Cum nuper oftendifsem hujus formulae integralis

$$\int \frac{x^{a-1} \partial x}{l x} \cdot \frac{(1 - x^b)(1 - x^c)}{1 - x^n}$$

a termino  $x = 0$  ad terminum  $x = 1$  extenfcae valorem ita exprimi, vt fit  $l \frac{P}{Q}$ , existente

P

$$P = \int \frac{x^{a+b-1} \partial x}{(1-x^n)^{1-\frac{c}{n}}} \text{ et } Q = \int \frac{x^{a-1} \partial x}{(1-x^n)^{1-\frac{c}{n}}}$$

quae integralia denuo ab  $x = 0$  ad  $x = 1$  sunt extendenda: manifestum est in hac forma generali plerasque integrationes supra inuentas contineri; quamobrem cum illis casibus valores integralium absolute exprimantur, operae pretium erit istam formam generalem ad illos casus applicare, vt ratio inter binas formulas integrales P et Q inde innotescat. Problema quidem primum et secundum huc plane non pertinent. Ex problemate igitur tertio et quarto eos perscrutemur casus, quos ad formam nostram generalem reuocare licet.

Evolutio formulae integralis supra §. 8. inuentae.

$$\int \frac{\partial x}{x \log x} \cdot \frac{x^{\nu-a} - x^a}{1+x^\nu} \left[ \begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \nu \end{array} \right] = l \cos. \frac{\alpha \pi}{2 \nu}.$$

§. 16. Quoniam hic denominator est  $1+x^\nu$ , vt is ad formam generalem reducatur, multiplicetur fractio supra et infra per  $1-x^\nu$ , et formula ista integralis hanc induet formam:

$$\int \frac{\partial x}{x \log x} \cdot \frac{(x^{\nu-a} - x^a)(1-x^\nu)}{1-x^{2\nu}}$$

Hic ante omnia dispiciendum est, vter exponentium  $\nu - a$  et  $a$  sit maior, vnde duos casus evolvi conveniet, prouti fuerit vel  $\nu - a < a$ , hoc est  $\nu < 2a$ , vel  $\nu - a > a$ , hoc est  $\nu > 2a$ .

§. 17. Sit igitur primo  $\nu < 2a$ , seu  $a > \frac{1}{2}\nu$ , atque formula integralis ita repraesentari poterit:

$$\int \frac{x^{\nu-a-1} \partial x}{\log x} \cdot \frac{(1-x^{2a-\nu})(1-x^\nu)}{1-x^{2\nu}}$$

Hinc iam comparatione cum forma generali instituta manifesto habebimus  $a = \nu - a$ ,  $b = 2a - \nu$  et  $c = \nu$ , denique  $n = 2\nu$ , ex quibus valoribus formabuntur sequentes formulae:

$$P = \int \frac{x^{\alpha-1} \partial x}{\sqrt{(1-x^{2\nu})}} \text{ et } Q = \int \frac{x^{\nu-\alpha-1} \partial x}{\sqrt{(1-x^{2\nu})}}$$

Ponere etiam potuiffemus  $b = \nu$  et  $c = 2\alpha - \nu$ , manentibus  $a = \nu - \alpha$  et  $n = 2\nu$ , hincque prodiffent valores

$$P = \int \frac{x^{2\nu-\alpha-1} \partial x}{(1-x^{2\nu})^{\frac{2\nu-2\alpha}{2\nu}}} \text{ et } Q = \int \frac{x^{\nu-\alpha-1} \partial x}{(1-x^{2\nu})^{\frac{2\nu-2\alpha}{2\nu}}}$$

vtrinque autem erit  $l \frac{P}{Q} = l \cot. \frac{\alpha\pi}{2\nu}$ .

