1793

Quatuor theoremata maxime notatu digna in calculo integrali

Leonhard Euler

QVATVOR THEOREMATA
MAXIME NOTATV DIGNA IN CALCVLO INTEGRALI.

Auctore

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Theorema Primum.

§. 1.

Denotante ϕ angulum quemcunque variabilem, si n significet numerum quemcunque, siue integrum, siue fractum, siue positiuum, siue negatiuum, tum vero statuatur $\partial s = \partial \phi\,(\sin.\phi)^{n-1}$, sequentes formulae integrales omnes algebraice exhiberi possunt:

I. $\int \partial s \sin.(n+1)\phi = \frac{\sin.\phi^n}{n} \sin.n\phi.$

II. $\int \partial s \sin.(n+3)\phi = \frac{\sin.\phi^n}{n+1}[\sin.(n+2)\phi + \frac{1}{n}\sin.n\phi].$

III. $\int \partial s \sin.(n+5)\phi = \frac{\sin.\phi^n}{n+2}[\sin.(n+4)\phi + \frac{2}{n+1}\sin.(n+2)\phi$

$+ \frac{2}{n+1} \cdot \frac{1}{n}\sin.n\phi].$

IV. $\int \partial s \sin.(n+7)\phi = \frac{\sin.\phi^n}{n+3}[\sin.(n+6)\phi + \frac{3}{n+2}\sin.(n+4)\phi$

$+ \frac{3}{n+2} \cdot \frac{2}{n+1}\sin.(n+2)\phi + \frac{3}{n+2} \cdot \frac{2}{n+1} \cdot \frac{1}{n}\sin.n\phi].$

V.

$$\text{V. } \int \partial s \,\text{fin.}(n+9)\Phi = \frac{\text{fin.}\Phi^n}{n+4}[\text{fin.}(n+8)\Phi + \tfrac{4}{n+3}\text{fin.}(n+6)\Phi$$
$$+ \tfrac{4}{n+3}\cdot\tfrac{3}{n+2}\text{fin.}(n+4)\Phi + \tfrac{4}{n+3}\cdot\tfrac{3}{n+2}\cdot\tfrac{2}{n+1}\text{fin.}(n+2)\Phi$$
$$+ \tfrac{4}{n+3}\cdot\tfrac{3}{n+2}\cdot\tfrac{2}{n+1}\cdot\tfrac{1}{n}\text{fin.}\,n\Phi].$$

$$\text{VI. } \int \partial s \,\text{fin.}(n+11)\Phi = \frac{\text{fin.}\Phi^n}{n+5}[\text{fin.}(n+10)\Phi + \tfrac{5}{n+4}\text{fin.}(n+8)\Phi$$
$$+ \tfrac{5}{n+4}\cdot\tfrac{4}{n+3}\text{fin.}(n+6)\Phi + \tfrac{5}{u+4}\cdot\tfrac{4}{n+3}\cdot\tfrac{3}{n+2}\text{fin.}(n+4)\Phi$$
$$+ \tfrac{5}{n+4}\cdot\tfrac{4}{n+3}\cdot\tfrac{3}{n+2}\cdot\tfrac{2}{n+1}\text{fin.}(n+2)\Phi + \tfrac{5}{n+4}\cdot\tfrac{4}{n+3}\cdot\tfrac{3}{n+2}\cdot\tfrac{2}{n+1}\cdot\tfrac{1}{n}\text{fin.}\,n\Phi].$$

etc. etc.

Vnde fi i denotet numerum pofitiuum quemcunque, generaliter habebimus

$$\int \partial s \,\text{fin.}(n+2i+1)\Phi = \frac{\text{fin.}\Phi^n}{n+i}[\text{fin.}(n+2i)\Phi$$
$$+ \tfrac{i}{n+i-1}.\text{fin.}(n+2i-2)\Phi + \tfrac{i}{n+i-1}\cdot\tfrac{i-1}{n+i-2}.\text{fin.}(n+2i-4)\Phi$$
$$+ \tfrac{i}{n+i-1}\cdot\tfrac{i-1}{n+i-2}\cdot\tfrac{i-2}{n+i-3}\text{fin.}(n+2i-4)\Phi$$
$$+ \tfrac{i}{n+i-1}\cdot\tfrac{i-1}{n+i-2}\cdot\tfrac{i-2}{n+i-3}\cdot\tfrac{i-3}{n+i-4}\text{fin.}(n+2i-6)\Phi$$
$$+ \tfrac{i}{n+i-1}\cdot\tfrac{i-1}{n+i-2}\cdot\tfrac{i-2}{n+i-3}\cdot\tfrac{i-3}{n+i-4}\cdot\tfrac{i-4}{n+i-5}\text{fin.}(n+2i-8)\Phi].$$

etc. etc.

quae terminorum progreffio quouis cafu fponte abrumpitur.

Demonftratio.

§. 2. Ad veritatem huius theorematis demonftrandam confideretur ifta formula: $Z = \text{fin.}\,\Phi^n\,\text{fin.}\,\lambda\Phi$, quae differentiata dat

$$\partial Z = \partial\Phi\,\text{fin.}\,\Phi^{n-1}(n\,\text{cof.}\,\Phi\,\text{fin.}\,\lambda\Phi + \lambda\,\text{fin.}\,\Phi\,\text{cof.}\,n\,\Phi).$$

At per reductiones cognitas eft

cof.

$$\cos.\phi \sin.\lambda\phi = \tfrac{1}{2}\sin.(\lambda-1)\phi + \tfrac{1}{2}\sin.(\lambda+1)\phi \text{ et}$$
$$\sin.\phi \cos.\lambda\phi = -\tfrac{1}{2}\sin.(\lambda-1)\phi + \tfrac{1}{2}\sin.(\lambda+1)\phi,$$

quibus valoribus ſubſtitutis, quoniam poſuimus $\partial\phi \sin.\phi^{n-1} = \partial s$, erit

$$2\partial Z = \partial s\,[(n-\lambda)\sin.(\lambda-1)\phi + (n+\lambda)\sin.(\lambda+1)\phi],$$

vnde denuo per partes integrando deducimus

$\int\partial s \sin.(\lambda+1)\phi = \frac{2Z}{\lambda+n} + \frac{\lambda-n}{\lambda+n}\int\partial s \sin.(\lambda-1)\phi$, ſiue

$$\int\partial s \sin.(\lambda+1)\phi = \frac{2\sin.\phi^n \sin.\lambda\phi}{\lambda+n} + \frac{\lambda-n}{\lambda+n}\int\partial s \sin.(\lambda-1)\phi.$$

§. 3. Stabilita igitur hac poſtrema reductione generali capiamus $\lambda = n$, vt adipiſcamur iſtam integrationem abſolutam:

$$\int\partial s \sin.(n+1)\phi = \tfrac{1}{n}\sin.\phi^n \sin.n\phi.$$

Nunc vero ſtatuamus $\lambda = n+2$, et forma illa generalis dabit

$$\int\partial s \sin.(n+3)\phi = \tfrac{1}{n+1}\sin.\phi^n \sin.(n+2)\phi + \tfrac{1}{n+1}\int\partial s \sin.(n+1)\phi,$$

ſicque haec integratio ad praecedentem eſt reducta. Jam ponamus $\lambda = n+4$, et forma generalis ſuppeditabit

$$\int\partial s \sin.(n+5)\phi = \tfrac{1}{n+2}\sin.\phi^n \sin.(n+4)\phi + \tfrac{2}{n+2}\int\partial s \sin.(n+3)\phi,$$

quae ergo integratio iterum ad praecedentem eſt reducta. Sit porro $\lambda = n+6$, et ex forma generali prodibit

$$\int\partial s \sin.(n+7)\phi = \tfrac{1}{n+3}\sin.\phi^n \sin.(n+6)\phi + \tfrac{3}{n+3}\int\partial s \sin.(n+5)\phi$$

ſicque augendis continuo valoribus ipſius λ binario, vlterius progredi licebit.

