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De formulis differentialibus quae per duas pluresve quantitates datas multiplicatae fiant integrabiles

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DE

FORMVLIS DIFFERENTIALIBVS,

QVAE PER DVAS PLVRESVE QVANTITATES DATAS MVLTIPLICATAE FIANT INTEGRABILES.

Auctore

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§. 1.

Iam saepius eiusmodi quaestiones tractaui, quibus curuae algebraicae requiruntur, quarum longitudo per datam formulam integralem exprimatur. Ita nuper (*) infinitas curuas algebraicas mihi quidem assignare licuit, quarum longitudo siue per arcus Parabolicos, siue Ellipticos, mensurari queat; tum vero etiam plures alias formulas, quibus longitudo curuae exprimatur, satis felici successu sum perscrutatus (**). Interim tamen ex his omnibus concludi debet, eam Analyseos partem, ad quam huiusmodi quaestiones sunt referendae, minime adhuc esse satis excultam, atque adeo etiamnunc quasi prima principia latere, vnde huiusmodi quaestionum solutionem peti oporteat. Plurimum igitur ad fines Analyseos promouendos conferre putandum est, si hoc argumentum Geometrae omni cura vlterius prosequi dignabuntur.

(*) C. Nov. Act. Acad. Tom. V. pro Anno 1787.
(**) C. Nov. Act. Acad. Tom. VI. pro Anno 1788.

§. 2. Quando autem quaeſtio proponitur de curuis Algebraicis inueniendis, quarum elementum indefinitum formula differentiali quadam praeſcripta ∂s exprimatur, totum negotium eo reducitur, vt angulus quispiam Φ inueſtigetur, ex quo hae duae formulae differentiales ∂s ſin. Φ et ∂s coſ. Φ integrabiles euadant. Quae inueſtigatio cum in genere ne ſuſcipi quidem queat, quaeſtionem inuerſam accuratius tractaſſe iuuabit, qua omnes eae formulae differentiales exquiruntur, quae tam per ſin. Φ quam per coſ. Φ multiplicatae reddantur integrabiles, cuius reſolutio cum nulla amplius laboret difficultate, eam in latiori ſenſu acceptam euoluamus, quo loco formularum ſin. Φ et coſ. Φ aliae quantitates quaecunque proponuntur. Quin etiam iſtam quaeſtionem ad tres pluresue huiusmodi quantitates extendamus. Quanquam autem methodum huiusmodi problemata ſoluendi iam ante complures annos adumbraui, qua noua quaedam pars Analyſeos Infinitorum, quam *indeterminatam* appellare liceat, conſtitui eſt cenſenda, tamen quoniam hoc argumentum tum nimis generaliter eſt tractatum, nunc operae pretium erit id maiori cura propius ad praeſens inſtitutum accommodare.

Problema 1.

Inueſtigare omnes formulas differentiales, quae per datas duas quantitates propoſitas multiplicatae reddantur integrabiles.

Solutio.

§. 3. Deſignemus formulam differentialem quaeſitam charactere ∂W, ſintque p et q bini illi multiplicatores dati, quibus haec formula integrabilis reddi debeat; ita vt hae duae formulae integrales: $\int p \partial W$ et $\int q \partial W$ euadant quantitates algebraicae. Denotent igitur P et Q iſtas quantitates algebraicas, vt ſit $\int p \partial W = P$ et $\int q \partial W = Q$, atque hinc duplici mo-

modo ſponte elicitur $\partial W = \frac{\partial P}{p}$ et $\partial W = \frac{\partial Q}{q}$. Nunc igitur quaeſtio perducta eſt ad duas quantitates algebraicas P et Q inueſtigandas, quarum differentialia inter ſe teneant datam rationem vt $p : q$, ſiue vt ſit $\frac{\partial P}{\partial Q} = \frac{p}{q}$.

§. 4. Iſta quidem conditio certo reſpectu facillime adimpleri poteſt, ita vt adeo relatio quaecunque inter quantitates P et Q ſtatui poſſit. Namque ſi curua $a\,q$ ita referat binas quantitates datas p et q, vt ſumta abſciſſa $a\,p = p$, applicata $p\,q$ fiat $= q$, ſuper eodem axe conſtruatur pro lubitu curua quaecunque A Q, ac ſumto in priori curua puncto quocunque q, ductaque chorda $a\,q$, in altera curua capiatur punctum Q, ad quod ducta tangens Q T illi cordae $a\,q$ fiat parallela; quo facto coordinatae huius alterius curuae A P et P Q exhibebunt ipſas quantitates quaeſitas P et Q. Si enim ponamus $AP = P$ et $PQ = Q$, in triangulo P Q T vtique erit $PQ : PT = \partial Q : \partial P$. Cum igitur hoc triangulum ſimile ſit triangulo $p\,q\,a$, erit $\partial Q : \partial P = q : p$, quae eſt ipſa proportio requiſita. Tab. I. Fig. 1.

§. 5. Verum haec conſtructio, licet facilis ac plana, ad inſtitutum noſtrum parum confert; propterea quod inuentio puncti Q poſtulat reſolutionem aequationum cuiusque ordinis, quae tamen neutiquam eſt in noſtra poteſtate. Nam ſi natura curvae A Q hac tantum aequatione exprimatur:

$$Q = \alpha P + \beta P^2 + \gamma P^6$$

hinc fiet

$$\frac{\partial Q}{\partial P} = \alpha + 2\beta P + 6\gamma P^5$$

fractioni $\frac{q}{p}$ aequalis ſtatuenda, ita vt quantitas P erui debeat ex hac aequatione ordinis quinti: $\alpha + 2\beta P + 6\gamma P^5 = \frac{q}{p}$, cuius reſolutio vtique vires Algebrae ſuperat. Multo maiorem autem difficultatem offendemus, ſi aequatio inter P et Q magis fuerit complicata, in eaque etiam altiores potestates ipsius Q occurrant; quamobrem ad inueniendas quantitates P et Q longe alia via nobis est ineunda.