§. 18. Hinc igitur duas nancifcimus integrationes notatu digniffimas. Cum enim fit  $\frac{P}{Q} = \cot. \frac{\alpha\pi}{2\nu}$ , hae duae integrationes ita fe habebunt:

$$\text{I. } \int \frac{x^{\alpha-1} \partial x}{\sqrt{1-x^{2\nu}}} : \int \frac{x^{\nu-\alpha-1} \partial x}{\sqrt{1-x^{2\nu}}} = \cot. \frac{\alpha\pi}{2\nu};$$

$$\text{II. } \int \frac{x^{2\nu-\alpha-1} \partial x}{(1-x^{2\nu})^{\frac{2\nu-2\alpha}{2\nu}}} : \int \frac{x^{\nu-\alpha-1} \partial x}{(1-x^{2\nu})^{\frac{2\nu-2\alpha}{2\nu}}} = \cot. \frac{\alpha\pi}{2\nu}.$$

§. 19. Sin autem fuerit  $\nu > 2\alpha$ , ipfa formula generalis mutatis signis ita debet repraesentari:

$$\int \frac{\partial x}{x l x} \cdot \frac{(x^\alpha - x^{\nu-\alpha})(1-x^\nu)}{1-x^{2\nu}} = l \text{ tang. } \frac{\alpha\pi}{2\nu},$$

cui aequationi nunc induamus hanc formam:

$$\int \frac{x^{\alpha-1} \partial x}{l x} \cdot \frac{(1-x^{\nu-2\alpha})(1-x^\nu)}{1-x^{2\nu}}$$

vnde iam manifesto habemus  $a = \alpha$ ,  $b = \nu - 2\alpha$ ,  $c = \nu$ , atque  $n = 2\nu$ , vnde deducuntur ifti valores:

$$P = \int \frac{x^{\nu-\alpha-1} \partial x}{\sqrt{(1-x^{2\nu})}} \text{ et } Q = \int \frac{x^{\alpha-1} \partial x}{\sqrt{(1-x^{2\nu})}}$$

Sin autem fumamus  $c = \nu - 2\alpha$  et  $b = \nu$ , manente  $a = \alpha$  et

$n = 2\nu$ , reperietur

$$P = \int \frac{x^{\alpha+\nu-1} \partial x}{(1-x^{2\nu})^{\frac{\nu+\alpha}{2\nu}}} \text{ et } Q = \int \frac{x^{\alpha-1} \partial x}{(1-x^{2\nu})^{\frac{\nu+\alpha}{2\nu}}}$$

§. 20. Cum nunc utrinque fit  $l \frac{P}{Q} = l \text{ tang. } \frac{\alpha\pi}{2\nu}$  ideoque  $\frac{P}{Q} = \text{tang. } \frac{\alpha\pi}{2\nu}$ , hinc adipiscimur iterum has duas integrationes:

$$\text{III. } \int \frac{x^{\nu-\alpha-1} \partial x}{\sqrt{1-x^{2\nu}}} : \int \frac{x^{\alpha-1} \partial x}{\sqrt{1-x^{2\nu}}} = \text{tang. } \frac{\alpha\pi}{2\nu},$$

quae quidem conuenit cum priore antecedentium, siquidem formulae P et Q tantum inter se permutantur; altera uero integratio est noua, scilicet

$$\text{IV. } \int \frac{x^{\alpha+\nu-1} \partial x}{(1-x^{2\nu})^{\frac{\nu+\alpha}{2\nu}}} : \int \frac{x^{\alpha-1} \partial x}{(1-x^{2\nu})^{\frac{\nu+\alpha}{2\nu}}} = \text{tang. } \frac{\alpha\pi}{2\nu}.$$

Euolutio formulae integralis §. 9. allatae:

$$\int \frac{x^{\theta-1} \partial x}{lx} \cdot \frac{1-x^{2\alpha-2\theta}}{1+x^{2\alpha}} \left[ \begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = l \text{ tang. } \frac{\theta\pi}{4\alpha}.$$

§. 21. Quo haec expressio ad formam praescriptam reducatur, multiplicetur supra et infra per  $1-x^{2\alpha}$ , ut habeamus hanc formam:

$$\int \frac{x^{\theta-1} \partial x}{lx} \cdot \frac{(1-x^{2\alpha-2\theta})(1-x^{2\alpha})}{1-x^{4\alpha}} = l \text{ tang. } \frac{\theta\pi}{4\alpha},$$

quae sponte ad formam generalem reuocatur, sumendo  $\alpha = \theta$ ,  $b = 2\alpha - 2\theta$ ,  $c = 2\alpha$  et  $n = 4\alpha$ , si modo fuerit  $\alpha > \theta$ . Si enim fuerit  $\theta > \alpha$ , alio modo comparatio institui debet, uti deinceps uidebimus. Ex his autem valoribus conficietur

$$P = \int \frac{x^{2\alpha-\theta-1} \partial x}{\sqrt{(1-x^{4\alpha})}} \text{ et } Q = \int \frac{x^{\theta-1} \partial x}{\sqrt{(1-x^{4\alpha})}},$$

vnde

vnde ergo deducitur

$$V. \int \frac{x^{2\alpha-\theta-1} \partial x}{\sqrt{(1-x^{2\alpha})}} : \int \frac{x^{\theta-1} \partial x}{\sqrt{(1-x^{2\alpha})}} = \text{tang. } \frac{\theta \pi}{4\alpha}.$$