§. 4. Quodſi iam ſingulos valores integrales antecedentes in ſequentibus ſubſtituamus, ſequentes orientur integrationes abſolutae:

I.

$$\text{I. } \int \partial s \,\text{ſin.}\, (n+1)\,\Phi = \frac{\text{ſin.}\,\Phi^n}{n}\,\text{ſin.}\, n\,\Phi.$$

$$\text{II. } \int \partial s \,\text{ſin.}\, (n+3)\,\Phi = \frac{\text{ſin.}\,\Phi^n}{n+1}\,[\text{ſin.}\,(n+2)\,\Phi + \tfrac{1}{n}\,\text{ſin.}\, n\,\Phi].$$

$$\text{III. } \int \partial s \,\text{ſin.}\, (n+5)\,\Phi = \frac{\text{ſin.}\,\Phi^n}{n+2}\,[\text{ſin.}\,(n+4)\,\Phi$$
$$+ \tfrac{2}{n+1}\,\text{ſin.}\,(n+2)\,\Phi + \tfrac{2}{n+1}\cdot\tfrac{1}{n}\,\text{ſin.}\, n\,\Phi].$$

$$\text{IV. } \int \partial s \,\text{ſin.}\, (n+7)\,\Phi = \frac{\text{ſin.}\,\Phi^n}{n+3}\,[\text{ſin.}\,(n+6)\,\Phi + \tfrac{3}{n+2}\,\text{ſin.}\,(n+4)\,\Phi$$
$$+ \tfrac{3}{n+2}\cdot\tfrac{2}{n+1}\,\text{ſin.}\,(n+2)\,\Phi + \tfrac{3}{n+2}\cdot\tfrac{2}{n+1}\cdot\tfrac{1}{n}\,\text{ſin.}\, n\,\Phi].$$

quae cum ſint eae ipſae formulae, quas in theoremate annunciauimus, eius veritas ſufficienter eſt euicta.

Theorema ſecundum.

§. 5. Denotante Φ angulum quemcunque variabilem, ſi n denotet numerum quemcunque, ac breuitatis gratia ponatur vt ante $\partial s = \partial \Phi\, \text{ſin.}\, \Phi^{n-1}$, etiam omnes ſequentes integrationes per algebraicos valores exhiberi poſſunt:

$$\text{I. } \int \partial s \,\text{coſ.}\, (n+1)\,\Phi = \frac{\text{ſin.}\,\Phi^n}{n}\,\text{coſ.}\, n\,\Phi.$$

$$\text{II. } \int \partial s \,\text{coſ.}\, (n+3)\,\Phi = \frac{\text{ſin.}\,\Phi^n}{n+1}\,[\text{coſ.}\,(n+2)\,\Phi + \tfrac{1}{n}\,\text{coſ.}\, n\,\Phi].$$

$$\text{III. } \int \partial s \,\text{coſ.}\, (n+5)\,\Phi = \frac{\text{ſin.}\,\Phi^n}{n+2}\,[\text{coſ.}\,(n+4)\,\Phi$$
$$+ \tfrac{2}{n+1}\,\text{coſ.}\,(n+2)\,\Phi + \tfrac{2}{n+1}\cdot\tfrac{1}{n}\,\text{coſ.}\, n\,\Phi].$$

$$\text{IV. } \int \partial s \,\text{coſ.}\, (n+7)\,\Phi = \frac{\text{ſin.}\,\Phi^n}{n+6}\,[\text{coſ.}\,(n+6)\,\Phi + \tfrac{3}{n+2}\,\text{coſ.}\,(n+4)\,\Phi$$
$$+ \tfrac{3}{n+2}\cdot\tfrac{2}{n+1}\,\text{coſ.}\,(n+2)\,\Phi + \tfrac{3}{n+2}\cdot\tfrac{2}{n+1}\cdot\tfrac{1}{n}\,\text{coſ.}\, n\,\Phi].$$

V. $\int \partial s \operatorname{cof.} (n+9)\Phi = \frac{\operatorname{fin.} \Phi^n}{n+4} [\operatorname{cof.} (n+8)\Phi + \frac{4}{n+3} \operatorname{cof.} (n+6)\Phi$

$+ \frac{4}{n+3} \cdot \frac{3}{n+2} \operatorname{cof.} (n+4)\Phi + \frac{4}{n+3} \cdot \frac{3}{n+2} \cdot \frac{2}{n+1} \operatorname{cof.} (n+2)\Phi$

$+ \frac{4}{n+3} \cdot \frac{3}{n+2} \cdot \frac{2}{n+1} \cdot \frac{1}{n} \operatorname{cof.} n\Phi].$

VI. $\int \partial s \operatorname{cof.} (n+11)\Phi = \frac{\operatorname{fin.} \Phi^n}{n+5} [\operatorname{cof.} (n+10)\Phi + \frac{5}{n+4} \operatorname{cof.} (n+8)\Phi$

$+ \frac{5}{n+4} \cdot \frac{4}{n+3} \operatorname{cof.} (n+6)\Phi + \frac{5}{n+4} \cdot \frac{4}{n+3} \cdot \frac{3}{n+2} \operatorname{cof.} (n+4)\Phi$

$+ \frac{5}{n+4} \cdot \frac{4}{n+3} \cdot \frac{3}{n+2} \cdot \frac{2}{n+1} \operatorname{cof.} (n+2)\Phi + \frac{5}{n+4} \cdot \frac{4}{n+3} \cdot \frac{3}{n+2} \cdot \frac{2}{n+1} \cdot \frac{1}{n} \operatorname{cof.} n\Phi].$

etc. etc.

Vnde patet, fi i denotet numerum pofitiuum quemcunque, fore in genere

$\int \partial s \operatorname{cof.} (n+2i+1)\Phi = \frac{\operatorname{fin.} \Phi^n}{n+2} [\operatorname{cof.} (n+2i)\Phi$

$+ \frac{i}{n+i-1} \operatorname{cof.} (n+2i-2)\Phi + \frac{i}{n+i-1} \cdot \frac{i-1}{n+i-2} \operatorname{cof.} (n+2i-4)\Phi$

$+ \frac{i}{n+i-1} \cdot \frac{i-1}{n+i-2} \cdot \frac{i-2}{n+i-3} \operatorname{cof.} (n+2i-6)\Phi].$

etc. etc.

quos terminos quouis cafu eo vsque continuari oportet, donec fponte euanefcant.

Demonftratio.

