1790

De motu oscillatorio penduli circa axem cylindricum plano horizontali incumbentem

Leonhard Euler

DE

MOTV OSCILLATORIO PENDVLI CIRCA AXEM CYLINDRICVM PLANO HORIZONTALI INCVMBENTEM.

Auctore
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§. 1.

Pendulum ſuperne traiectum concipiatur cylindro, qui vtroque termino plano horizontali incumbat, quod tantam frictionem opponat, vt nullus motus reptorius ſe admiſcere queat. Dum igitur tale pendulum oſcillationes peragit, cylindrus ſuper plano horizontali alternatim voluendo procedet et recedet, ſicque pendulum motu ſuo nullam aliam reſiſtentiam patietur praeter aërem. Quanquam autem talis motus iam ſaepius eſt tractatus, tamen hic praecipue in vires ſum inquiſiturus, quibus axis cylindricus planum horizontale inter oſcillandum premit, et quantam vim frictio exſerere debeat, vt motus reptorius penitus coërceatur.

§. 2. Sit igitur C centrum axis cylindrici, qui tempore quocunque, ab initio motus elapſo $= t$, plano horizontali QV in O incumbat, ita vt recta COP ſit verticalis. Nunc vero pendulum talem ſitum obliquum teneat, vt, ducta ex C Tab. V. Fig. 1.

 ad

ad eius centrum grauitatis G recta C G, angulus obliquitatis sit O C G $= \Phi$. Ponamus porro radium axis cylindrici C O $= c$, distantiam centri grauitatis a puncto C, hoc est C G $= a$, massam seu pondus totius penduli $=$ M eiusque momentum inertiae respectu axis per G transeuntis et axi cylindri paralleli $= Mkk$. Hinc ergo ducta horizontali G P et verticali G Q erit G P $=$ sin. Φ, C P $= a$ cos. Φ et G Q $= a$ cos. $\Phi - c$.

§. 3. Hoc igitur penduli statu erit arcus O A $= c\Phi$, cui in recta horizontali Q V aequale capiatur interuallum O A $= c\Phi$, eritque A punctum, vbi cylindrus initio motus, quo rectam C G verticalem fuisse assumemus, plano horizontali incumbebat, quod ergo punctum erit fixum, quippe a quo punctum O durante motu modo recedit modo accedit, atque adeo in contrariam partem excurrit, cui motui aequalis erit motus ipsius puncti C, quod in statu penduli verticali ipsi puncto A imminebit et durante motu oscillatorio ab hoc situ modo dextrorsum modo sinistrorsum recedit, quem ergo motum, vna cum proprio motu penduli, accurate determinari oportet, vt totius motus perfectam cognitionem acquiramus. Euidens autem est hoc modo istum motum nondum esse exploratum.

§. 4. Cum igitur non punctum O sed punctum A tanquam fixum spectari debeat, ad id situm puncti G referri conueniet per coordinatas A Q $= x$ et Q G $= y$, atque euidens est fore $x = a$ sin. $\Phi - c\Phi$ et $y = a$ cos. $\Phi - c$, vnde differentiando erit $\partial x = a\partial\Phi$ cos. $\Phi - c\partial\Phi$ et $\partial y = - a\partial\Phi$ sin. Φ, denuoque differentiando

$$\partial\partial x = \quad a\partial\partial\Phi \text{ cos. } \Phi - a\partial\Phi^2 \text{ sin. } \Phi - c\partial\partial\Phi \text{ et}$$

$$\partial\partial y = - a\partial\partial\Phi \text{ cos. } \Phi - a\partial\Phi^2 \text{ cos. } \Phi,$$

vbi sumto elemento temporis ∂t constante erit $\frac{\partial\Phi}{\partial t}$ celeritas angu-

angularis, qua pendulum a situ verticali recedit, et $\frac{\partial\partial\Phi}{\partial t^2}$ erit acceleratio huius motus angularis.