§. 22. Possimus etiam valores litterarum  $b$  et  $c$  inter se permutare, vt fit  $b = 2\alpha$  et  $c = 2\alpha - 2\theta$ , manentibus  $a = \theta$  et  $n = 4\alpha$ ; tum autem fiet

$$P = \int \frac{x^{2\alpha+\theta-1} \partial x}{(1-x^{2\alpha})^{\frac{\alpha+\theta}{2\alpha}}} \text{ et } Q = \int \frac{x^{\theta-1} \partial x}{(1-x^{2\alpha})^{\frac{\alpha+\theta}{2\alpha}}},$$

hincque deducitur reductio

$$VI. \int \frac{x^{2\alpha+\theta-1} \partial x}{(1-x^{2\alpha})^{\frac{\alpha+\theta}{2\alpha}}} : \int \frac{x^{\theta-1} \partial x}{(1-x^{2\alpha})^{\frac{\alpha+\theta}{2\alpha}}} = \text{tang. } \frac{\theta \pi}{4\alpha},$$

quae autem, aequae ac praecedens, locum non habet, nisi fit  $\alpha > \theta$ .

§. 23. Quod si autem  $\theta$  superet  $\alpha$ , aequationem nostram in aliam formam transfundi oportet, signa vtrinque mutando, vnde prodibit

$$\int \frac{x^{2\alpha-\theta-1} \partial x}{l x} \cdot \frac{(1-x^{2\theta-2\alpha})(1-x^{2\alpha})}{1-x^{2\alpha}} = l \cot. \frac{\theta \pi}{4\alpha}.$$

Hic iam iterum duplex comparatio institui potest: primo scilicet sumamus  $a = 2\alpha - \theta$ ,  $b = 2\theta - 2\alpha$ ,  $c = 2\alpha$  et  $n = 4\alpha$ , vnde formamus

$$P = \int \frac{x^{\theta-1} \partial x}{\sqrt{(1-x^{2\alpha})}} \text{ et } Q = \int \frac{x^{2\alpha-\theta-1} \partial x}{\sqrt{(1-x^{2\alpha})}}$$

hincque oritur septima relatio haec:

$$VII. \int \frac{x^{\theta-1} \partial x}{\sqrt{(1-x^{2\alpha})}} : \int \frac{x^{2\alpha-\theta-1} \partial x}{\sqrt{(1-x^{2\alpha})}} = \cot. \frac{\theta \pi}{4\alpha},$$

quae manifesto cum quinta congruit.

§. 24. Noua autem reductio obtinebitur, si statuamus  $b = 2\alpha$  et  $c = 2\theta - 2\alpha$ , manentibus  $a = 2\alpha - \theta$  et  $n = 4\alpha$ ; tum igitur erit

$$P = \int \frac{x^{4\alpha - \theta - 1} \partial x}{(1 - x^{4\alpha})^{\frac{3\alpha - \theta}{2\alpha}}} \text{ et } Q = \int \frac{x^{2\alpha - \theta - 1} \partial x}{(1 - x^{4\alpha})^{\frac{3\alpha - \theta}{2\alpha}}}.$$

Hinc vero colligitur reductio octaua

$$\text{VIII. } \int \frac{x^{4\alpha - \theta - 1} \partial x}{(1 - x^{4\alpha})^{\frac{3\alpha - \theta}{2\alpha}}} : \int \frac{x^{2\alpha - \theta - 1} \partial x}{(1 - x^{4\alpha})^{\frac{3\alpha - \theta}{2\alpha}}} = \cot. \frac{\theta \pi}{4\alpha}.$$

§. 25. Hic autem probe notandum est, quaternas posteriores reductiones ex quatuor prioribus oriri, si in istis loco  $\alpha$  scribatur  $\theta$ , at  $2\alpha$  loco  $\nu$ , ita ut quatuor posteriores reductiones iam in prioribus contineantur; quamobrem siue quatuor priores, siue posteriores, penitus omittere licebit, ita ut nobis tantum quatuor relinquantur, inter quas porro, quoniam tertia non discrepat a prima, tantum tres supererunt huiusmodi reductiones, quae quidem ex problemate tertio sunt natae.