§. 6. Ad veritatem horum integralium demonftrandam confideretur ifta formula: $Z = \operatorname{fin.} \Phi^n \operatorname{cof.} \lambda \Phi$, cuius differentiatio praebet

$$\partial Z = \partial \Phi \operatorname{fin.} \Phi^{n-1} (n \operatorname{cof.} \Phi \operatorname{cof.} \lambda \Phi - \lambda \operatorname{fin.} \Phi \operatorname{fin.} \lambda \Phi),$$

quae expreffio, ob

$$\operatorname{cof.} \Phi \operatorname{cof.} \lambda \Phi = \tfrac{1}{2} \operatorname{cof.} (\lambda - 1)\Phi + \tfrac{1}{2} \operatorname{cof.} (\lambda + 1)\Phi \text{ et}$$

$$\operatorname{fin.} \Phi \operatorname{fin.} \lambda \Phi = \tfrac{1}{2} \operatorname{cof.} (\lambda - 1)\Phi - \tfrac{1}{2} \operatorname{cof.} (\lambda + 1)\Phi$$

fi

si loco $\partial \Phi \sin. \Phi^{n-1}$ valorem assumtum ∂s scribamus, fiet

$$2 \partial Z = \partial s [(n-\lambda) \cos. (\lambda - 1) \Phi + (n+\lambda) \cos. (\lambda + 1) \Phi],$$

vnde iterum per partes integrando erit

$$2 \sin. \Phi^n \cos. \lambda \Phi = (n-\lambda) \int \partial s \cos. (\lambda - 1) \Phi + (n+\lambda) \int \partial s \cos. (\lambda + 1) \Phi,$$

atque hinc deducimus sequentem reductionem generalem:

$$\int \partial s \cos. (\lambda + 1) \Phi = \tfrac{2}{\lambda + n} \sin. \Phi^n \cos. \lambda \Phi + \tfrac{\lambda - n}{\lambda + n} \int \partial s \cos. (\lambda - 1) \Phi.$$

§. 7. Ponamus igitur primo $\lambda = n$, vt obtineamus hanc integrationem absolutam:

$$\int \partial s \cos. (n+1) \Phi = \tfrac{1}{n} \sin. \Phi^n \cos. n \Phi.$$

Fiat iam $\lambda = n + 2$, et forma generalis dabit

$$\int \partial s \cos. (n+3) \Phi = \tfrac{1}{n+1} \sin. \Phi^n \cos. (n+2) \Phi + \tfrac{1}{n+1} \int \partial s \cos. (n+1) \Phi.$$

Statuatur porro $\lambda = n + 4$, et consequemur

$$\int \partial s \cos. (n+5) \Phi = \tfrac{1}{n+2} \sin. \Phi^n \cos. (n+4) \Phi + \tfrac{2}{n+2} \int \partial s \cos. (n+3) \Phi.$$

Ponamus vlterius $\lambda = n + 6$, ac reperiemus

$$\int \partial s \cos. (n+7) \Phi = \tfrac{1}{n+3} \sin. \Phi^n \cos. (n+6) \Phi + \tfrac{3}{n+3} \int \partial s \cos. (n+5) \Phi.$$

Faciamus simili modo vlterius $\lambda = n + 8$, ac nanciscemur

$$\int \partial s \cos. (n+9) \Phi = \tfrac{1}{n+4} \sin. \Phi^n \cos. (n+8) \Phi + \tfrac{4}{n+4} \int \partial s \cos. (n+7) \Phi.$$

etc. etc.

§. 8. Quodsi iam singulos valores integrales praecedentes in sequentes introducamus, perueniemus ad istas integrationes absolutas:

I. $\int \partial s \cos. (n+1) \Phi = \tfrac{1}{n} \sin. \Phi^n \cos. n \Phi.$

II. $\int \partial s \cos. (n+3) \Phi = \tfrac{1}{n+1} \sin. \Phi^n [\cos. (n+2) \Phi + \tfrac{1}{n} \cos. n \Phi].$

III. $\int \partial s \cos. (n+5) \Phi = \tfrac{1}{n+2} \sin. \Phi^n [\cos. (n+4) \Phi$
$+ \tfrac{2}{n+1} \cos. (n+2) \Phi + \tfrac{2}{n+1} \cdot \tfrac{1}{n} \cos. n \Phi].$

IV. $\int \partial s \cos. (n + 7) \phi = \frac{1}{n+3} \sin. \phi^n [\cos. (n + 6) \phi$
$+ \frac{3}{n+2} \cos. (n + 4) \phi + \frac{3}{n+2} \cdot \frac{2}{n+1} \cos. (n + 2) \phi$
$+ \frac{3}{n+2} \cdot \frac{2}{n+1} \cdot \frac{1}{n} \cos. n \phi]$.
etc. etc.

quae manifesto sunt eae ipsae formulae, quas in theoremate produximus, quarum ergo veritas nunc solide est demonstrata.

Corollarium.

§. 9. Haec duo theoremata combinata inseruire possunt ad innumerabiles curuas algebraicas inueniendas, quarum arcus indefiniti s omnes per eandem formulam integralem $\int \partial \phi \sin. \phi^{n-1}$ exprimantur. Cum enim elementum curuae sit $\partial s = \partial \phi \sin. \phi^{n-1}$, omnes plane curuae huic conditioni satisfacientes ita generaliter exhiberi possunt, vt earum coordinatae sint $x = \int \partial s \cos. \omega$ et $y = \int \partial s \sin. \omega$. Nunc autem videmus ambas istas expressiones reuera fore algebraicas, si angulus ω ita accipiatur, vt sit $\omega = (n + 2i + 1) \phi$, vbi loco i numerum quemcunque integrum positiuum accipere licet. Quamobrem numerum talium curuarum algebraicarum in infinitum augere licebit: curua autem simplicissima sine dubio prodibit, ponendo $i = 0$. Hoc argumentum iam nuper fusius pertractavimus.

Theorema tertium.

§. 10. Denotante ϕ angulum quemcunque variabilem, si n significet numerum quemcunque siue integrum, siue fractum, siue positiuum, siue negatiuum, tum vero statuatur $\partial s = \partial \phi \cos. \phi^{n-1}$, sequentes formulae integrales omnes algebraice exhiberi possunt:

I. $\int \partial s \cos. (n + 1) \phi = \frac{1}{n} \cos. \phi^n \sin. n \phi$.

II. $\int \partial s \cos. (n + 3) \phi = \frac{1}{n+1} \cos. \phi^n [\sin. (n+2) \phi - \frac{1}{n} \sin. n \phi]$.

III.