§. 5. His constitutis consideremus vires, quibus pendulum hoc motu sollicitatur. Ac primo quidem ob proprium pondus sustinebit vim $= M$, cuius directio per ipsum centrum grauitatis G deorsum tendit. Deinde vero ob pressionem axis cylindrici in planum horizontale QV vis aderit verticalis axem sursum secundum OC vrgens, quae vis, cum sit adhuc incognita, ponatur $= \Pi$, in directione OC sollicitans, haecque vis in statu aequilibrii vtique aequaretur ponderi totius penduli, foretque $\Pi = M$; durante autem motu oscillatorio mox videbimus, eam modo fore maiorem, modo minorem. Denique ne punctum O super plano horizontali prorepat, necesse est vt in puncto O certa quadam vi horizontaliter vrgeatur, quae cum pariter sit incognita, concipiamus eam in directione OC agere quantitate $= \Theta$, quae etiam erit variabilis atque ab ipso motu penduli potissimum pendebit. Ad perfectam autem totius motus cognitionem plurimum intererit istas duas vires Π et Θ accurate inuestigasse.

§. 6. Secundum praecepta igitur in Mechanica tradita primo inuestigemus motum progressiuum centri grauitatis G, tum vero etiam motum gyratorium circa ipsum centrum grauitatis, quippe cui aequalis erit motus angularis circa punctum C. Motum autem progressiuum secundum binas directiones fixas coordinatarum x et y indagari oportet, vnde pro directione AQ, quia sola vis Θ in ipsa hac directione agere assumitur, principia motus hanc suppeditant formulam: $\frac{\partial\partial x}{2g\,\partial t^2} = \frac{\Theta}{M}$, vbi g est altitudo lapsus grauium vno minuto secundo. Simili modo pro directione QG habetur primo vis promouens, seu ipsum pondus $= M$, praeterea vero vis reagens $= \Pi$, vnde prodit

$\frac{\partial \partial y}{2 g \partial t^2} = \frac{M - \Pi}{M} = 1 - \frac{\Pi}{M}$; pro hoc enim motu progreſſiuo definiendo omnes vires ipſi centro grauitatis G applicatae ſunt concipiendae. Subſtitutis igitur loco $\partial \partial x$ et $\partial \partial y$ valoribus aſſignatis, per differentialia ſecundi gradus binae vires incognitae Π et Θ ita determinabuntur:

$$\frac{\Pi}{M} = 1 + \frac{a \partial \partial \phi \, \text{fin.} \, \phi + a \partial \phi^2 \, \text{cof.} \, \phi}{2 g \partial t^2},$$

$$\frac{\Theta}{M} = \frac{a \partial \partial \phi \, \text{cof.} \, \phi - a \partial \phi^2 \, \text{fin.} \, \phi - c \partial \partial \phi}{2 g \partial t^2}.$$

§. 7. Nunc ad motum angularem determinandum momenta virium reſpectu centri grauitatis G capi oportet, vnde quia primae vis, ſeu ponderis M, directio per ipſum centrum grauitatis tranſit, eius momentum erit nullum. Secunda vis Π, in directione OC agens, momentum habebit $\Pi \cdot GP$, quod ergo erit Πa fin. ϕ, et quia huius vis actione angulus ϕ minuetur, eius momentum negatiue ſumi debet. Tertia vis Θ in directione OQ agens momentum dabit

$$\Theta \cdot GQ = \Theta (a \, \text{cof.} \, \phi - c),$$

cuius actione pariter angulus ϕ minuetur, ideoque negatiue accipi oportet. Quoniam igitur momentum virium per momentum inertiae diuiſum praebet accelerationem angularem, inde orietur iſta aequatio:

$$\frac{\partial \partial \phi}{2 g \partial t^2} = - \frac{\Pi a \, \text{fin.} \, \phi - \Theta (a \, \text{cof.} \, \phi - c)}{M k k},$$

hocque modo nacti ſumus tres aequationes, ex quibus non ſolum ad quoduis tempus angulus ϕ, ſed etiam ambae vires incognitae Π et Θ definiri debebunt.