Eolutio formulae integralis §. 12. allatae:

$$\int \frac{\partial x}{x \sqrt{x}} \cdot \frac{(x^\theta - x^\alpha)(1 - x^{\nu - \alpha - \theta})}{1 - x^\nu} \left[ \begin{array}{l} \text{ab } x=0 \\ \text{ad } x=1 \end{array} \right] = \int \frac{\text{fin. } \frac{\theta \pi}{\nu}}{\text{fin. } \frac{\alpha \pi}{\nu}}.$$

§. 26. Ista expressio iam congruit cum forma nostra generali, neque idcirco ulteriori transformatione indiget. Hic quidem duo casus essent distinguendi, prouti fuerit vel  $\theta > \alpha$ , vel  $\theta < \alpha$ ; verum hac etiam distinctione carere possumus, propterea quod binae litterae  $\alpha$  et  $\theta$  inter se sunt permutabiles: iis enim permutatis signa utrinque inuertuntur. Hanc ob causam, quoscunque valores habuerint ambae litterae  $\alpha$  et  $\theta$ , minorem semper littera  $\theta$ , maiorem vero littera  $\alpha$  designare licebit, vnde aequatio nostra ita repraesentabitur:

$$\int \frac{x^{\theta-1} \partial x}{l x} \cdot \frac{(1-x^{\alpha-\theta})(1-x^{\nu-\alpha-\theta})}{1-x^{\nu}} = \int \frac{\text{fin. } \frac{\theta \pi}{\nu}}{\text{fin. } \frac{\alpha \pi}{\nu}}$$

§. 27. Nihilo vero minus duo casus distinguendi etiam hic occurrunt, prouti fuerit vel  $\nu > \alpha + \theta$ , vel  $\nu < \alpha + \theta$ . Sit igitur primo  $\nu > \alpha + \theta$ , et forma exposita manebit invariata, quae denuo duplicem comparisonem cum generali admittit. Primo igitur statuamus  $a = \theta$ ,  $b = \alpha - \theta$ ,  $c = \nu - \alpha - \theta$  et  $n = \nu$ , qui valores nobis suppeditant

$$P = \int \frac{x^{\alpha-1} \partial x}{(1-x^{\nu})^{\frac{\alpha+\theta}{\nu}}} \quad \text{et} \quad Q = \int \frac{x^{\theta-1} \partial x}{(1-x^{\nu})^{\frac{\alpha-\theta}{\nu}}}$$

ficque ex hac evolutione habebimus sequentem reductionem:

$$I. \int \frac{x^{\alpha-1} \partial x}{(1-x^{\nu})^{\frac{\alpha+\theta}{\nu}}} : \int \frac{x^{\theta-1} \partial x}{(1-x^{\nu})^{\frac{\alpha-\theta}{\nu}}} = \frac{\text{fin. } \frac{\theta \pi}{\nu}}{\text{fin. } \frac{\alpha \pi}{\nu}}$$

§. 28. Secunda nascetur reductio permutandis litteris  $b$  et  $c$ , ita ut fit  $a = \theta$ ,  $b = \nu - \alpha - \theta$ ,  $c = \alpha - \theta$ , et  $n = \nu$ , vnde formantur hae formulae:

$$P = \int \frac{x^{\nu-\alpha-1} \partial x}{(1-x^{\nu})^{\frac{\nu-\alpha+\theta}{\nu}}} \quad \text{et} \quad Q = \int \frac{x^{\theta-1} \partial x}{(1-x^{\nu})^{\frac{\nu-\alpha+\theta}{\nu}}}$$

quare secunda reductio hinc orta erit

$$II. \int \frac{x^{\nu-\alpha-1} \partial x}{(1-x^{\nu})^{\frac{\nu-\alpha+\theta}{\nu}}} : \int \frac{x^{\theta-1} \partial x}{(1-x^{\nu})^{\frac{\nu-\alpha+\theta}{\nu}}} = \frac{\text{fin. } \frac{\theta \pi}{\nu}}{\text{fin. } \frac{\alpha \pi}{\nu}}$$

quae duae reductiones postulant ut fit  $\nu > \alpha + \theta$ .

