1790

Solutio facilis problematis, quo quaeritur circulus, qui datos tres circulos tangat

Leonhard Euler

SOLVTIO FACILIS FROBLEMATIS, QVO QVAERITVR CIRCVLVS, QVI DATOS TRES CIRCVLOS TANGAT.

Auctore
L. EVLERO.

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§. 1.

Sint A, B, C centra trium circulorum datorum, pro quorum situ sit distantia $CA = a$, $CB = b$ et angulus $ACB = C$. Deinde sint α, β, γ radii horum trium circulorum, O centrum circuli omnes tres tangentis, cuius radius ponatur $= x$. Ductis igitur rectis OA, OB, OC, quae per puncta contactus transibunt, habebimus $OA = x + \alpha$, $OB = x + \beta$ et $OC = x + \gamma$. Hic vero ante omnia probe notandum est, contactus singulorum circulorum non solum intus, sed etiam extus euenire posse. Ita si circulus C extus tangeretur, foret $OC = x - \gamma$; vnde patet, radios horum circulorum tam negatiue, quam affirmatiue accipi posse. Hinc euidens est octo solutiones locum habere posse, quae scilicet ex variatione signorum litterarum α, β, γ oriuntur. Interim tamen in solutione has litteras α, β, γ vt positiuas spectabimus, calculo vero expedito nihil impedit, quo minus earum vni vel alteri valor negatiuus tribuatur. Tab. I. Fig. 5.

§. 2.

§. 2. Statuamus autem breuitatis gratia $OC = x + \gamma = z$ vt fit $x = z - \gamma$, tum vero fiat $\alpha - \gamma = f$ et $\beta - \gamma = g$, vt prodeat $OA = z + f$ et $OB = z + g$. Quodfi iam vocentur anguli $OCA = p$ et $OCB = q$, ita vt fit $p + q = C$, ex triangulo OCA, cuius latera funt $OC = z$; $OA = z + f$ et $AC = a$, colligitur:

$$\text{cof.}\, p = \frac{aa + zz - (z+f)^2}{2az} = \frac{aa - 2fz + ff}{2az}.$$

Simili modo ex triangulo OCB, cuius latera funt $OC = z$; $OB = z + g$ et $CB = b$, colligitur:

$$\text{cof.}\, q = \frac{bb + zz - (z+g)^2}{2bz} = \frac{bb - 2gz - gg}{2bz}.$$

Quoniam igitur angulorum p et q fumma datur $= C$, euidens eft hinc incognitam z determinari poffe.

§. 3. Cum igitur fit $p + q = C$, erit

$$\text{cof.}\, C = \text{cof.}\, p\, \text{cof.}\, q - \text{fin.}\, p\, \text{fin.}\, q,$$

tum vero

$$\text{fin.}\, C = \text{fin.}\, p\, \text{cof.}\, q + \text{cof.}\, p\, \text{fin.}\, q,$$

cuius formulae quadratum praebet:

$$\text{fin.}\, C^2 = \text{fin.}\, p^2\, \text{cof.}\, q^2 + \text{cof.}\, p^2\, \text{fin.}\, q^2 + 2\, \text{fin.}\, p\, \text{cof.}\, q\, \text{cof.}\, p\, \text{fin.}\, q.$$

At vero ex priori formula eft

$$\text{fin.}\, p\, \text{fin.}\, q = \text{cof.}\, p\, \text{cof.}\, q - \text{cof.}\, C,$$

quo valore fubftituto prodit

$$\text{fin.}\, C^2 = \text{fin.}\, p^2\, \text{cof.}\, q^2 + \text{cof.}\, p^2\, \text{fin.}\, q^2 + 2\, \text{cof.}\, p^2\, \text{cof.}\, q^2 - 2\, \text{cof.}\, p\, \text{cof.}\, q\, \text{cof.}\, C,$$

hinc iam cum fit

$$\text{fin.}\, p^2 = 1 - \text{cof.}\, p^2 \text{ et } \text{fin.}\, q^2 = 1 - \text{cof.}\, q^2,$$

haec aequatio induet iftam formam:

$$\text{fin.}\, C^2 = \text{cof.}\, p + \text{cof.}\, q - 2\, \text{cof.}\, p\, \text{cof.}\, q\, \text{cof.}\, C.$$

§. 4.