III. $\int \partial s \,\text{cof.}\, (n+5)\Phi = \frac{1}{n+2} \,\text{cof.}\, \Phi^n \,[\text{fin.}\, (n+4)\Phi$
$- \frac{2}{n+1} \,\text{fin.}\, (n+2)\Phi + \frac{2}{n+1} \cdot \frac{1}{n} \,\text{fin.}\, n\Phi].$

IV. $\int \partial s \,\text{cof.}\, (n+7)\Phi = \frac{1}{n+3} \,\text{cof.}\, \Phi^n \,[\text{fin.}\, (n+6)\Phi$
$- \frac{3}{n+2} \,\text{fin.}\, (n+4)\Phi + \frac{3}{n+2} \cdot \frac{2}{n+1} \,\text{fin.}\, (n+2)\Phi$
$- \frac{3}{n+2} \cdot \frac{2}{n+1} \cdot \frac{1}{n} \,\text{fin.}\, n\Phi].$

V. $\int \partial s \,\text{cof.}\, (n+9)\Phi = \frac{1}{n+4} \,\text{cof.}\, \Phi^n \,[\text{fin.}\, (n+8)\Phi$
$- \frac{4}{n+3} \,\text{fin.}\, (n+6)\Phi + \frac{4}{n+3} \cdot \frac{3}{n+2} \,\text{fin.}\, (n+4)\Phi$
$- \frac{4}{n+3} \cdot \frac{3}{n+2} \cdot \frac{2}{n+1} \,\text{fin.}\, (n+2)\Phi + \frac{4}{n+3} \cdot \frac{3}{n+2} \cdot \frac{2}{n+1} \cdot \frac{1}{n} \,\text{fin.}\, n\Phi].$

VI. $\int \partial s \,\text{cof.}\, (n+11)\Phi = \frac{1}{n+5} \,\text{cof.}\, \Phi^n \,[\text{fin.}\, (n+10)\Phi$
$- \frac{5}{n+4} \,\text{fin.}\, (n+8)\Phi + \frac{5}{n+4} \cdot \frac{4}{n+3} \,\text{fin.}\, (n+6)\Phi$
$- \frac{5}{n+4} \cdot \frac{4}{n+3} \cdot \frac{3}{n+2} \,\text{fin.}\, (n+4)\Phi + \frac{5}{n+4} \cdot \frac{4}{n+3} \cdot \frac{3}{n+2} \cdot \frac{2}{n+1} \,\text{fin.}\, (n+2)\Phi$
$- \frac{5}{n+4} \cdot \frac{4}{n+3} \cdot \frac{3}{n+2} \cdot \frac{2}{n+1} \cdot \frac{1}{n} \,\text{fin.}\, n\Phi].$

Ex quibus concluditur fore generaliter, denotante i numerum integrum positiuum quemcunque:

$$\int \partial s \,\text{cof.}\, (n+2i+1)\Phi = \frac{1}{n+i} \,\text{cof.}\, \Phi^n \,[\text{fin.}\, (n+2i)\Phi$$
$$- \frac{i}{n+i-1} \,\text{fin.}\, (n+2i-2)\Phi + \frac{i}{n+i-1} \cdot \frac{i-1}{n+i-2} \,\text{fin.}\, (n+2i-4)\Phi$$
$$- \frac{i}{n+i-1} \cdot \frac{i-1}{n+i-2} \cdot \frac{i-2}{n+i-3} \,\text{fin.}\, (n+2i-6)\Phi + \text{etc.}\,]$$

Demonstratio.

§. 11. Ad veritatem huius theorematis demonstrandam consideretur ista formula: $Z = \text{cof.}\, \Phi^n \,\text{fin.}\, \lambda\Phi$, quae differentiata dat

$$\partial Z = \partial\Phi \,\text{cof.}\, \Phi^{n-1} (-n \,\text{fin.}\, \Phi \,\text{fin.}\, \lambda\Phi + \lambda \,\text{cof.}\, \Phi \,\text{cof.}\, \lambda\Phi),$$

quae per reductiones ante adhibitas transformatur in hanc formam:

$$2\partial Z = \partial s\,[(\lambda-n)\,\text{cof.}\, (\lambda-1)\Phi + (\lambda+n)\,\text{cof.}\, (\lambda+1)\Phi]$$

 vnde

vnde iterum per partes integrando nanciſcimur

$$2Z = (\lambda - n)\int \partial s \operatorname{cof.}(\lambda - 1)\Phi + (\lambda + n)\int \partial s \operatorname{cof.}(\lambda + 1)\Phi,$$

hincque deducimus iſtam integrationem generalem:

$$\int \partial s \operatorname{cof.}(\lambda + 1)\Phi = \tfrac{2}{\lambda + n}\operatorname{cof.}\Phi^n \operatorname{fin.}\lambda\Phi - \tfrac{(\lambda - n)}{\lambda + n}\int \partial s \operatorname{cof.}(\lambda - 1)\Phi.$$

§. 12. Sumamus nunc primo $\lambda = n$, vt poſterius integrale tollatur, ac prodibit

$$\int \partial s \operatorname{cof.}(n + 1)\Phi = \tfrac{1}{n}\operatorname{cof.}\Phi^n \operatorname{fin.} n\Phi.$$

Nunc autem porro ponamus $\lambda = n + 2$, et forma noſtra generalis nobis praebebit

$$\int \partial s \operatorname{cof.}(n+3)\Phi = \tfrac{1}{n+1}\operatorname{cof.}\Phi^n \operatorname{fin.}(n+2)\Phi - \tfrac{1}{n+1}\int \partial s \operatorname{cof.}(n+1)\Phi,$$

vbi ergo poſterius integrale iam eſt inuentum. Fiat vlterius $\lambda = n + 4$, et habebimus

$$\int \partial s \operatorname{cof.}(n+5)\Phi = \tfrac{1}{n+2}\operatorname{cof.}\Phi^n \operatorname{fin.}(n+4)\Phi - \tfrac{2}{n+2}\int \partial s \operatorname{cof.}(n+3)\Phi,$$

quod poſtremum integrale itidem iam patet. Sumamus nunc $\lambda = n + 6$, et forma generalis dabit

$$\int \partial s \operatorname{cof.}(n+7)\Phi = \tfrac{1}{n+3}\operatorname{cof.}\Phi^n \operatorname{fin.}(n+6)\Phi - \tfrac{3}{n+3}\int \partial s \operatorname{cof.}(n+5)\Phi.$$

Simili modo ſi faciamus $\lambda = n + 8$, obtinebimus

$$\int \partial s \operatorname{cof.}(n+9)\Phi = \tfrac{1}{n+4}\operatorname{cof.}\Phi^n \operatorname{fin.}(n+8)\Phi - \tfrac{4}{n+4}\int \partial s \operatorname{cof.}(n+7)\Phi.$$

Hocque modo vlterius progrediendo, perpetuo ſequentia integralia per praecedentia exprimere licebit.

§. 13. Quodſi ergo valores integrales praecedentes in ſequentibus ſubſtituamus, conſequemur iſtas integrationes abſolutas:

I. $\int \partial s \operatorname{cof.}(n + 1)\Phi = \tfrac{1}{n}\operatorname{cof.}\Phi^n \operatorname{fin.} n\Phi.$

II. $\int \partial s \operatorname{cof.}(n + 3)\Phi = \tfrac{1}{n+1}\operatorname{cof.}\Phi^n [\operatorname{fin.}(n+2)\Phi - \tfrac{1}{n}\operatorname{fin.} n\Phi].$

III. $\int \partial s \operatorname{cof.}(n + 5)\Phi = \tfrac{1}{n+2}\operatorname{cof.}\Phi^n [\operatorname{fin.}(n+4)\Phi$

$$- \tfrac{2}{n+1}\operatorname{fin.}(n+2)\Phi + \tfrac{2}{n+1}\cdot\tfrac{1}{n}\operatorname{fin.} n\Phi].$$

IV.