§. 6. Cum igitur haec aequatio resoluenda proponatur $\frac{\partial Q}{\partial P} = \frac{q}{p}$; vbi quantitates p et q vt datae spectantur: ponamus breuitatis gratia $\frac{q}{p} = t$, vt esse debeat $\frac{\partial Q}{\partial P} = t$, ideoque $\partial Q = t \partial P$, quae formula cum integrabilis esse debeat, ob $Q = \int t \partial P$ per reductiones notissimas habebimus $Q = tP - \int P \partial t$; ita vt tantum formula $\int P \partial t$ integrabilis sit reddenda, id quod facillime praestatur, ponendo $\int P \partial t = T$. Hinc enim fiet $P = \frac{\partial T}{\partial t}$; vnde, quaecunque functio algebraica ipsius t pro T accipiatur, semper idoneum valorem pro quantitate P adipiscimur, scilicet $P = \frac{\partial T}{\partial t}$; ex quo porro elicimus $Q = \frac{t \partial T}{\partial t} - T$, sicque plene satisfactum erit conditioni requisitae: $\frac{\partial Q}{\partial P} = \frac{q}{p} = t$. Sumto enim elemento ∂t constanti, erit

$$\partial P = \frac{\partial \partial T}{\partial t} \text{ et } \partial Q = \partial T + \frac{t \partial \partial T}{\partial t} - \partial T = \frac{t \partial \partial T}{\partial t}$$

vnde manifesto prodit $\frac{\partial Q}{\partial P} = t$, vti requiritur. Eadem autem aequalitas prodiisset, etiamsi elementum ∂t non fuisset constans assumtum; tum enim prodiisset

$$\partial P = \frac{\partial t . \partial \partial T - \partial T \partial \partial t}{\partial t^2} \text{ et}$$

$$\partial Q = \frac{\partial t^2 \partial T + t \partial t \partial \partial T - t \partial T \partial \partial t}{\partial t^2} - \partial T = \frac{t \partial t \partial \partial T - t \partial T \partial \partial t}{\partial t^2}$$

vnde iterum colligitur $\frac{\partial Q}{\partial P} = t$, vt ante.

§. 7. Inuentis autem duabus quantitatibus P et Q ipsa formula differentialis quaesita ∂W duplici modo expressa habetur, scilicet vel $\partial W = \frac{\partial P}{p}$ vel $\partial W = \frac{\partial Q}{q}$, quae autem necessario ad eandem expressionem deducere debent; ex vtraque enim colligitur fore $\partial W = \frac{\partial t \partial \partial T - \partial T \partial \partial t}{p \partial t^2}$. Cum autem hic sit fit $t = \frac{q}{p}$, erit $\partial t = \frac{p\,\partial q - q\,\partial p}{pp}$; vnde cum fit

$$P = \frac{\partial T}{\partial t}, \text{ erit } P = \frac{pp\,\partial T}{p\,\partial q - q\,\partial p},$$

vnde reperimus

$$\partial P = \frac{pp\,\partial\partial T}{p\,\partial q - q\,\partial p} + \frac{2p\,\partial p\,\partial T}{p\,\partial q - q\,\partial p} - \frac{pp\,\partial T(p\,\partial\partial q - q\,\partial\partial p)}{(p\,\partial q - q\,\partial p)^2}$$

hincque denique ipsa formula differentialis quaesita erit

$$\partial W = \frac{p\,\partial\partial T}{p\,\partial q - q\,\partial p} + \frac{2\,\partial p\,\partial T}{p\,\partial q - q\,\partial p} - \frac{p\,\partial T(p\,\partial\partial q - q\,\partial\partial p)}{(p\,\partial q - q\,\partial p)^2}$$

tum autem necessario fiet, vti constituimus

$$\int p\,\partial W = P = \frac{pp\,\partial T}{p\,\partial q - q\,\partial p} \text{ et}$$

$$\int q\,\partial W = Q = \frac{pq\,\partial T}{p\,\partial q - q\,\partial p} - T.$$

Alia Solutio.

§. 8. Quoniam ambo multiplicatores praescripti q et p aequaliter in computum ingredi debebant, quod tamen in solutione inuenta longe secus euenit, vbi altera harum quantitatum p longe alia ratione inest atque altera q, operae pretium erit eiusmodi solutionem tradere, in quam ambae quantitates p et q pari ratione ingrediantur, ita vt, facta earum permutatione, formula pro ∂W inuenta nullam alterationem patiatur, quandoquidem haec circumstantia ad elegantiam solutionis pertinere est censenda, licet solutio ante inuenta in se spectata quaestioni pariter perfecte satisfaciat.

§. 9. Maneant igitur in praecedente solutione omnia eadem vsque ad introductionem litterae T; et quoniam peruenimus ad hanc aequationem: $Q = tP - \int P\,\partial t$, vbi ob $t = \frac{q}{p}$ est $\partial t = \frac{p\,\partial q - q\,\partial p}{pp}$, quae formula denominatorem habet pp, statuamus $\int P\,\partial t = \frac{v}{p}$, vt differentiando prodeat

$$\frac{P(p\,\partial q - q\,\partial p)}{pp} = \frac{p\,\partial v - v\,\partial p}{pp}$$

sicque

ſicque obtinebimus

$$P = \frac{p\,\partial v - v\,\partial p}{p\,\partial q - q\,\partial p},$$

ex quo valore porro deducimus

$$Q = \frac{q}{p} \cdot \frac{p\,\partial v - v\,\partial p}{p\,\partial q - q\,\partial p} - \frac{v}{p},$$

quae expreſſio reducitur ad hanc:

$$Q = \frac{q\,\partial v - v\,\partial q}{p\,\partial q - q\,\partial p}$$

quae alteri P perfecte eſt analoga, dum valor ipſius Q ex P ſponte prodit permutatione litterarum p et q, ſolo ſigno excepto.