§. 8. Subſtituamus nunc loco $\frac{\Pi}{M}$ et $\frac{\Theta}{M}$ valores modo ante aſſignatos, et multiplicando per $2 g k k \partial t^2$ reſultabit ſequens aequatio:

k k

$$kk\partial\partial\Phi = -2ag\partial t^2 \text{fin.}\Phi - (aa+cc)\partial\partial\Phi$$
$$+ 2ac\partial\partial\Phi \text{cof.}\Phi - ac\partial\Phi \text{fin.}\Phi,$$

quae reducitur ad hanc formam:

$$\partial\partial\Phi(kk+aa+cc-2ac\,\text{cof.}\Phi)$$
$$+ ac\partial\Phi^2 \text{fin.}\Phi = -2ag\partial t^2 \text{fin.}\Phi,$$

haecque aequatio commode integrabilis redditur, ſi modo multiplicetur per $2\partial\Phi$; tum enim integrale reperitur:

$$\partial\Phi^2(kk+aa+cc-2ac\,\text{cof.}\Phi) = 4ag\partial t^2 \text{cof.}\Phi + C\partial t^2.$$

§. 9. Quia $\frac{\partial\Phi}{\partial t}$ exprimit celeritatem angularem, aequatio inuenta, per ∂t^2 diuiſa, erit:

$$\frac{\partial\Phi^2}{\partial t^2}(kk+aa+cc-2ac\,\text{cof.}\Phi) = C + 4ag\,\text{cof.}\Phi,$$

vbi ad conſtantem rite determinandam conſideretur maxima penduli excurſio a ſitu verticali, quae fiat per angulum $=\zeta$, et cum in hoc ſitu celeritas prorſus euaneſcat, conſtans C erit ita definienda, vt poſito $\Phi=\zeta$ fiat $\frac{\partial\Phi}{\partial t}=0$, ex qua conditione conſtans iſta per integrationem ingreſſa ita definietur, vt ſit $C=-4ag\,\text{cof.}\zeta$, ſicque noſtra aequatio motum penduli determinans erit

$$\partial\Phi^2(kk+aa+cc-2ac\,\text{cof.}\Phi) = 4ag\partial t^2(\text{cof.}\Phi - \text{cof.}\zeta),$$

ex qua aequatione ad quoduis tempus t ſtatus penduli definiri poterit. Cum enim radice extracta hinc prodeat

$$\partial t = \frac{\partial\Phi\sqrt{(kk+aa+cc-2ac\,\text{cof.}\Phi)}}{2\sqrt{ag(\text{cof.}\Phi-\text{cof.}\zeta)}},$$

determinatio totius motus perducta eſt ad integrationem iſtius formulae differentialis. Inuento autem eius integrali tempus ſtatim in minutis ſecundis expreſſum reperitur, vnde ſi eius integrale a termino $\Phi=0$ vsque ad terminum $\Phi=\zeta$ extendatur, habebitur tempus vnius ſemi-oſcillationis penduli, cuius ergo duplum dabit tempus vnius oſcillationis.

§. 10. Facile autem patet iſtud integrale in genere aliter exhiberi non poſſe, niſi praefixo ſigno integrationis: erit igitur tempus vnius dimidiae oſcillationis

$$t = \frac{1}{2\sqrt{ag}} \int \frac{\sqrt{(kk + aa + cc - 2ac\,\mathrm{cof.}\,\Phi)}}{\sqrt{(\mathrm{cof.}\,\Phi - \mathrm{cof.}\,\zeta)}} . \partial\Phi \left[\begin{array}{l} \text{a } \Phi = 0 \\ \text{ad } \Phi = \zeta \end{array}\right].$$

Verum ſi oſcillationes fuerint valde paruae, iſtud tempus facile ad quadraturam circuli reuocatur, quem ergo caſum hic accuratius euoluamus. Cum ſit $\mathrm{cof.}\,\Phi = 1 - 2\,\mathrm{fin.}\,\tfrac{1}{2}\Phi^2$, ſimilique modo $\mathrm{cof.}\,\zeta = 1 - 2\,\mathrm{fin.}\,\tfrac{1}{2}\zeta^2$, ponamus $\mathrm{fin.}\,\tfrac{1}{2}\zeta = b$ et $\mathrm{fin.}\,\tfrac{1}{2}\Phi = s$, vt ſit $\mathrm{cof.}\,\zeta = 1 - 2bb$ et $\mathrm{cof.}\,\Phi = 1 - 2ss$, tum vero erit $\partial\Phi = \frac{2\partial s}{\sqrt{(1-ss)}}$. Quod ſi iam breu. gr. ſtatuamus $kk + (a-c)^2 = bb$ formula noſtra induet hanc formam:

$$t = \frac{1}{\sqrt{2ag}} \int \frac{\partial s \sqrt{(bb + 4acss)}}{\sqrt{(1-ss)(bb-ss)}},$$

vbi notetur eſſe b quantitatem valde paruam, et tempus ſemi-oſcillationis repertum iri, ſi integrale ab $s = 0$ vsque ad $s = b$ extendatur.

§. 11. Cum igitur b reſpectu vnitatis ſit fractio valde exigua, et variabilis s fractionem b nunquam excedere queat, proxime erit

$$\frac{1}{\sqrt{(1-ss)}} = 1 + \tfrac{1}{2}ss \text{ et}$$

$$\sqrt{(bb + 4acss)} = b + \frac{2acss}{b},$$

ex his valoribus erit

$$t = \frac{1}{\sqrt{2ag}} \int \frac{\partial s}{\sqrt{(bb-ss)}} \left(b + \frac{4ac + bb}{2b} ss\right),$$

ſicque integrale conſtat ex duabus partibus, quarum poſterior prae priore eſt quaſi infinite parua ideoque ita repraeſentari poteſt:

$$t = \frac{b}{\sqrt{2ag}} \int \frac{\partial s}{\sqrt{(bb-ss)}} + \frac{4ac+bb}{2b\sqrt{2ag}} \int \frac{ss\,\partial s}{\sqrt{(bb-ss)}},$$

vbi manifeſtum eſt eſſe $\int \frac{\partial s}{\sqrt{(bb-ss)}} = A\,\mathrm{fin.}\,\frac{s}{b}$, vnde, ſumto

$s = b$

$s = b$, pro vna femi-ofcillatione erit $\int \frac{\partial s}{\sqrt{(bb - ss)}} = \frac{\pi}{2}$. Praeterea facile patet effe

$$\int \frac{ss\partial s}{\sqrt{(bb-ss)}} = -\tfrac{1}{2}s\sqrt{(bb-ss)} + \tfrac{1}{2}bb\int \frac{\partial s}{\sqrt{(bb-ss)}}.$$

Hinc ergo fumto $s = b$ erit iftud integrale $= \frac{\pi bb}{4}$, quocirca tempus vnius dimidiae ofcillationis erit

$$t = \frac{b\pi}{2\sqrt{2ag}} + \frac{\pi b b(4ac + bb)}{8b\sqrt{2ag}},$$

ideoque tempus integrae ofcillationis

$$= \frac{\pi b}{\sqrt{2ag}} + \frac{\pi bb(4ac+bb)}{4b\sqrt{2ag}}.$$

Vnde patet, quo maiores fuerint penduli excurfiones, tempora ofcillationum eo maiora incrementa accipere.

§. 12. Quodfi ponamus axem cylindricum infinite effe gracilem, ita vt fit $c = 0$, durante motu ofcillatorio punctum C immotum manebit, et habebitur cafus penduli vulgaris; tum igitur erit $bb = kk + aa$ et tempus vnius ofcillationis fiet $\frac{\pi\sqrt{(kk+aa)}}{\sqrt{2ag}} + \frac{\pi bb\sqrt{(kk+aa)}}{4\sqrt{2ag}}$. Quodfi praeterea fuerit $kk = 0$, fiue corpus ofcillans infinite paruum, habebitur cafus penduli fimplicis, longitudinis $CG = a$, pro quo ergo tempus ofcillationis erit $\frac{\pi\sqrt{a}}{\sqrt{2g}} + \frac{\pi bb\sqrt{a}}{4\sqrt{2g}}$, vnde viciffim tempora omnium ofcillationum ad pendulum fimplex reduci poterunt, fiue femper hinc longitudo penduli fimplicis affignari poterit, quod eodem tempore ofcillationes fuas peragat.