IV. $\int \partial s \cos. (n + 7) \Phi = \frac{1}{n+3} \cos. \Phi^n [\sin. (n + 6) \Phi$

$- \frac{3}{n+2} \sin. (n + 4) \Phi + \frac{3}{n+2} \cdot \frac{2}{n+1} \sin. (n + 2) \Phi$

$- \frac{3}{n+2} \cdot \frac{2}{n+1} \cdot \frac{1}{n} \sin. n \Phi]$.

V. $\int \partial s \cos. (n + 9) \Phi = \frac{1}{n+4} \cos. \Phi^n [\sin. (n + 8) \Phi$

$- \frac{4}{n+3} \sin. (n + 6) \Phi + \frac{4}{n+3} \cdot \frac{3}{n+2} \sin. (n + 4) \Phi$

$- \frac{4}{n+3} \cdot \frac{3}{n+2} \cdot \frac{2}{n+1} \sin. (n+2) \Phi + \frac{4}{n+3} \cdot \frac{3}{n+2} \cdot \frac{2}{n+1} \cdot \frac{1}{n} \sin. n \Phi]$.

etc. etc.

vnde veritas noſtri theorematis abunde elucet.

Theorema quartum.

§. 14. Denotante Φ angulum quemcunque variabilem, ſi n ſignificet numerum quemcunque, ſiue integrum, ſiue fractum, ſiue poſitiuum, ſiue negatiuum, tum vero ſtatuatur $\partial s = \partial \Phi \cos. \Phi^{n-n}$; ſequentes formulae integrales omnes algebraice exprimi poterunt.

I. $\int \partial s \sin. (n + 1) \Phi = - \frac{1}{n} \cos. \Phi^n \cos. n \Phi$

II. $\int \partial s \sin. (n + 3) \Phi = - \frac{1}{n+1} \cos. \Phi^n [\cos. (n + 2) \Phi$

$- \frac{1}{n} \cos. n \Phi]$.

III. $\int \partial s \sin. (n + 5) \Phi = - \frac{1}{n+2} \cos. \Phi^n [\cos. (n + 4) \Phi$

$- \frac{2}{n+1} \cos. (n + 2) \Phi + \frac{2}{n+1} \cdot \frac{1}{n} \cos. n \Phi]$.

IV. $\int \partial s \sin. (n + 7) \Phi = - \frac{1}{n+3} \cos. \Phi^n [\cos. (n + 6) \Phi$

$- \frac{3}{n+2} \cos. (n + 4) \Phi + \frac{3}{n+2} \cdot \frac{2}{n+1} \cos. (n + 2) \Phi$

$- \frac{3}{n+2} \cdot \frac{2}{n+1} \cdot \frac{1}{n} \cos. n \Phi]$.

V. $\int \partial s \sin. (n + 9) \Phi = - \frac{1}{n+4} \cos. \Phi^n [\cos. (n + 8) \Phi$

$- \frac{4}{n+3} \cos. (n + 6) \Phi + \frac{4}{n+3} \cdot \frac{3}{n+2} \cos. (n + 4) \Phi$

$- \frac{4}{n+3} \cdot \frac{3}{n+2} \cdot \frac{2}{n+1} \cos. (n + 2) \Phi + \frac{4}{n-3} \cdot \frac{3}{n+2} \cdot \frac{2}{n+1} \cdot \frac{1}{n} \cos. n \Phi]$.

VI.

VI. $\int \partial s \text{ fin. } (n+11)\Phi = -\frac{1}{n+5} \text{ cof. } \Phi^n [\text{cof. } (n+10)\Phi$
$-\frac{5}{n+4} \text{ cof. } (n+8)\Phi + \frac{5}{n+4} \cdot \frac{4}{n+3} \text{ cof. } (n+6)\Phi$
$-\frac{5}{n+4} \cdot \frac{4}{n+3} \cdot \frac{3}{n+2} \text{ cof. } (n+4)\Phi$
$+\frac{5}{n+4} \cdot \frac{4}{n+3} \cdot \frac{3}{n+2} \cdot \frac{2}{n+1} \text{ cof. } (n+2)\Phi$
$-\frac{5}{n+4} \cdot \frac{4}{n+3} \cdot \frac{3}{n+2} \cdot \frac{2}{n+1} \cdot \frac{1}{n} \text{ cof. } n\Phi]$.

Vnde manifesto patet, si i denotet numerum quemcunque integrum positiuum, fore in genere

$$\int \partial s \text{ fin. } (n+2i+1)\Phi = -\frac{1}{n+i} \text{ cof. } \Phi^n [\text{cof. } (n+2i)\Phi$$
$$-\frac{i}{n+i-1} \text{ cof. } (n+2i-2)\Phi$$
$$+\frac{i}{n+i-1} \cdot \frac{i-1}{n+i-2} \text{ cof. } (n+2i-4)\Phi$$
$$-\frac{i}{n+i-1} \cdot \frac{i-1}{n+i-2} \cdot \frac{i-2}{n+i-3} \text{ cof. } (n+2i-6)\Phi$$
$$+\frac{i}{n+i-1} \cdot \frac{i-1}{n+i-2} \cdot \frac{i-2}{n+i-3} \cdot \frac{i-3}{n+i-4} \text{ cof. } (n+2i-8)\Phi \text{ etc.}$$

Demonstratio.

§. 15. Ad hoc theorema demonstrandum consideretur formula $Z = \text{cof. } \Phi^n \text{ cof. } \lambda\Phi$, quae differentiata praebet

$$\partial Z = -\partial\Phi \text{ cof. } \Phi^{n-1} (n \text{ fin. } \Phi \text{ cof. } \lambda\Phi + \lambda \text{ cof. } \Phi \text{ fin. } \lambda\Phi),$$

quae per notas reductiones reducitur ad hanc formam:

$$2\partial Z = -\partial s [(\lambda+n) \text{ fin. } (\lambda+1)\Phi + (\lambda-n) \text{ fin. } (\lambda-1)\Phi],$$

quae iterum per partes integrata dat

$$2Z = -(\lambda+n)\int \partial s \text{ fin. } (\lambda+1)\Phi - (\lambda-n)\int \partial s \text{ fin. } (\lambda-1)\Phi,$$

vnde deducitur ista integratio generalis:

$$\int \partial s \text{ fin. } (\lambda+1)\Phi = -\frac{2}{\lambda+n} \text{ cof. } \Phi^n \text{ cof. } \lambda\Phi - \frac{(\lambda-n)}{\lambda+n}\int \partial s \text{ fin. } (\lambda-1)\Phi.$$

§. 16. Vt membrum integrale postremum e medio tollatur, capiamus $\lambda = n$ et forma generalis dabit

$$\int \partial s \text{ fin. } (n+1)\Phi = -\frac{1}{n} \text{ cof. } \Phi^n \text{ cof. } n\Phi.$$

Statuamus nunc porro $\lambda = n+2$, ac proueniet

$$\int \partial s \sin.(n+3)\Phi = -\tfrac{1}{n+1}\cos.\Phi^n \cos.(n+2)\Phi$$
$$-\tfrac{1}{n+1}\int \partial s \sin.(n+1)\Phi.$$

Fiat porro $\lambda = n+4$, vt oriatur

$$\int \partial s \sin.(n+5)\Phi = -\tfrac{1}{n+2}\cos.\Phi^n \cos.(n+4)\Phi$$
$$-\tfrac{2}{n+2}\int \partial s \sin.(n+3)\Phi.$$

Sit iam $\lambda = n+6$, fiet

$$\int \partial s \sin.(n+7)\Phi = -\tfrac{1}{n+3}\cos.\Phi^n \cos.(n+6)\Phi$$
$$-\tfrac{3}{n+3}\int \partial s \sin.(n+5)\Phi.$$

Simili modo fit $\lambda = n+8$, ac refultabit

$$\int \partial s \sin.(n+9)\Phi = -\tfrac{1}{n+4}\cos.\Phi^n \cos.(n+8)\Phi$$
$$-\tfrac{4}{n+4}\int \partial s \sin.(n+7)\Phi.$$

etc. etc.

vbi pariter fequentia integralia per praecedentia definiuntur.