§. 29. Sin autem fuerit  $\nu < \alpha + \theta$ , ipsa aequationis forma hoc modo immutari debet:

$$\int \frac{x^{\nu-\alpha-1} \partial x}{l x} \cdot \frac{(1-x^{\alpha-\theta})(1-x^{\alpha+\theta-\nu})}{1-x^{\nu}} = \int \frac{\text{fin. } \frac{\alpha \pi}{\nu}}{\text{fin. } \frac{\theta \pi}{\nu}}$$

vbi iterum gemina comparatio institui potest. Sit igitur primo  $a = \nu - \alpha$ ,  $b = \alpha - \theta$ ,  $c = \alpha + \theta - \nu$  et  $n = \nu$ , vnde oriuntur hae formulae:

$$P = \int \frac{x^{\nu-\theta-1} \partial x}{(1-x^\nu)^{\frac{2\nu-\alpha-\theta}{\nu}}} \text{ et } Q = \int \frac{x^{\nu-\alpha-1} \partial x}{(1-x^\nu)^{\frac{2\nu-\alpha-\theta}{\nu}}}.$$

Hinc igitur concluditur tertia reductio:

$$\text{III. } \int \frac{x^{\nu-\theta-1} \partial x}{(1-x^\nu)^{\frac{2\nu-\alpha-\theta}{\nu}}} : \int \frac{x^{\nu-\alpha-1} \partial x}{(1-x^\nu)^{\frac{2\nu-\alpha-\theta}{\nu}}} = \frac{\text{fin. } \frac{\alpha\pi}{\nu}}{\text{fin. } \frac{\theta\pi}{\nu}}.$$

§. 30. Denique statuamus  $a = \nu - \alpha$ ,  $b = \alpha + \theta - \nu$ ,  $c = \alpha - \theta$  et  $n = \nu$ , et formulae hinc sequentes nascentur:

$$P = \int \frac{x^{\theta-1} \partial x}{(1-x^\nu)^{\frac{\nu-\alpha+\theta}{\nu}}} \text{ et } Q = \int \frac{x^{\nu-\alpha-1} \partial x}{(1-x^\nu)^{\frac{\nu-\alpha+\theta}{\nu}}}$$

ita vt quarta hinc oriatur reductio:

$$\text{IV. } \int \frac{x^{\theta-1} \partial x}{(1-x^\nu)^{\frac{\nu-\alpha+\theta}{\nu}}} : \int \frac{x^{\nu-\alpha-1} \partial x}{(1-x^\nu)^{\frac{\nu-\alpha+\theta}{\nu}}} = \frac{\text{fin. } \frac{\alpha\pi}{\nu}}{\text{fin. } \frac{\theta\pi}{\nu}}.$$

§. 31. Quatuor igitur hic nacti sumus formularum integralium paria, quae eandem inter se tenent rationem ac sinus duorum angulorum; dum evolutiones praecedentes tantum tria huiusmodi paria praebuerant, quarum ratio  $P:Q$  tangenti cuiuspiam anguli aequatur, vbi quidem evidens est secundam et quartam inter se conuenire. Cum igitur huiusmodi reductiones altioris sint indaginis, ac sine dubio insignem vsum habere queant, opere pretium erit eas clarius ob oculos exponere.

### Problema.

§. 32. Inuenire binas formulas integrales  $P$  et  $Q$  ab  $x = 0$  ad  $x = 1$  extensas, vt fiat  $\frac{P}{Q} = \text{tang. } \frac{m\pi}{2n}$ .