§. 10. Cum igitur inuenerimus $P = \frac{p\,\partial v - v\,\partial p}{p\,\partial q - q\,\partial p}$, per differentiationem nanciſcimur

$$\partial P = \frac{(p\,\partial q - q\,\partial p)(p\,\partial\partial v - v\,\partial\partial p) - (p\,\partial v - v\,\partial p)(p\,\partial\partial q - q\,\partial\partial p)}{(p\,\partial q - q\,\partial p)^2}$$

quae expreſſio, facta euolutione, reducitur ad hanc:

$$\partial P = \frac{p\,\partial\partial v(p\,\partial q - q\,\partial p) - p\,\partial v(p\,\partial\partial q - q\,\partial\partial p) + p v(\partial p\,\partial\partial q - \partial q\,\partial\partial p)}{(p\,\partial q - q\,\partial p)^2}$$

Hinc igitur formula differentialis quaeſita ∂W ita exprimetur, vt ſit

$$\partial W = \frac{\partial\partial v(p\,\partial q - q\,\partial p) - \partial v(p\,\partial\partial q - q\,\partial\partial p) + v(\partial p\,\partial\partial q - \partial q\,\partial\partial p)}{(p\,\partial q - q\,\partial p)^2};$$

vbi ambae quantitates p et q manifeſto ſunt permutabiles, ſi quidem mutatio ſignorum nullum diſcrimen afferre eſt cenſenda.

§. 11. Iſta ſolutio non ſolum antecedentem ſupereminet inſigni elegantia, ſed etiam pariter eſt maxime generalis, quandoquidem quantitas v arbitrio noſtro penitus relinquitur, ideoque eius loco omnes plane functiones ipſarum p et q accipi poſſunt. At vero iſta expreſſio conditiones praeſcriptas ita adimplet, vt inde fiat

$$\int p\,\partial W = P = \frac{p\,\partial v - v\,\partial p}{p\,\partial q - q\,\partial p} \text{ et}$$

$$\int q\,\partial W = Q = \frac{q\,\partial v - v\,\partial q}{p\,\partial q - q\,\partial p}.$$

Quae

Quae ambae expreſſiones vtique ſunt algebraicae, dummodo pro v functiones algebraicae ipſarum p et q accipiantur.

§. 12. Iſti valores integrales nobis inſuper duas inſignes proprietates formulae differentialis inuentae declarant, quae in eo conſiſtunt, vt iſta formula ad nihilum redigatur, tam poſito $v = p$ quam $v = q$. Cum enim formula integralis $\int p\,\partial W$ manifeſto euaneſcat poſito $v = p$, neceſſe eſt vt etiam formula differentialis eodem caſu euaneſcat; quod idem de altera formula integrali eſt tenendum, quae caſu $v = q$ euaneſcit.

Corollarium.

§. 13. Quoniam ſolutio huius problematis eo eſt perducta, vt binae quantitates P et Q inveſtigentur, quarum differentialia ∂P et ∂Q datam inter ſe teneant rationem, vt $p : q$, operae pretium erit poſteriorem ſolutionem ſub forma theorematis memoriae imprimi.

Theorema.

§. 14. Si duae quantitates P et Q deſiderentur, quarum differentialia ∂P et ∂Q eandem inter ſe teneant rationem quam duae quantitates datae p et q, ita vt eſſe debeat $\frac{\partial P}{\partial Q} = \frac{p}{q}$, huic requiſito generaliſſime ſatisfiet, ſumendo

$$P = \frac{p\,\partial v - v\,\partial p}{p\,\partial q - q\,\partial p} \quad \text{et} \quad Q = \frac{q\,\partial v - v\,\partial q}{p\,\partial q - q\,\partial p}$$

vbi quantitas v penitus arbitrio noſtro eſt relicta.

Exemplum 1.

§. 15. *Inuenire formulam differentialem ∂W, quae tam per ſinum quam per coſinum cuiuspiam anguli variabilis ϕ multiplicata euadat integrabilis.*

Hic ergo erit $p = \text{fin.}\,\Phi$ et $q = \text{cof.}\,\Phi$, hincque differentiando (vbi quidem elementum $\partial\Phi$ conftans affumamus) prodibit

$$\partial p = \partial\Phi\,\text{cof.}\,\Phi \text{ et } \partial q = -\partial\Phi\,\text{fin.}\,\Phi$$

porroque

$$\partial\partial p = -\partial\Phi^2\,\text{fin.}\,\Phi \text{ et } \partial\partial q = -\partial\Phi^2\,\text{cof.}\,\Phi$$

vnde formulae, quae in expreffionem ipfius ∂W ingrediuntur, fequentes valores fortientur:

I. $p\,\partial q - q\,\partial p = -\partial\Phi$,

II. $p\,\partial\partial q - q\,\partial\partial p = 0$,

III. $\partial p\,\partial\partial q - \partial q\,\partial\partial p = -\partial\Phi^3$,

ex quibus valoribus ergo concluditur formula differentialis quaefita

$$\partial W = -\frac{\partial\partial v}{\partial\Phi} - v\,\partial\Phi.$$

§. 16. Quod haec formula prodiit negatiua, negotium nullo modo turbat, ac tuto ftatuere poterimus

$$\partial W = v\,\partial\Phi + \frac{\partial\partial v}{\partial\Phi}$$

fumto fcilicet elemento $\partial\Phi$ conftante; tum autem pariter, mutatis fignis, erit

$$\int\partial W\,\text{fin.}\,\Phi = \frac{\partial v\,\text{fin.}\,\Phi}{\partial\Phi} - v\,\text{cof.}\,\Phi \text{ et}$$

$$\int\partial W\,\text{cof.}\,\Phi = \frac{\partial v\,\text{cof.}\,\Phi}{\partial\Phi} + v\,\text{fin.}\,\Phi.$$

Tum vero etiam euidens eft ipfam formulam ∂W euanefcere, tam cafu $v = \text{fin.}\,\Phi$ quam cafu $v = \text{cof.}\,\Phi$.

§. 17. Quodfi ergo formula ∂W exprimat elementum cuiuspiam lineae curuae ∂s, vt fit $\partial s = v\,\partial\Phi + \frac{\partial\partial v}{\partial\Phi}$, quaecunque functio algebraica fuerit v, femper curua algebraica exhiberi poterit. Conftitutis enim coordinatis orthogonalibus x et y, fi si sumatur $\partial x = \partial s \cos. \phi$ et $\partial y = \partial s \sin. \phi$, vt fiat $\partial x^2 + \partial y^2 = \partial s^2$, quia ambae hae formulae sunt integrabiles, coordinatae curuae quaesitae erunt

$$x = \frac{\partial v \cos. \phi}{\partial \phi} + v \sin. \phi \text{ et } y = \frac{\partial v \sin. \phi}{\partial \phi} - v \cos. \phi,$$

vbi notasse iuuabit fore $xx + yy = \frac{\partial v^2}{\partial \phi^2} + vv$. Praeterea vero si formula $\int v \partial \phi$ integrationem admittat, tum curua ipsa erit rectificabilis; fiet enim $s = \int v \partial \phi + \frac{\partial v}{\partial \phi}$.

Scholion.

§. 18. Quo insignis vsus huius tractationis vberius ob oculos ponatur, tam huic exemplo, quam sequentibus, cuique problema speciale adiungamus, in quo integratio cuiuspiam aequationis differentialis secundi gradus perficiatur; quod saepissime egregium vsum habere poterit.

Problema speciale 1.

§. 19. *Si Φ denotet functionem quamcunque ipsius ϕ, resoluere istam aequationem differentialem secundi gradus:*

$$v \partial \phi + \frac{\partial \partial v}{\partial \phi} = \Phi \partial \phi$$

in qua elementum $\partial \phi$ constans est assumtum, siue per integrationem inuenire valorem ipsius v.

Quia iam inuenimus huius aequationis membrum sinistrum integrabile fieri duobus casibus, dum vel per sin. ϕ vel cos. ϕ multiplicatur, membrum autem dextrum iam est functio ipsius ϕ tantum; eius integratio nulla laborat difficultate. Primo enim haec aequatio in sin. ϕ ducta et integrata dabit

$$\frac{\partial v \sin. \phi}{\partial \phi} - v \cos. \phi = \int \Phi \partial \phi \sin. \phi;$$

at vero multiplicatio per cos. ϕ praebebit

$$\frac{\partial v \cos. \phi}{\partial \phi} + v \sin. \phi = \int \Phi \partial \phi \cos. \phi.$$

§. 20.

§. 20. Cum igitur iam geminam habeamus aequationem primi gradus, fine vlla vlteriori integratione valorem quantitatis v elicere poterimus: prior enim in cof. ϕ ducta et a posteriori in fin. ϕ ducta ablata perducet ad hanc aequationem:

$$v = \text{fin.}\,\phi \int \Phi\, \partial\phi\, \text{cof.}\,\phi - \text{cof.}\,\phi \int \Phi\, \partial\phi\, \text{fin.}\,\phi,$$

quod integrale vtique est completum, propterea quod, ob binas formulas integrales, geminam constantem arbitrariam inuoluit.

§. 21. Quoniam per reductiones notissimas est

$$\int \Phi\, \partial\phi\, \text{fin.}\,\phi = -\Phi\, \text{cof.}\,\phi + \int \partial\Phi\, \text{cof.}\,\phi \text{ et}$$

$$\int \Phi\, \partial\phi\, \text{cof.}\,\phi = \Phi\, \text{fin.}\,\phi - \int \partial\Phi\, \text{fin.}\,\phi,$$

fi hi valores fubstituantur, valor ipsius v etiam hoc modo exprimi poterit:

$$v = \Phi - \text{fin.}\,\phi \int \partial\Phi\, \text{fin.}\,\phi - \text{cof.}\,\phi \int \partial\Phi\, \text{cof.}\,\phi.$$

Exemplum 2.