Inueftigatio virium Π & Θ.

§. 13. Valores harum virium iam fupra per differentialia fecundi gradus expreffos dedimus:

$$\frac{\Pi}{M} = \frac{2g\partial t^2 + a\partial\partial\Phi\, fin.\Phi + a\partial\Phi^2\, cof.\Phi}{2g\partial t^2},$$

$$\frac{\Theta}{M} = \frac{a\partial\partial\Phi\, cof.\Phi - a\partial\Phi^2\, fin.\Phi - c\partial\partial\Phi}{2g\partial t^2},$$

vnde tantum opus eft, vt ifta differentialia ad quantitates finitas

tas reuocentur. Iam vero ex integratione generali §. 9. colligitur fore

$$\frac{\partial \Phi^2}{2g\,\partial t^2} = \frac{2a(\text{cof.}\,\Phi - \text{cof.}\,\zeta)}{kk + aa + cc - 2ac\,\text{cof.}\,\Phi};$$

ex aequatione autem differentiali fecundi gradus fupra inuenta fiet

$$\frac{\partial\partial \Phi}{2g\,\partial t^2} = \frac{-ac\,\partial\Phi^2\,\text{fin.}\,\Phi}{2g\,\partial t^2(kk + aa + cc - 2ac\,\text{cof.}\,\Phi} - \frac{a\,\text{fin.}\,\Phi}{kk + aa + cc - 2ac\,\text{cof.}\,\Phi},$$

vnde per meras quantitates finitas erit

$$\frac{\partial\partial \Phi}{2g\,\partial t^2} = \frac{-2aac\,\text{fin.}\,\Phi(\text{cof.}\,\Phi - \text{cof.}\,\zeta)}{(kk + aa + cc - 2ac\,\text{cof.}\,\Phi)^2} - \frac{a\,\text{fin.}\,\Phi}{kk + aa + cc - 2ac\,\text{cof.}\,\Phi},$$

fiue

$$\frac{\partial\partial \Phi}{2g\,\partial t^2} = \frac{-(kk + aa + cc - 2ac\,\text{cof.}\,\zeta)\,a\,\text{fin.}\,\Phi}{(kk + aa + cc - 2ac\,\text{cof.}\,\Phi)^2},$$

quae expreffiones cum fint fatis complicatae, ponamus br. gr. $\frac{\partial \Phi^2}{2g\,\partial t^2} = R$ et $\frac{\partial\partial \Phi}{2g\,\partial t^2} = -S$. Ita vt fit

$$R = \frac{2a(\text{cof.}\,\Phi - \text{cof.}\,\zeta)}{kk + aa + cc - 2ac\,\text{cof.}\,\Phi} \quad \text{et}$$

$$S = \frac{a(kk + aa + cc - 2ac\,\text{cof.}\,\zeta)\,\text{fin.}\,\Phi}{(kk + aa + cc - 2ac\,\text{cof.}\,\Phi)}.$$

§. 14. His ergo valoribus fubftitutis erit per meras quantitates finitas.

$$\frac{\Pi}{M} = 1 - aS\,\text{fin.}\,\Phi + aR\,\text{cof.}\,\Phi,$$

$$\frac{\Theta}{M} = S(c - a\,\text{cof.}\,\Phi) - aR\,\text{fin.}\,\Phi,$$

vnde patet durante motu ofcillatorio has vires continuo variari. Hinc ergo primo quaeramus iftas vires pro fitu penduli verticali, vbi $\Phi = 0$, ac reperietur fore $\frac{\Pi}{M} = 1 + aR$ et $\frac{\Theta}{M} = (c - a)S$. Hoc autem cafu fit $R = \frac{2a(1 - \text{cof.}\,\zeta)}{kk + (a - c)^2}$ et $S = 0$, confequenter pro fitu verticali erit

$$\frac{\Pi}{M} = 1 = \frac{2aa(1 - \text{cof.}\,\zeta)}{kk + (a - c)^2},$$

$$\frac{\Theta}{M} = 0;$$

vnde patet hoc fitu preffionem effe maiorem quam pondus.

§. 15.