§. 17. Quamobrem fi vbique valores integrales praecedentes fubftituantur, orientur fequentes integrationes abfolutae:

I. $\int \partial s \sin.(n+1)\Phi = -\tfrac{1}{n}\cos.\Phi^n \cos.n\Phi.$

II. $\int \partial s \sin.(n+3)\Phi = -\tfrac{1}{n+1}\cos.\Phi^n [\cos.(n+2)\Phi$
$-\tfrac{1}{n}\cos.n\Phi].$

III. $\int \partial s \sin.(n+5)\Phi = -\tfrac{1}{n+2}\cos.\Phi^n [\cos.(n+4)\Phi$
$-\tfrac{2}{n+1}\cos.(n+2)\Phi + \tfrac{2}{n+1}\cdot\tfrac{1}{n}\cos.n\Phi].$

IV. $\int \partial s \sin.(n+7)\Phi = -\tfrac{1}{n+3}\cos.\Phi^n [\cos.(n+6)\Phi$
$-\tfrac{3}{n+2}\cos.(n+4)\Phi + \tfrac{3}{n+2}\cdot\tfrac{2}{n+1}\cos.(n+2)\Phi$
$-\tfrac{3}{n+2}\cdot\tfrac{2}{n+1}\cdot\tfrac{1}{n}\cos.n\Phi].$

V. $\int \partial s \sin. (n+9) \phi = -\frac{1}{n+4} \cos. \phi^n [\cos. (n+8) \phi$
$- \frac{4}{n+3} \cos. (n+6) \phi + \frac{4}{n+3} \cdot \frac{3}{n+2} \cos. (n+4) \phi$
$- \frac{4}{n+3} \cdot \frac{3}{n+2} \cdot \frac{2}{n+1} \cos. (n+2) \phi$
$+ \frac{4}{n+3} \cdot \frac{3}{n+2} \cdot \frac{2}{n+1} \cdot \frac{1}{n} \cos. n \phi]$.

etc. etc.

ſicque veritas theorematis propoſiti ſufficienter eſt euicta.

Corollarium 1.

§. 18. Si $\partial s = \partial \phi \cos. \phi^{n-1}$ denotet elementum cuiuspiam lineae curuae, cuius coordinatae orthogonales ſint x et y, ita vt ſit $\partial s^2 = \partial x^2 + \partial y^2$, huic conditioni generatim ſatisfiet, ſumendo $\partial x = \partial s \cos. \omega$ et $\partial y = \partial s \sin. \omega$. Nunc igitur ex binis poſterioribus theorematibus patet, innumerabiles huiusmodi curuas algebraicas exhiberi poſſe, ſi ſcilicet capiatur $\omega = (n + 2i + 1) \phi$, quandoquidem hinc valores ipſarum x et y algebraice exprimi poſſunt; ac ſimpliciſſima quidem curua prodibit ponendo $i = 0$, tum enim fiet

$$x = \int \partial s \cos. (n+1) \phi = -\frac{1}{n} \cos. \phi^n \sin. n \phi \text{ et}$$
$$y = \int \partial s \sin. (n+1) \phi = -\frac{1}{n} \cos. \phi^n \cos. n \phi.$$

Corollarium 2.

§. 19. Quodſi ſumatur $n = 1$, vt fieri debeat $\partial s = \partial \phi$, ideoque $s = \phi$, hoc eſt arcui circulari aequalis, tum facile oſtendi poteſt, quicunque valor numero i tribuatur, curuas reſultantes omnes fore circulos, ita vt hoc caſu praeter circulum nulla alia curua algebraica ſatisfaciat, id quod pro caſu $i = 3$ oſtendiſſe ſufficiat. Tum enim erit

$$x = \int \partial s \cos. 8 \phi = \tfrac{1}{4} \cos. \phi (\sin. 7 \phi - \sin. 5 \phi + \sin. 3 \phi - \sin. \phi)$$

quae

quae forma per reductiones abit in hanc: $x = \frac{1}{8}$ fin. $8\,\Phi$. Tum vero habebitur fimili modo

$$y = \int \partial s \,\text{fin.}\, 8\,\Phi = -\tfrac{1}{4}\,\text{cof.}\,\Phi\,(\text{cof.}\,7\,\Phi - \text{cof.}\,5\,\Phi + \text{cof.}\,3\,\Phi - \text{cof.}\,\Phi),$$

quae per fimiles reductiones praebet

$$y = \tfrac{1}{8}(1 - \text{cof.}\,8\,\Phi) \text{ ideoque } \tfrac{1}{8} - y = \tfrac{1}{8}\,\text{cof.}\,8\,\Phi.$$

Ex his iam valoribus conjunctis manifeftum eft fore $x\,x + (\frac{1}{8} - y)^2 = \frac{1}{64}$, quae vtique eft aequatio pro circulo. Eodem modo oftendi poteft, quicunque valor numero i tribuatur, femper quoque circulum effe proditurum.

Corollarium 3.

§. 20. Cafus quoque, quo $n = -\frac{1}{2}$, omni attentione eft dignus, pro quo curua fimpliciffima erit

$$x = \int \partial s \,\text{cof.}\, \tfrac{1}{2}\Phi = \frac{2\,\text{fin.}\,\frac{1}{2}\Phi}{\sqrt{\text{cof.}\,\Phi}} \text{ et}$$

$$y = \int \partial s \,\text{fin.}\, \tfrac{1}{2}\Phi = \frac{2\,\text{cof.}\,\frac{1}{2}\Phi}{\sqrt{\text{cof.}\,\Phi}};$$

ita vt elementum huius curuae futurum fit $\partial s = \frac{\partial \Phi}{\text{cof.}\,\Phi\,\sqrt{\text{cof.}\,\Phi}}$. Iam ad angulum Φ eliminandum, quoniam eft

$$\text{cof.}\,\tfrac{1}{2}\Phi^2 - \text{fin.}\,\tfrac{1}{2}\Phi^2 = \text{cof.}\,\Phi$$

habebimus $y\,y - x\,x = 4$, fiue $y\,y = 4 + x\,x$, quae eft aequatio pro Hyperbola aequilatera, fiue rectangula.

Scholion 1.