§. 22. *Inuenire formulam differenticlem ∂W, quae tam per tangentem, quam fecantem cuiuspiam anguli variabilis ϕ multiplicata euadat integrabilis.*

Hic igitur esto $p = \text{tang.}\,\phi$ et $q = \text{fec.}\,\phi$, unde fequitur

$$\partial p = \frac{\partial\phi}{\text{cof.}\,\phi^2} \text{ et } \partial q = \frac{\partial\phi\, \text{fin.}\,\phi}{\text{cof.}\,\phi^2},$$

porro vero

$$\partial\partial p = \frac{2\,\partial\phi^2\, \text{fin.}\,\phi}{\text{cof.}\,\phi^3} \text{ et } \partial\partial q = \frac{\partial\phi^2}{\text{cof.}\,\phi} + \frac{2\,\partial\phi^2\, \text{fin.}\,\phi^2}{\text{cof.}\,\phi^3}, \text{ fiue}$$

$$\partial\partial q = \frac{2\,\partial\phi^2}{\text{cof}\,\phi^3} - \frac{\partial\phi^2}{\text{cof.}\,\phi}.$$

Hinc igitur colliguntur fequentes aequationes:

I. $p\,\partial q - q\,\partial p = -\frac{\partial\phi}{\text{cof.}\,\phi}$.

II. $p\,\partial\partial q - q\,\partial\partial p = -\frac{\partial\phi^2}{\text{cof.}\,\phi}\,\text{tang.}\,\phi = -\frac{\partial\phi^2\, \text{fin.}\,\phi}{\text{cof.}\,\phi^2}$.

III. $\partial p\,\partial\partial q - \partial q\,\partial\partial p = \frac{\partial\phi^3}{\text{cof.}\,\phi^3}$.

Ex quibus ergo valoribus concluditur formula differentialis quaesita

$$\partial W = -\frac{\partial \partial v \cos. \Phi}{\partial \Phi} + \partial v \sin. \Phi + \frac{v \partial \Phi}{\cos. \Phi}.$$

§. 23. Ternos autem illos valores facilius hoc modo reperire licet. Primo enim cum sit $\frac{q}{p} = \frac{1}{\sin. \Phi}$, erit differentiando

$$\frac{p \partial q - q \partial p}{p p} = -\frac{\partial \Phi \cos. \Phi}{\sin. \Phi^2},$$

vnde per $p p = \frac{\sin. \Phi^2}{\cos. \Phi^2}$ multiplicando oritur

$$p \partial q - q \partial p = -\frac{\partial \Phi}{\cos. \Phi},$$

quae denuo differentiata dat

$$p \partial \partial q - q \partial \partial p = -\frac{\partial \Phi^2 \sin. \Phi}{\cos. \Phi^2}.$$

Deinde cum sit $\frac{\partial q}{\partial p} = \sin. \Phi$, erit differentiando

$$\frac{\partial p \partial \partial q - \partial q \partial \partial p}{\partial p^2} = \partial \Phi \cos. \Phi,$$

quae per $\partial p^2 = \frac{\partial \Phi^2}{\cos. \Phi^4}$ multiplicata dat

$$\partial p \partial \partial q - \partial q \partial \partial p = \frac{\partial \Phi^3}{\cos. \Phi^3}.$$

Cum igitur sit

$$\partial W = -\frac{\partial \partial v \cos. \Phi}{\partial \Phi} + \partial v \sin. \Phi + \frac{v \partial \Phi}{\cos. \Phi}, \text{ erit}$$

$$\int p \partial W = P = \frac{v}{\cos. \Phi} - \frac{\partial v \sin. \Phi}{\partial \Phi} \text{ et}$$

$$\int q \partial W = Q = \frac{v \sin. \Phi}{\cos. \Phi} - \frac{\partial v}{\partial \Phi}.$$

Problema speciale 2.

§. 24. *Denotante* $\varPhi$ *functionem quamcunque anguli* Φ, *resolvere istam aequationem secundi gradus:*

$$-\frac{\partial \partial v \cos. \Phi}{\partial \Phi} + \partial v \sin. \Phi + \frac{v \partial \Phi}{\cos. \Phi} = \varPhi \partial \Phi.$$

Hic primo per tang. Φ multiplicando et integrando obtinetur haec aequatio primi gradus:

$$\frac{v}{\cos. \Phi} - \frac{\partial v \sin. \Phi}{\partial \Phi} = \int \varPhi \partial \Phi \text{ tang. } \Phi;$$

at

at vero per sec. Φ multiplicando et integrando prodit

$$\frac{v\,\sin.\Phi}{\cos.\Phi} - \frac{\partial v}{\partial\Phi} = \int \Phi\,\partial\Phi \,\sec.\Phi.$$

Haec posterior ducta in sin. Φ et a priori subtracta dat

$$v \cos.\Phi = \int \Phi\,\partial\Phi\,\mathrm{tang.}\Phi - \sin.\Phi \int \Phi\,\partial\Phi\,\sec.\Phi$$

ideoque

$$v = \sec.\Phi \int \Phi\,\partial\Phi\,\mathrm{tg.}\Phi - \mathrm{tg.}\Phi \int \Phi\,\partial\Phi\,\sec.\Phi$$

quae est integratio completa aequationis propositae.

§. 25. Denique hic annotasse iuuabit, si Φ denotet angulum, quem curuae cuiuspiam elementum ∂s cum elemento abscissae ∂x constituit, atque ∂W exprimat ipsum elementum abscissae ∂x, tum fore elementum applicatae $\partial y = \partial x$ tang. $\Phi = p\,\partial W$, elementum vero curuae $\partial s = \partial x$ sec. $\Phi = q\,\partial W$, unde ergo habebitur ipsa applicata

$$y = \frac{v}{\cos.\Phi} - \frac{\partial v \sin.\Phi}{\partial\Phi},$$

atque ipsa curuae longitudo erit

$$s = \frac{v \sin.\Phi}{\cos.\Phi} - \frac{\partial v}{\partial\Phi}.$$

Cum igitur sit

$$\partial x = -\frac{\partial\partial v \cos.\Phi}{\partial\Phi} + \partial v \sin.\Phi + \frac{v\,\partial\Phi}{\cos.\Phi},$$

si modo haec formula etiam integrationem admittat, tum prodibit curua algebraica, simulque rectificabilis. Jam vero integrando, qua fieri licet, prodit

$$x = -\frac{\partial v}{\partial\Phi}\cos.\Phi - \int \partial v \sin.\Phi + \int \partial v \sin.\Phi + \int \frac{v\,\partial\Phi}{\cos.\Phi} = -\frac{\partial v \cos.\Phi}{\partial\Phi} + \int \frac{v\,\partial\Phi}{\cos.\Phi}$$