§. 4. Quo iam hanc aequationem facilius resoluamus, statuamus breuitatis gratia $aa - ff = 2F$ et $bb - gg = 2G$, vt sit $\cos. p = \frac{F - fz}{az}$ et $\cos. q = \frac{G - gz}{bz}$, quibus valoribus substitutis nostra aequatio erit

$$\sin. C^2 = \frac{(F - fz)^2}{aazz} + \frac{(G - gz)^2}{bbzz} - 2\,\frac{(F - fz)(G - gz)\cos. C}{abzz},$$

quae aequatio in ordinem redacta euadet:

$$\begin{aligned} aabbzz \sin. C^2 = {} & bbF^2 - 2bbFfz + bbffzz \\ & + aaG^2 - 2aaGgz + aaggzz \\ & - 2abFG \cos. C + 2abGfz \cos. C \\ & - 2abfgzz \cos. C + 2abFgz \cos. C. \end{aligned}$$

Aequatio igitur inuenta sub hac forma repraesentari potest: $Lzz - 2Mz + N = 0$, existente

$$\begin{aligned} L &= bbff + aagg - aabb \sin. C^2 - 2abfg \cos. C, \\ M &= bbFf + aaGg - ab \cos. C\,(Gf + Fg), \\ N &= bbF^2 + aaG^2 - 2abFG \cos. C. \end{aligned}$$

Quoniam igitur valores litterarum L, M, N sunt cogniti, ex resolutione huius aequationis quadraticae colligitur fore $z = \frac{M \pm \sqrt{(M^2 - LN)}}{L}$. Vnde patet binas solutiones prodire, atque adeo ambas reales, si modo fuerit $M^2 > LN$; consequenter ob illas octo signorum variationes, quae supra sunt commemoratae, omnino sedecim diuersae videntur dari solutiones, id quod eo magis notari meretur, quod omnes istae solutiones tantum ex aequatione quadratica deducuntur. Praeterea vero imprimis hoc obseruandum est, dari etiam casus, quibus solutio prorsus fit impossibilis, id quod euenit, quando fuerit $LN > M^2$.

§. 5. Hoc igitur modo solutio nostri problematis ad calculum non nimis complexum est deducta, vnde, quicunque

 valo-

valores pro quantitatibus cognitis fuerint assumpti, per calculum numericum valor ipsius z facile assignabitur.

§. 6. Quanquam autem applicatio ad quosuis casus propositos nulla laborat difficultate, tamen, quando aequatio quadratica radicem negatiuam inuoluit, ita vt littera z negatiuum obtineat valorem, non tam facile perspicitur, quo modo tales casus sint repraesentandi. Omnis autem difficultas euanescet, si ostenderimus, valores negatiuos pro z inuentos proprie non ad ipsum casum propositum pertinere, sed ad eum casum, quo radiis circulorum α, β, γ contraria signa tribuuntur; vnde fit, vt, etiamsi quaelibet aequatio quadratica duas complectatur radices, tamen omnino non plures quam octo solutiones locum habere queant.

§. 7. Quod quo clarius perspiciatur, ante omnia obseruasse iuuabit, cum sit $\alpha - \gamma = f$ et $\beta - \gamma = g$, mutatis signis litterarum α, β, γ etiam f et g contraria signa recipere; nihilo vero minus litterae deriuatae $F = \frac{aa - ff}{2}$ et $G = \frac{bb - gg}{2}$, nullam patiuntur mutationem. Hinc cum sit

$$L = bbff + aagg - aabb \sin. C^2 - abfg \cos. C,$$

euidens est, mutatis litterarum f et g signis litteram L eandem manere; at quia est

$$M = bbFf + aaGg - ab \cos. C (Gf + Fg),$$

mutatis litterarum f et g signis etiam signum litterae M immutatur; denique vero ob

$$N = bbF^2 + aaG^2 - 2abFG \cos. C$$

etiam littera N idem signum conseruabit. Quamobrem mutatis signis radiorum α, β, γ solutio perducet ad hanc aequationem:

$$Lzz + 2Mz + N = 0,$$

quae

quae eadem aequatio prodit, si z negatiue capiatur. Ex quo sequitur, quoties valor ipsius z prodit negatiuus, tum eum respondere casui quo radiis α, β, γ contraria signa tribuuntur; ita vt bini casus α, β, γ et $-\alpha$, $-\beta$, $-\gamma$ ad eandem aequationem quadraticam perducant, cuius radix positiua priori casui conuenit, negatiua vero posteriori, quae autem tum vt positiua est spectanda. Hinc igitur manifestum est, tantum quatuor diuersas aequationes quadraticas prodire posse, quarum quaelibet binos casus complectatur, vti hoc schema declarat:

$$\text{I.}\begin{cases}\alpha+\beta+\gamma\\ -\alpha-\beta-\gamma\end{cases};\ \text{II.}\begin{cases}\alpha+\beta-\gamma\\ -\alpha-\beta+\gamma\end{cases};\ \text{III.}\begin{cases}\alpha-\beta+\gamma\\ -\alpha+\beta-\gamma\end{cases};\ \text{IV.}\begin{cases}-\alpha+\beta+\gamma\\ +\alpha-\beta-\gamma\end{cases}.$$

Hoc obseruato adplicatio ad quaeuis exempla nulli amplius difficultati erit obnoxia.