So-



### Solutio.

Triplici igitur modo hoc fieri potest, secundum evolutionem primam supra institutam. I. Ex prima enim reductione, cum sit  $\cot. \frac{\alpha \pi}{2 \nu} = \text{tang.} \frac{(\nu - \alpha) \pi}{2 \nu}$ , fiet  $\nu - \alpha = m$  et  $\nu = n$ , ita ut sit  $\alpha = n - m$ . Hinc igitur erit

$$P = \int \frac{x^{n-m-1} \partial x}{\sqrt{(1-x^{2n})}} \text{ et } Q = \int \frac{x^{m-1} \partial x}{\sqrt{(1-x^{2n})}},$$

quae ergo est solutio prima. II. Secunda reductio supra allata erat  $\frac{P}{Q} = \cot. \frac{\alpha \pi}{2 \nu} = \text{tang.} \frac{(\nu - \alpha) \pi}{2 \nu}$ , ubi ergo iterum est  $\alpha = n - m$  et  $\nu = n$ , sicque secunda solutio huius problematis constabit his formulis:

$$P = \int \frac{x^{m+n-1} \partial x}{(1-x^{2n})^{\frac{m+n}{2n}}} \text{ et } Q = \int \frac{x^{m-1} \partial x}{(1-x^{2n})^{\frac{m+n}{2n}}}.$$

Hae autem formulae tantum valent, quando fuerit  $m < \frac{1}{2}a$ , ideoque ipse angulus  $\frac{m \pi}{2n}$  minor semirecto. III. Quoniam tertia reductio ibi allata cum prima conuenit, ex quarta, ubi erat  $\frac{P}{Q} = \text{tang.} \frac{\alpha \pi}{2 \nu}$ , ideoque pro nostro casu  $\alpha = m$  et  $\nu = n$ , tertia solutio ita se habebit:

$$P = \int \frac{x^{m+n-1} \partial x}{(1-x^{2n})^{\frac{m+n}{2n}}} \text{ et } Q = \int \frac{x^{m-1} \partial x}{(1-x^{2n})^{\frac{m+n}{2n}}},$$

qui valores quoniam a praecedentibus non sunt diuersi, duas tantum adipiscimur solutiones nostri problematis, quarum secunda limitatione quadam indiget, scilicet  $m < \frac{1}{2}n$ , prior vero ad omnes angulos recto non maiores patet. Hae ergo duae solutiones ita repraesententur:

$$\text{I. } P = \int \frac{x^{n-m-1} \partial x}{\sqrt{(1-x^{2n})}}, \quad Q = \int \frac{x^{m-1} \partial x}{\sqrt{(1-x^{2n})}},$$

II.

$$\text{II. } P = \int \frac{x^{m+n-1} \partial x}{(1-x^{2n})^{\frac{m+n}{2n}}}, \quad Q = \int \frac{x^{m-1} \partial x}{(1-x^{2n})^{\frac{m+n}{2n}}},$$

ex vtraque igitur erit  $\frac{P}{Q} = \text{tang. } \frac{m\pi}{2n}$ .

### Problema.

§. 33. *Inuenire binas formulas integrales P et Q, ut fiat*  
 $\frac{P}{Q} = \frac{\text{fin. } \frac{p\pi}{2n}}{\text{fin. } \frac{q\pi}{2n}}$ , *siquidem ambo illa integralia ab*  $x=0$  *ad*  $x=1$  *extendantur.*

### Solutio.

Ad hanc igitur formam transferamus quatuor illas reductiones in euolutione tertia traditas, et cum pro prima et secunda esset  $\frac{P}{Q} = \frac{\text{fin. } \frac{\theta\pi}{v}}{\text{fin. } \frac{\alpha\pi}{v}}$ , pro forma hic praescripta erit  $\theta = p$ ,  $\alpha = q$  et  $v = 2n$ , quamobrem hinc nanciscimur duas sequentes solutiones:

$$\text{I. } P = \int \frac{x^{q-1} \partial x}{(1-x^{2n})^{\frac{p+q}{2n}}} \quad \text{et} \quad Q = \int \frac{x^{p-1} \partial x}{(1-x^{2n})^{\frac{p+q}{2n}}},$$

$$\text{II. } P = \int \frac{x^{2n-q-1} \partial x}{(1-x^{2n})^{\frac{2n-q+p}{2n}}} \quad \text{et} \quad Q = \int \frac{x^{p-1} \partial x}{(1-x^{2n})^{\frac{2n-q+p}{2n}}}.$$