§. 21. Quanquam autem in his quatuor theorematibus infinitae formulae integrabiles funt exhibitae, tamen occurrere poffunt certi cafus, quibus integralia affignata euadunt incongrua, atque adeo naturam quantitatum algebraicarum penitus amittunt. Tales cafus oriuntur, quoties exponens n vel euanefcit, vel

vel numero integro negatiuo fit aequalis. Hoc enim casu fieri potest, vt quispiam factor in denominatoribus in nihilum abeat, ideoque ipsi termini in infinitum excrescere videntur. Etiamsi enim hoc incommodum adiectione constantium pariter infinitarum euitari posset, tamen ipsi termini inde resultantes non amplius forent algebraici. Ita si esset $n = 0$, omnia prorsus integralia ibi exhibita penitus tollerentur. Si autem esset $n = -1$, tum tantum primae formulae relinquerentur, sequentes omnes autem euaderent inutiles. Si esset $n = -2$, tum binae priores formae tantum subsistere possent; solae autem ternae, si esset $n = -3$, etc. His autem casibus exceptis, quicunque valores exponenti n tribuantur, singula theoremata innumerabiles suppeditant formulas integrabiles.

Scholion.

§. 22. Quemadmodum binis prioribus theorematibus iam sum vsus ad innumerabiles curuas algebraicas inueniendas, quarum longitudo s hoc valore exprimatur: $s = \int \partial \Phi \sin. \Phi^{n-1}$; ita etiam bina posteriora theoremata innumerabilibus curuis algebraicis inueniendis inseruire possunt, quarum longitudo sit $s = \int \partial \Phi \cos. \Phi^{n-1}$. Etiamsi enim hi duo casus prorsus inter se conueniant, si quidem, loco Φ scribendo $90° - \Phi$, altera formula in alteram transformatur; unde quis suspicari posset, duo posteriora theoremata tuto omitti potuisse; tamen hos casus non tam plane ex prioribus deducere licet, quippe qui veritates per se notatu dignissimas inuoluere sunt censendi. Quin etiam omnia haec quatuor theoremata iunctim sumpta viam sternunt ad infinitas curuas algebraicas inuestigandas, quarum longitudo s formula multo magis complicata exprimatur; ad quod ostendendum ante oculos exponamus integrationes generales, ad quas singula theoremata nos duxerunt.

I.

$$\text{I. } \int \partial \Phi \sin.\Phi^{n-1} \sin.(n+2i+1)\Phi = \tfrac{1}{n+i} \sin.\Phi^n \, [\sin.(n+2i)\Phi$$
$$+ \tfrac{i}{n+i-1} \sin.(n+2i-2)\Phi + \tfrac{i}{n+i-1}\cdot\tfrac{i-1}{n+i-2} \sin.(n+i-4)\Phi$$
$$+ \tfrac{i}{n+i-1}\cdot\tfrac{i-1}{n+i-2}\cdot\tfrac{i-2}{n+i-3} \sin.(n+2i-6)\Phi$$
$$+ \tfrac{i}{n+i-1}\cdot\tfrac{i-1}{n+i-2}\cdot\tfrac{i-2}{n+i-3}\cdot\tfrac{i-3}{n+i-4} \sin.(n+2i-8)\Phi \text{ etc.}]$$

$$\text{II. } \int \partial \Phi \sin.\Phi^{n-1} \cos.(n+2i+1)\Phi = \tfrac{1}{n+i} \sin.\Phi^n \, [\cos.(n+2i)\Phi$$
$$+ \tfrac{i}{n+i-1} \cos.(n+2i-2)\Phi + \tfrac{i}{n+i-1}\cdot\tfrac{i-1}{n+i-2} \cos.(n+2i-4)\Phi$$
$$+ \tfrac{i}{n+i-1}\cdot\tfrac{i-1}{n+i-2}\cdot\tfrac{i-2}{n+i-3} \cos.(n+2i-6)\Phi$$
$$+ \tfrac{i}{n+i-1}\cdot\tfrac{i-1}{n+i-2}\cdot\tfrac{i-2}{n+i-3}\cdot\tfrac{i-3}{n+i-4} \cos.(n+2i-8)\Phi \text{ etc.}]$$

$$\text{III. } \int \partial \Phi \cos.\Phi^{n-1} \cos.(n+2i+1)\Phi = \tfrac{1}{n+i} \cos.\Phi^n \, [\sin.(n+2i)\Phi$$
$$- \tfrac{i}{n+i-1} \sin.(n+2i-2)\Phi + \tfrac{i}{n+i-1}\cdot\tfrac{i-1}{n+i-2} \sin.(n+2i-4)\Phi$$
$$- \tfrac{i}{n+i-1}\cdot\tfrac{i-1}{n+i-2}\cdot\tfrac{i-2}{n+i-3} \sin.(n+2i-6)\Phi$$
$$+ \tfrac{i}{n+i-1}\cdot\tfrac{i-1}{n+i-2}\cdot\tfrac{i-2}{n+i-3}\cdot\tfrac{i-3}{n+i-4} \sin.(n+2i-8)\Phi \text{ etc.}]$$

$$\text{IV. } \int \partial \Phi \cos.\Phi^{n-1} \sin.(n+2i+1)\Phi = -\tfrac{1}{n+i} \cos.\Phi^n \, [\cos(n+2i)\Phi$$
$$- \tfrac{i}{n+i-1} \cos.(n+2i-2)\Phi + \tfrac{i}{n+i-1}\cdot\tfrac{i-1}{n+i-2} \cos.(n+2i-4)\Phi$$
$$- \tfrac{i}{n+i-1}\cdot\tfrac{i-1}{n+i-2}\cdot\tfrac{i-2}{n+i-3} \cos.(n+2i-6)\Phi$$
$$+ \tfrac{i}{n+i-1}\cdot\tfrac{i-1}{n+i-2}\cdot\tfrac{i-2}{n+i-3}\cdot\tfrac{i-3}{n+i-4} \cos.(n+2i-8)\Phi \text{ etc.}]$$

Problema ſingulare.

Inuenire innumerabiles curuas algebraicas, quarum arcus indefiniti s iſta formula integrali exprimantur:

$$s = \int \partial \Phi \surd (a\,a \sin.\Phi^{2n-2} + b\,b \cos.\Phi^{2n-2}).$$

Solutio.

§. 23. Cum igitur elementum huius curuae ſit

$$\partial s = \partial \Phi \surd (a\,a \sin.\Phi^{2n-2} + b\,b \cos.\Phi^{2n-2}),$$

 euidens

euidens eſt huic conditioni ſatisfieri, ſi elementa coordinatarum, quae primo ſint X et Y, ita conſtituantur:

$$\partial X = a\,\partial \Phi \text{ ſin. } \Phi^{n-1} \text{ et } \partial Y = b\,\partial \Phi \text{ coſ. } \Phi^{n-1};$$

quandoquidem hinc manifeſto fit $\partial X^2 + \partial Y^2 = \partial s^2$. Verum quia hae formulae, pauciſſimis caſibus exceptis, non forent integrabiles, eae noſtro inſtituto minus inſeruiunt; at vero ex iis alias coordinatas, quae ſint x et y, formare licebit, vbi integratio certe ſuccedet. Quodſi enim in genere ſtatuamus

$$\partial x = \partial X \text{ coſ. } \omega - \partial Y \text{ ſin. } \omega \text{ et}$$
$$\partial y = \partial X \text{ ſin. } \omega + \partial Y \text{ coſ. } \omega$$