§. 8. Verum quia hoc problema est geometricum, non tam calculus numericus, quam constructio geometrica desiderari solet, quae vtique nimis prodiret operosa, si eam ex aequatione quadratica $Lzz - 2Mz + N = 0$, deriuare vellemus; quamobrem hic plurimum iuuabit talem constructionem ex ipsis calculi principiis deriuasse.

Constructio problematis propositi.

§. 9. Ponatur angulus datus $ACB = 2\zeta$, ac statuatur angulus $ACO = p = \zeta + \Phi$ & $BCO = q = \zeta - \Phi$, calculumque ita instruamus, vt angulus Φ fiat nostra incognita. Habebimus ergo

$$\text{cos.}(\zeta + \Phi) = \frac{F}{az} - \frac{f}{a} \text{ et}$$

$$\text{cos.}(\zeta - \Phi) = \frac{G}{bz} - \frac{g}{b},$$

hinc eliminando z erit

$$\frac{G}{b}\,\text{cof.}\,(\zeta+\phi)-\frac{F}{a}\,\text{cof.}\,(\zeta-\phi)=\frac{Fg-Gf}{ab},\ \text{fiue}$$
$$a\,G\,\text{cof.}\,(\zeta+\phi)-b\,F\,\text{cof.}\,(\zeta-\phi)=F\,g-G\,f,$$

quae aequatio porro euoluta dat

$$\left.\begin{array}{l} a\,G\,\text{cof.}\,\zeta\,\text{cof.}\,\phi-a\,G\,\text{fin.}\,\zeta\,\text{fin.}\,\phi \\ -b\,F\,\text{cof.}\,\zeta\,\text{cof.}\,\phi-b\,F\,\text{fin.}\,\zeta\,\text{fin.}\,\phi \end{array}\right\}=F\,g-G\,f,$$

fiue

$$G\,f-F\,g=(b\,F-a\,G)\,\text{cof.}\,\zeta\,\text{cof.}\,\phi+(a\,G+b\,F)\,\text{fin.}\,\zeta\,\text{fin.}\,\phi.$$

Ponatur hic breuitatis gratia, $Gf-Fg=L$; $bF-aG=M$ et $aG+bF=N$, habebimus hanc aequationem:

$$L=M\,\text{cof.}\,\zeta\,\text{cof.}\,\phi+N\,\text{fin.}\,\zeta\,\text{fin.}\,\phi,$$

cuius conftructio fequenti modo erui poterit. Introducatur angulus θ, vt fit tang. $\theta=\frac{M\,cof.\,\zeta}{N\,fin.\,\zeta}$, vt adeo ifte angulus θ fit cognitus; hinc ergo fiet

$$\text{fin.}\,\theta=\frac{M\,cof.\,\zeta}{\sqrt{(M^2\,cof.\,\zeta^2+N^2\,fin.\,\zeta^2)}}\ \text{et}$$
$$\text{cof.}\,\theta=\frac{N\,fin.\,\zeta}{\sqrt{(M^2\,cof.\,\zeta^2+N^2\,fin.\,\zeta^2)}}.$$

Cum igitur fit

$$N\,\text{fin.}\,\zeta=\text{cof.}\,\theta\,\sqrt{(M^2\,\text{cof.}\,\zeta^2+N^2\,\text{fin.}\,\zeta^2)}\ \text{et}$$
$$M\,\text{cof.}\,\zeta=\text{fin.}\,\theta\,\sqrt{(M^2\,\text{cof.}\,\zeta^2+N^2\,\text{fin.}\,\zeta^2)},$$

his fubftitutis erit

$$\frac{L}{\sqrt{(M^2\,cof.\,\zeta^2+N^2\,fin.\,\zeta^2)}}=\text{fin.}\,\theta\,\text{cof.}\,\phi+\text{cof.}\,\theta\,\text{fin.}\,\phi=\text{fin.}\,(\theta+\phi).$$

Quodfi iam ille angulus, cuius finus eft $\frac{L}{\sqrt{(M^2\,cof.\,\zeta^2+N^2\,fin.\,\zeta^2)}}$, ponatur $=\lambda$, habebitur ifta aequatio: fin. $\lambda=$ fin. $(\theta+\phi)$, vnde quia etiam eft fin. $\lambda=$ fin. $(180^\circ-\lambda)$, habebimus duplicem aequationem: altera dat $\theta+\phi=\lambda$, ideoque $\phi=\lambda-\theta$; altera vero aequatio dat $\phi=180^\circ-\lambda-\theta$. Si autem loco L, M et N valores affumpti reftituantur, prodibit tag. $\theta=\frac{(b\,F-a\,G)\,cof.\,\zeta}{(a\,G+b\,F)\,fin.\,\zeta}$, vnde angulus θ facile colligitur.