Quam ob rem necesse est vt formula $\int \frac{v\,\partial\Phi}{\cos.\Phi}$ integrationem admittat, veluti euenit, si sumatur $v = \cos.\Phi^2$, tum enim erit $\int \frac{v\,\partial\Phi}{\cos.\Phi} = \sin.\Phi$, atque ob

$$\partial v = -2\,\partial\Phi \sin.\Phi \cos.\Phi$$

prodibit

$$x = 2 \sin. \phi \cos. \phi^2 + \sin. \phi = \sin. \phi (1 + 2 \cos. \phi^2),$$

tum vero erit

$$y = \cos. \phi (1 + 2 \sin. \phi^2)$$

et arcus curuae

$$s = 3 \sin. \phi \cos. \phi.$$

Exemplum 3.

§. 26. *Inuenire formulam differentialem* ∂W, *quae siue multiplicata siue diuisa per datam quantitatem* t *euadat integrabilis.*

Hic ergo hae duae formulae $t \partial W$ et $\frac{\partial W}{t}$ reddi debent integrabiles. Pro hoc igitur exemplo erit $p = t$ et $q = \frac{1}{t}$, unde fit $\partial p = \partial t$ et $\partial q = -\frac{\partial t}{tt}$. Cum ergo sit $\frac{q}{p} = \frac{1}{tt}$, erit

$$\frac{p \partial q - q \partial p}{pp} = -\frac{2 \partial t}{t^3},$$

unde per $pp = tt$ multiplicando fit

$$p \partial q - q \partial p = -\frac{2 \partial t}{t};$$

quae formula porro differentiata, sumendo ∂t constans, praebet

$$p \partial \partial q - q \partial \partial p = + \frac{2 \partial t^2}{tt}.$$

Deinde quia est $\frac{\partial q}{\partial p} = -\frac{1}{tt}$, fiet

$$\frac{\partial p \partial \partial q - \partial q \partial \partial p}{\partial p^2} = \frac{2 \partial t}{t^3},$$

quae aequatio per ∂p^2 multiplicata praebet

$$\partial p \partial \partial q - \partial q \partial \partial p = \frac{2 \partial t^3}{t^3},$$

ex quibus valoribus colligitur formula quaesita

$$\partial W = -\frac{t \partial \partial v}{2 \partial t} - \frac{1}{2} \partial v + \frac{v \partial t}{2t}$$

siue hunc valorem duplicando et signa mutando statui potest

$$\partial W = \frac{t \partial \partial v}{\partial t} + \partial v - \frac{v \partial t}{t}.$$

§. 27. Multiplicemus igitur hanc formam per t vt fiat

$t \partial$

$$t\,\partial W = \frac{tt\,\partial\partial v}{\partial t} + t\,\partial v - v\,\partial t,$$

vbi primi membri integratio dat

$$\int t\,\partial W = \frac{tt\,\partial v}{\partial t} - \int(t\,\partial v + v\,\partial t)$$

ſicque prodit

$$\int t\,\partial W = \frac{tt\,\partial v}{\partial t} - t\,v.$$

Simili modo cum ſit

$$\frac{\partial W}{t} = \frac{\partial\partial v}{\partial t} + \frac{\partial v}{t} - \frac{v\,\partial t}{tt},$$

erit integrando

$$\int\frac{\partial W}{t} = \frac{\partial v}{\partial t} + \int\frac{(t\,\partial v - v\,\partial t)}{tt} = \frac{\partial v}{\partial t} + \frac{v}{t}.$$

Problema ſpeciale 3.

§. 28. *Denotante T functionem quamcunque ipſius t, reſolvere aequationem ſecundi gradus:*

$$\frac{t\,\partial\partial v}{\partial t} + \partial v - \frac{v\,\partial t}{t} = T\,\partial t.$$

Haec reſolutio per praecedentia facile expeditur; primo enim per t multiplicando et integrando prodit

$$\frac{tt\,\partial v}{\partial t} - t\,v = \int T\,t\,\partial t.$$

At vero per t diuidendo et integrando fit

$$\frac{\partial v}{\partial t} + \frac{v}{t} = \int\frac{T\,\partial t}{t},$$

vnde eliminando terminum $\frac{\partial v}{\partial t}$ oritur

$$v = \tfrac{1}{2}\,t\int\frac{T\,\partial t}{t} - \frac{1}{2t}\int T\,t\,\partial t$$

quae expreſſio quomodo ſatisfaciat, videamus. Primo hinc erit

$$\frac{\partial v}{\partial t} = \tfrac{1}{2}\int\frac{T\,\partial t}{t} + \frac{1}{2tt}\int T\,t\,\partial t$$

quae aequatio denuo differentiata dat

$$\frac{\partial\partial v}{\partial t^2} = \frac{T}{t} - \frac{1}{t^3}\int T\,t\,\partial t.$$

His

His igitur valoribus ſubſtitutis prodit

$$\frac{t\,\partial\partial v}{\partial t} = T\,\partial t - \frac{\partial t}{tt}\int T\,t\,\partial t$$
$$+\,\partial v = \frac{\partial t}{2}\int\frac{T\,\partial t}{t} + \frac{\partial t}{2tt}\int T\,t\,\partial t$$
$$-\,\frac{v\,\partial t}{t} = -\,\frac{\partial t}{2}\int\frac{T\,\partial t}{t} + \frac{\partial t}{2tt}\int T\,t\,\partial t$$

Summa $= T\,\partial t$, prorſus vt ante.