hinc vtique fiet

$$\partial x^2 + \partial y^2 = \partial X^2 + \partial Y^2 = \partial s^2.$$

Hae autem ſingulae partes reuera integrationem admittent, ſi capiamus $\omega = (n + 2i + 1)\Phi$; quamobrem, ſi loco ∂X et ∂Y valores aſſumtos reſtituamus, ambae coordinatae x et y ita algebraice exprimentur, vt ſit

$$x = a \int \partial \Phi \text{ ſin. } \Phi^{n-1} \text{ coſ. } (n + 2i + 1)\Phi$$
$$- b \int \partial \Phi \text{ coſ. } \Phi^{n-1} \text{ ſin. } (n + 2i + 1)\Phi \text{ et}$$
$$y = a \int \partial \Phi \text{ ſin. } \Phi^{n-1} \text{ ſin. } (n + 2i + 1)\Phi$$
$$+ b \int \partial \Phi \text{ coſ. } \Phi^{n-1} \text{ coſ. } (n + 2i + 1)\Phi$$

vbi hae quatuor formulae integrales ope noſtrorum theorematum algebraice exhiberi poterunt, ita vt, dum pro i omnes numeros integros poſitiuos, non excepta zyphra, aſſumere licet, infinitae curuae algebraicae problemati ſatisfacientes aſſignari poterunt, quarum ſimpliciſſima, ſumendo $i = 0$, erit his formulis contenta:

$$x = \tfrac{a}{n} \text{ ſin. } \Phi^n \text{ coſ. } n\Phi + \tfrac{b}{n} \text{ coſ. } \Phi^n \text{ coſ. } n\Phi \text{ et}$$
$$y = \tfrac{a}{n} \text{ ſin. } \Phi^n \text{ ſin. } n\Phi + \tfrac{b}{n} \text{ coſ. } \Phi^n \text{ ſin. } n\Phi,$$

quae

quae ergo ſuccincte ita referri poſſunt, vt ſit

$$x = \frac{1}{n} \text{cof.}\, n\,\Phi\, (a \text{ fin. } \Phi^n + b \text{ cof. } \Phi^n) \text{ et}$$

$$y = \frac{1}{n} \text{fin.}\, n\,\Phi\, (a \text{ fin. } \Phi^n + b \text{ cof. } \Phi^n).$$

Hinc patet fore

$$\frac{y}{x} = \text{tang.}\, \Phi \text{ et } \sqrt{(x\,x + y\,y)} = \frac{1}{n} (a \text{ fin. } \Phi^n + b \text{ cof. } \Phi^n).$$

Vnde haud difficile erit pro quouis caſu aequationem inter ipſas coordinatas x et y elicere.

Corollarium 1.

§. 24. Elementum curuae

$$\partial s = \partial \Phi \sqrt{(a\,a \text{ fin. } \Phi^{2n-2} + b\,b \text{ cof. } \Phi^{2n-2})}$$

in plures alias formas notatu dignas transfundere licet. Veluti ſi ponatur fin. $\Phi = v$, ob $\partial \Phi = \frac{\partial v}{\sqrt{(1 - v\,v)}}$, erit

$$\partial s = \frac{\partial v}{\sqrt{(1-v\,v)}} \sqrt{[a\,a\,v^{2n-2} + b\,b\,(1 - v\,v)^{n-1}]},$$

vbi operae pretium eſt notaſſe, caſu $n = 2$ fieri

$$\partial s = \frac{\partial v}{\sqrt{(1-v\,v)}} \sqrt{[(a\,a - b\,b)\,v\,v + b\,b]}$$

qua forma elementum Ellipſeos exprimitur, ita vt ope huius problematis infinitae curuae algebraicae reperiri queant, quarum longitudinem per arcus ellipticos metiri liceat.

Corollarium 2.

§. 25. Pro alia transformatione ponamus

$$\text{fin. } \Phi = \sqrt{\frac{1-v}{2}} \text{ et cof. } \Phi = \sqrt{\frac{1+v}{2}},$$

eritque $\partial \Phi = -\frac{\partial v}{2\sqrt{(1-v\,v)}}$, hincque ergo fiet

$$\partial s = -\frac{\partial v}{2\sqrt{(1-v\,v)}} \sqrt{\left[\frac{a\,a\,(1-v)^{n-1} + b\,b\,(1+v)^{n-1}}{2^{n-1}}\right]},$$

quae formula caſu $n = 2$ abit in hanc:

∂ s

$$\partial s = -\frac{\partial v}{2\sqrt{(1-vv)}}\sqrt{\frac{aa+bb+(bb-aa)v}{2}},$$

qua itidem elementum ellipticum exprimitur.

Corollarium 3.

§. 26. Quodſi porro ponamus tang. $\Phi = t$, erit

$$\text{ſin.}\,\Phi = \frac{t}{\sqrt{(1+tt)}} \text{ et cof.}\,\Phi = \frac{1}{\sqrt{(1+tt)}},$$

tum vero $\partial\Phi = \frac{\partial t}{1+tt}$, quibus ſubſtitutis elementum curuae noſtrae erit

$$\partial s = \frac{\partial t}{1+tt}\sqrt{\frac{aa\,t^{2n+2}+bb}{(1+tt)^{n-1}}}, \text{ ſiue}$$

$$\partial s = \partial t\sqrt{\frac{aa\,t^{2n-2}+bb}{(1+tt)^{n+1}}},$$

vnde ſumendo $n = 2$ iterum prodit elementum ellipticum

$$\partial s = \partial t\sqrt{\frac{aatt+bb}{(1+tt)^3}}.$$

Scholion.

§. 27. Caeterum quoniam in noſtris theorematibus infiniti factores ſunt indicati, per quos quaepiam formula differentialis multiplicata reddatur integrabilis; meminiſſe iuuabit, in elementis calculi integralis methodum tradi ſolere, qua ex cognito vno tali factore innumerabiles alii reperiri poſſunt. Veluti ſi formula differentialis $v\,\partial x$, ducta in quantitatem p, praebeat integrale $\int p\,v\,\partial x = q$, tum, denotante Q functionem quamcunque ipſius q, etiam multiplicator Qp formulam propoſitam $v\,\partial x$ reddet integrabilem. Cum enim ſit $p\,v\,\partial x = \partial q$, erit $Q\,p\,v\,\partial x = Q\,\partial q$; unde quoties formula $\int Q\,\partial q$ eſt integrabilis, etiam factor ille Qp formulam propoſitam $v\,\partial x$ reddet integrabilem. Verum perſpicuum eſt, hunc caſum toto coelo

coelo discrepare a formulis illis integralibus, quas in nostris theorematis attulimus. Nam cum formula $\partial \phi \, \text{sin.} \phi^{n-1}$, ducta in sin. $(n+1)\phi$, praebeat integrale $\frac{1}{n}$ sin. ϕ^n sin. $n\phi$, hinc nemo certe secundum methodum memoratam reliquos multiplicatores idoneos, qui sunt

$$\text{sin.}\,(n+3)\phi;\ \text{sin.}\,(n+5)\phi;\ \text{sin.}\,(n+7)\phi;\ \text{etc.}$$

tum vero etiam

$$\text{cos.}\,(n+1)\phi;\ \text{cos.}\,(n+3)\phi;\ \text{cos.}\,(n+5)\phi;\ \text{etc.}$$

elicere valebit, quamobrem illa theoremata tanto magis omni attentione digna sunt censenda.