Porro

Porro vero erit

$$\sqrt{(M^2 \cos.\zeta^2 + N^2 \sin.\zeta^2)} = \sqrt{(bbF^2 + aaG^2 - 2abFG\cos.2\zeta)},$$

quae formula radicalis ergo facile conſtruitur per triangulum PQR, cuius angulus PQR $= 2\zeta$ et latera PQ $= bF$ et PR $= aG$; tum enim erit tertium latus Tab. I. Fig. 6.

$$QR = \sqrt{(bbF^2 + aaG^2 - 2abFG\cos.2\zeta)}.$$

Denique erit $\sin.\lambda = \frac{Gf - Fg}{QR}$, ex quo etiam angulus λ facile conſtruitur, vnde determinatio anguli ϕ, ideoque rectae CO innoteſcit, nec non ipſa quantitas

$$CO = z = \frac{F}{f + a\cos.(\zeta + \phi)}.$$

Fig. 1.

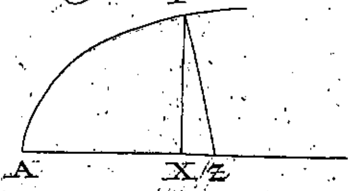


Fig. 2.

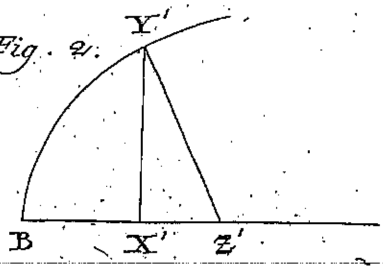


Fig. 3.

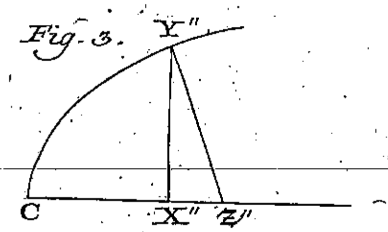
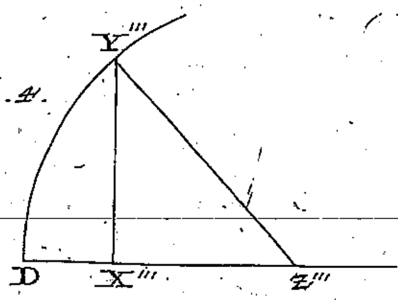


Fig. 4.



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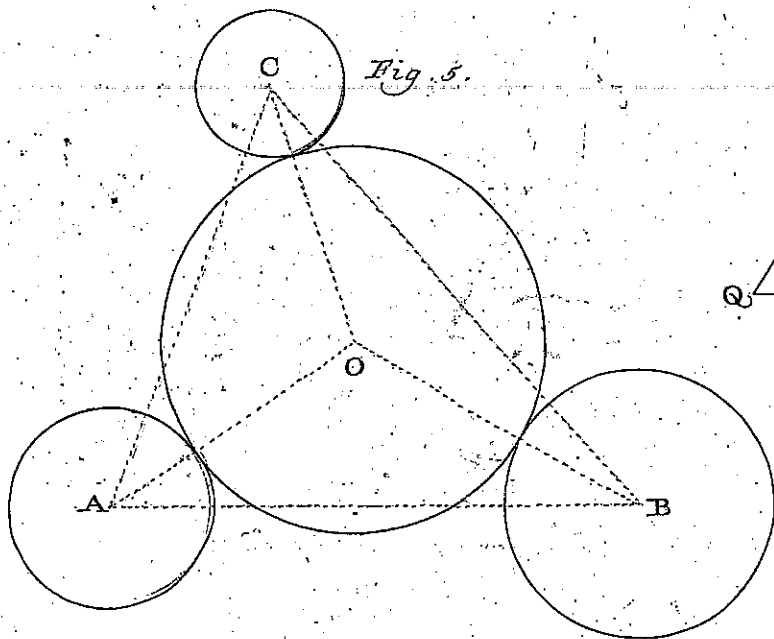

Fig. 6.