Scholion.

§. 29. Duo priora exempla, quae hic attulimus, inſignem vſum praeſtant in curuarum indole reſpectu rectificationis exploranda, vti oſtendimus; tertium autem exemplum ideo eſt notatu dignum, quod in huiusmodi inueſtigationibus ſaepius eiusmodi formulae differentiales occurrunt, quas, per eandem quantitatem tam multiplicatas quam diuiſas, reddi oportet integrabiles. Veluti ſi in ſuperficie cylindri recti praeter rectas axi parallelas aliae lineae duci debeant, quae ſint rectificabiles, quaeſtio ad inuentionem eiusmodi quantitatis algebraicae t reducitur, per quam iſta formula differentialis $\frac{\partial v}{\sqrt{(1-vv)}}$ tam multiplicata quam diuiſa integrationem admittat. Poſtquam autem plurimum nequicquam in hoc negotio deſudaſſem, aſſeuerare non dubito, nullam plane dari eiusmodi quantitatem t, qua hae duae formulae: $\frac{t\,\partial v}{\sqrt{(1-vv)}}$ et $\frac{\partial v}{t\sqrt{(1-vv)}}$ ſimul fiant integrabiles. Praeterea vero omnino certum mihi videtur, praeter ſimplices poteſtates ipſius v nullas alias eius functiones t dari, unde hae duae formulae differentiales: $\frac{t\,\partial v}{v}$ et $\frac{\partial v}{t\,v}$ ſimul euadant integrabiles.

Problema.

§. 30. *Si formula differentialis ∂W vtcunque fuerit compoſita ex quantitatibus variabilibus p et q (inter quas quidem cer-*

ta relatio dari assumitur) definire quantitatem v ex hac aequatione differentiali secundi gradus:

$$\partial W = \frac{\partial\partial v(p\,\partial q - q\,\partial p) - \partial v(p\,\partial\partial q - q\,\partial\partial p) + v(\partial p\,\partial\partial q - \partial q\,\partial\partial p)}{(p\,\partial q - q\,\partial p)^2}.$$

Solutio.

Quia nouimus hanc aequationem integrabilem reddi, si multiplicetur tam per p quam per q: haec duplex integratio nobis duas suppeditat aequationes differentiales primi gradus, quae sunt:

$$\int p\,\partial W = \frac{p\,\partial v - v\,\partial p}{p\,\partial q - q\,\partial p} \text{ et } \int q\,\partial W = \frac{q\,\partial v - v\,\partial q}{p\,\partial q - q\,\partial p},$$

quarum posterior, ducta in p, si subtrahatur a priore ducta in q, praebet sequentem aequationem: $q\int p\,\partial W - p\int q\,\partial W = v$; sicque innotescit valor quaesitus quantitatis v, qui, quoniam geminam integrationem inuoluit, ob duplicem constantem arbitrariam pro integrali completo aequationis differentio-differentialis est habendus.

Problema 2.

Inuenire formulam differentialem ∂W, quae per tres quantitates variabiles datas p, q et r multiplicata fiat integrabilis.

Solutio.

§. 31. Ponamus haec tria integralia, quae prodire debent, esse

$$1^\circ.\int p\,\partial W = P,\quad 2^\circ.\int q\,\partial W = Q,\quad 3^\circ.\int r\,\partial W = R,$$

vnde triplici modo formula differentialis quaesita ∂W exprimetur

$$1^\circ.\ \partial W = \frac{\partial P}{p};\quad 2^\circ.\ \partial W = \frac{\partial Q}{q};\ \text{et}\quad 3^\circ.\ \partial W = \frac{\partial R}{r}$$

§. 32. Jam supra §. 14. vidimus duabus prioribus conditionibus, quibus requiritur vt sit $\frac{\partial P}{\partial Q} = \frac{p}{q}$, satisfieri, si capiatur

$$P = \frac{p\,\partial v - v\,\partial p}{p\,\partial q - q\,\partial p} \text{ et } Q = \frac{q\,\partial v - v\,\partial q}{p\,\partial q - q\,\partial p}.$$

Simili modo conditionibus primae et tertiae, qua requiritur vt ſit $\frac{\partial P}{\partial R} = \frac{p}{r}$, ſatisfiet, loco v aliam quantitatem u ſcribendo, his duobus valoribus:

$$P = \frac{p\,\partial u - u\,\partial p}{p\,\partial r - r\,\partial p} \text{ et } R = \frac{r\,\partial u - u\,\partial r}{p\,\partial r - r\,\partial p}.$$

Nunc igitur totum negotium eo eſt reductum, vt ambo valores pro P inventi ad aequalitatem perducantur; vbi ergo quaeritur, quales quantitates pro v et u accipi debeant, vt iſtae duae formulae pro P inuentae inter ſe euadant aequales, quippe quo facto ſimul etiam bini reliqui valores Q et R innoteſcent.