§. 15. At vero pro excurſionibus maximis, vbi $\Phi = \zeta$, habebimus $R = 0$ et

$$S = \frac{a \sin. \zeta}{kk + aa + cc - 2ac \cos. \zeta}$$

hincque pro hoc ſtatu reperitur $\frac{\Pi}{M} = 1$, ſiue preſſio ipſi ponderi exacte eſt aequalis. Deinde erit

$$\frac{\Theta}{M} = \frac{-a(a-c)\sin. \zeta}{kk + aa + cc - 2ac \cos. \zeta},$$

vnde patet iſtam vim non in plagam O A, vti finximus, ſed in plagam contrariam A V eſſe directam, cui ergo coërcendae frictio ſuper plano horizontali maior eſſe debet. Ceterum ex noſtris formulis haud difficile erit pro quouis penduli ſtatu medio ambas vires Π et Θ aſſignare, quae autem ſecundum nullam legem ſimplicem repraeſentari poſſunt.

Fig. 1.

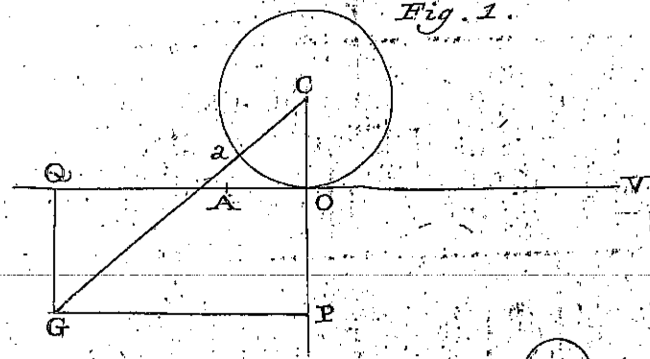

Fig. 3.

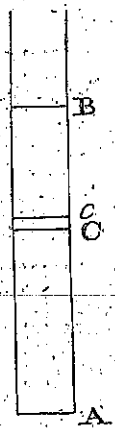

Fig. 2.

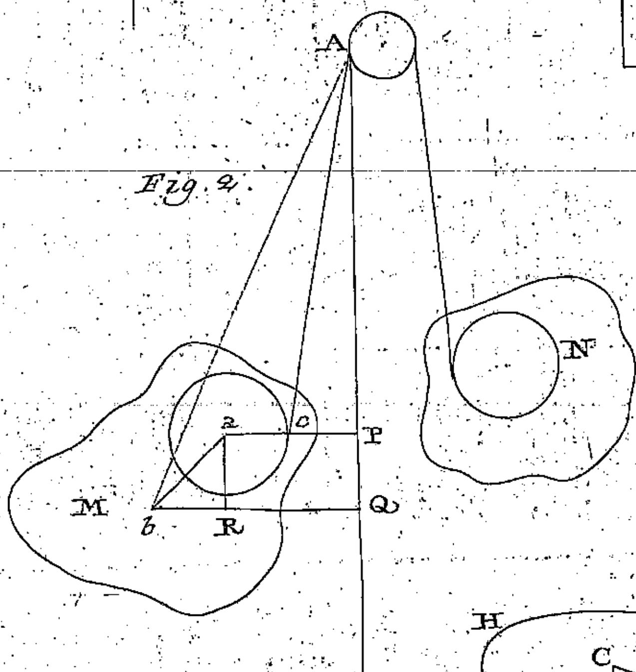

Fig. 4.

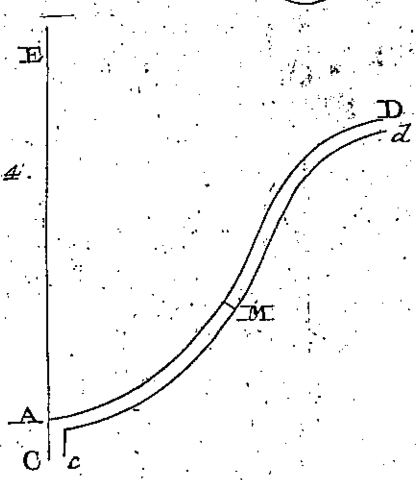

Fig. 5.