Tertia vero et quarta reductio habebant  $\frac{P}{Q} = \frac{\text{fin. } \frac{\alpha\pi}{v}}{\text{fin. } \frac{\theta\pi}{v}}$ , pro qua igitur erit  $\alpha = p$ ,  $\theta = q$ ,  $v = 2n$ , vnde ambae solutiones sequentes deducuntur:

$$\text{III. } P = \int \frac{x^{2n-q-1} \partial x}{(1-x^{2n})^{\frac{p-q-q}{2n}}} \quad \text{et} \quad Q = \int \frac{x^{2n-p-1} \partial x}{(1-x^{2n})^{\frac{4n-p-q}{2n}}},$$

IV.

$$\text{IV. } P = \int \frac{x^{q-1} \partial x}{(1-x^{2n})^{\frac{2n-p+q}{2n}}} \text{ et } Q = \int \frac{x^{2n-p-1} \partial x}{(1-x^{2n})^{\frac{2n-p+q}{2n}}}$$

Hinc igitur patet quadruplici modo fieri posse  $\frac{P}{Q} = \frac{\sin. \frac{p\pi}{2n}}{\sin. \frac{q\pi}{2n}}$

### Corollarium 1.

§. 34. Si assumamus  $q = n$ , vt fiat  $\sin. \frac{q\pi}{2n} = 1$ , ideoque prodire debeat  $\frac{P}{Q} = \sin. \frac{p\pi}{2n}$ ; pro hoc casu quatuor inuentae solutiones dabunt

$$\text{I. } P = \int \frac{x^{n-1} \partial x}{(1-x^{2n})^{\frac{p+n}{2n}}} \text{ et } Q = \int \frac{x^{p-1} \partial x}{(1-x^{2n})^{\frac{p+n}{2n}}}$$

$$\text{II. } P = \int \frac{x^{n-1} \partial x}{(1-x^{2n})^{\frac{n+p}{2n}}} \text{ et } Q = \int \frac{x^{p-1} \partial x}{(1-x^{2n})^{\frac{n+p}{2n}}}$$

$$\text{III. } P = \int \frac{x^{n-1} \partial x}{(1-x^{2n})^{\frac{3n-p}{2n}}} \text{ et } Q = \int \frac{x^{2n-p-1} \partial x}{(1-x^{2n})^{\frac{3n-p}{2n}}}$$

$$\text{IV. } P = \int \frac{x^{n-1} \partial x}{(1-x^{2n})^{\frac{3n-p}{2n}}} \text{ et } Q = \int \frac{x^{2n-p-1} \partial x}{(1-x^{2n})^{\frac{3n-p}{2n}}}$$

vbi ergo solutio prima cum secunda et tertia cum quarta conuenit.

### Corollarium 2.

§. 35. Sumamus nunc esse  $q = n - p$ , vt fiat  $\sin. \frac{q\pi}{2n} = \cos. \frac{p\pi}{2n}$ , ideoque prodire debeat  $\frac{P}{Q} = \tan. \frac{p\pi}{2n}$ . Pro hoc ergo casu quatuor solutiones inuentae euadent

$$\text{I. } P = \int \frac{x^{n-p-1} \partial x}{\sqrt{(1-x^{2n})}} \text{ et } Q = \int \frac{x^{p-1} \partial x}{\sqrt{(1-x^{2n})}}$$

$$\text{II. } P = \int \frac{x^{n+p-1} \partial x}{(1-x^{2n})^{\frac{n+p}{2n}}} \text{ et } Q = \int \frac{x^{p-1} \partial x}{(1-x^{2n})^{\frac{n+p}{2n}}},$$

$$\text{III. } P = \int \frac{x^{n+p-1} \partial x}{(1-x^{2n})^{\frac{3}{2}}} \text{ et } Q = \int \frac{x^{2n-p-1} \partial x}{(1-x^{2n})^{\frac{3}{2}}},$$

$$\text{IV. } P = \int \frac{x^{n-p-1} \partial x}{(1-x^{2n})^{\frac{3n-p}{2n}}} \text{ et } Q = \int \frac{x^{2n-p-1} \partial x}{(1-x^{2n})^{\frac{3n-p}{2n}}},$$

hincque erit  $\frac{P}{Q} = \text{tang. } \frac{p\pi}{2n},$

vbi prima et secunda forma cum iis quas in praecedente problemate inuenimus profus conueniunt; tertia autem forma, ob  $(1-x^{2n})^{\frac{3}{2}}$ , fit incongrua, quia inde P et Q in infinitum excrescerent; quarta autem nouam formam dare videtur.