§. 33. Cum igitur debeat eſſe

$$\frac{p\,\partial v - v\,\partial p}{p\,\partial q - q\,\partial p} = \frac{p\,\partial u - u\,\partial p}{p\,\partial r - r\,\partial p},$$

quo hoc facilius effici poſſit, ſtatuamus $v = Vp$ et $u = Up$, et conditio adimplenda erit

$$\frac{\partial V}{p\,\partial q - q\,\partial p} = \frac{\partial U}{p\,\partial r - r\,\partial p}, \text{ ideoque } \frac{\partial V}{\partial U} = \frac{p\,\partial q - q\,\partial p}{p\,\partial r - r\,\partial p}.$$

Quare ſi theorema ſupra §. 14. datum in ſubſidium vocemus, loco P et Q nunc habemus V et U, at vero loco p et q nunc habemus $p\,\partial q - q\,\partial p$ et $p\,\partial r - r\,\partial p$, ideoque loco v introducendo quantitatem Z haec conditio adimplebitur, ſi ſtatuamus:

$$V = \frac{(p\,\partial q - q\,\partial p)\,\partial Z - Z(p\,\partial\partial q - q\,\partial\partial p)}{(p\,\partial q - q\,\partial p)(p\,\partial\partial r - r\,\partial\partial p) - (p\,\partial r - r\,\partial p)(p\,\partial\partial q - q\,\partial\partial p)};$$

$$U = \frac{(p\,\partial r - r\,\partial p)\,\partial Z - Z(p\,\partial\partial r - r\,\partial\partial p)}{(p\,\partial q - q\,\partial p)(p\,\partial\partial r - r\,\partial\partial p) - (p\,\partial r - r\,\partial p)(p\,\partial\partial q - q\,\partial\partial p)};$$

vbi notetur denominatorem reuocari poſſe ad ſequentem formam ſatis concinnam:

$$p\,\partial\partial p(q\,\partial r - r\,\partial q) + p\,\partial\partial q(r\,\partial p - p\,\partial r) + p\,\partial\partial r(p\,\partial q - q\,\partial p)$$

in qua praeter factorem communem p ternae litterae p, q et r ſunt inter ſe permutabiles.

§. 34. Solutio ergo noſtri problematis ſequenti modo abſoluetur:

1°. Sumta pro lubitu quantitate quacunque variabili Z, quaeratur quantitas V, vt ſit

$$Vp = \frac{(p\partial q - q\partial p)\partial Z - Z(p\partial\partial q - q\partial\partial p)}{\partial\partial p(q\partial r - r\partial q) + \partial\partial q(r\partial p - p\partial r) + \partial\partial r(p\partial q - q\partial p)}$$

qui ſimul eſt valor litterae v.

2°. Eodem modo colligatur valor

$$Up = \frac{(p\partial r - r\partial p)\partial Z - Z(p\partial\partial r - r\partial\partial p)}{\partial\partial p(q\partial r - r\partial q) + \partial\partial q(r\partial p - p\partial r) + \partial\partial r(p\partial q - q\partial p)}$$

qui ſimul eſt valor ipſius u.

3°. Inuentis autem his valoribus pro v et u formentur porro iſti valores:

$$P = \frac{p\partial v - v\partial p}{p\partial q - q\partial p}, \text{ vel etiam } P = \frac{p\partial u - u\partial p}{p\partial r - r\partial p}$$

quippe qui ambo valores ad aequalitatem ſunt perducti; praeterea vero ſumatur

$$Q = \frac{q\partial v - v\partial q}{p\partial q - q\partial p} \text{ et } R = \frac{r\partial u - u\partial r}{p\partial r - r\partial p}.$$

4°. Denique hae ternae formulae $\frac{\partial P}{p}$, $\frac{\partial Q}{q}$, $\frac{\partial R}{r}$ producent expreſſiones inter ſe prorſus aequales, atque adeo valorem formulae differentialis quaeſitae ∂W.

§. 35. Solutio haec adhuc magis contrahi poteſt, ſi, poſitis vt ante $v = Vp$ et $u = Up$, inſuper ſtatuatur $q = px$ et $r = py$, vbi quia q et r tanquam functiones ipſius p ſpectari poſſunt, etiam x et y erunt functiones cognitae ipſius p; tum autem fiet $\frac{\partial V}{\partial U} = \frac{\partial x}{\partial y}$. Quare cum x et y ſint functiones ipſius p, ponatur $\partial x = X\partial p$ et $\partial y = Y\partial p$, ita vt etiam X et Y futurae ſint functiones cognitae ipſius p, hinc autem habebimus hanc aequationem reſolvendam $\frac{\partial V}{\partial U} = \frac{X}{Y}$.

§. 36. Nunc igitur introducendo nouam quantitatem variabilem indefinitam Z theorema noſtrum §. 14. nobis dabit

V

$$V = \frac{X \partial Z - Z \partial X}{X \partial Y - Y \partial X} \text{ et } U = \frac{Y \partial Z - Z \partial Y}{X \partial Y - Y \partial X}.$$

Inuentis autem valoribus V et U ſimul habentur litterae $v = Vp$ et $u = Up$, ex quibus vt ante determinabuntur valores P, Q et R, hincque tandem ipſa formula differentialis quaeſita ∂W.

Corollarium.

§. 37. Quoniam litterae V et U duabus conſtant partibus, altera per Z, altera vero per ∂Z affecta, etiam quantitates v et u duabus huiusmodi partibus conſtabunt; unde earum differentialia inſuper partem ſecundo differentiali $\partial \partial Z$ affectam continebunt. Huiusmodi ergo tres partes in litteris P, Q et R occurrent, quae cum denuo differentiari debeant, vt formula differentialis ∂W eliciatur, euidens eſt in expreſſione ∂W differentialia ipſius Z ad tertium gradum aſſurgere, unde pro ∂W huiusmodi prodibit expreſſio:

$$\partial W = AZ + B \partial Z + C \partial \partial Z + D \partial^3 Z$$

vbi litteras A, B, C et D pro formulis per euolutionem valorum ſupra aſſignatorum poſuimus. Quibus praeceptis applicationi ad caſus particulares non immorari eſt opus.