1790

De duabus pluribusve curvis algebraicis in quibus si a terminis fixis aequales arcus abscindantur eorum amplitudines datam inter se teneant rationem

Leonhard Euler

DE DVABVS PLVRIBVSVE CVRVIS ALGEBRAICIS, IN QVIBVS, SI A TERMINIS FIXIS AEQVALES ARCVS ABSCINDANTVR, EORVM AMPLITVDINES DATAM INTER SE TENEANT RATIONEM.

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§. 1.

Vnamquamque earum linearum curuarum, de quibus hic agitur, veluti curuam AY, ita ad suum axem AZ referri concipiamus, vt ipsi in A normaliter insistat, existente puncto A eo termino fixo, a quo arcus AY abscindantur. Hinc posito arcu $AY = s$, ductaque ad arcum in Y normali YZ, angulus AZY metietur amplitudinem arcus AY, quam vocemus $= \omega$; tum vero si ex Y ad axem ducatur normalis YX, vocenturque coordinatae $AX = x$ et $XY = y$, primo quidem erit $\partial x^2 + \partial y^2 = \partial s^2$; deinde quia angulus $AYX = AZY = \omega$, manifestum est fore $\partial x = \partial s \sin. \omega$ et $\partial y = \partial s \cos. \omega$. Quamobrem cum de curuis algebraicis hic agatur, eiusmodi relationem inter elementum arcus ∂s et ampli- Tab. I. Fig. 1.

amplitudinem ω conſtitui neceſſe eſt, vt ambae iſtae formulae differentiales: $\partial x = \partial s \text{ ſin. } \omega$ et $\partial y = \partial s \text{ coſ. } \omega$ reddantur integrabiles.

§. 2. Huic autem conditioni ſatisfiet, ſi ſtatuatur $\partial s = v \partial \omega + \frac{\partial \partial v}{\partial \omega}$, denotante v functionem quamcunque algebraicam altitudinis, vbi ſcilicet ratione differentialium altiorum elementum $\partial \omega$ conſtans eſt acceptum. Hinc igitur ambas coordinatas x et y algebraice exprimere licebit; cum enim ſit

$$\partial x = v \partial \omega \text{ ſin. } \omega + \frac{\partial \partial v \text{ ſin. } \omega}{\partial \omega} \text{ et}$$

$$\partial y = v \partial \omega \text{ coſ. } \omega + \frac{\partial \partial v \text{ coſ. } \omega}{\partial \omega},$$

per notam integralium reductionem facile reperietur fore

$$x = \frac{\partial v}{\partial \omega} \text{ ſin. } \omega - v \text{ coſ. } \omega \text{ et}$$

$$y = \frac{\partial v}{\partial \omega} \text{ coſ. } \omega - v \text{ ſin. } \omega,$$

id quod ſumendis differentialibus ſtatim patebit. Quare cum ambo hi valores pro x et y algebraicae exprimantur, ipſa curva vtique erit algebraica, quaecunque etiam functio algebraica amplitudinis ω loco v accipiatur. Cum porro ſit $\partial s = v \partial \omega + \frac{\partial \partial v}{\partial \omega}$, erit integrando ipſa curuae longitudo $AY = s = \int v \partial \omega + \frac{\partial v}{\partial \omega}$; ex quo patet curuam adeo fore rectificabilem, ſi modo formula $\int v \partial \omega$ integrationem admittat; ſin autem haec formula non fuerit integrabilis, curuae rectificatio a certa quadam pendebit quadratura arbitrio noſtro relicta.

§. 3. Manifeſtum hic eſt, plures huiusmodi formulas pro elemento ∂s aſſumendas inuicem coniungi poſſe. Veluti ſi ſtatuamus:

$$\partial s = v \partial \omega + \frac{\partial \partial v}{\partial \omega} + u \partial \omega + \frac{\partial \partial u}{\partial \omega} + w \partial \omega + \frac{\partial \partial w}{\partial \omega},$$

exiſtentibus v, u et w functionibus quibuscunque algebraicis ipſius ω, ſimili modo patebit fore

$x =$

$$x = \frac{\partial v}{\partial \omega} \text{fin.}\, \omega - v \,\text{cof.}\, \omega + \frac{\partial u}{\partial \omega} \text{fin.}\, \omega - u \,\text{cof.}\, \omega$$
$$+ \frac{\partial w}{\partial \omega} \text{fin.}\, \omega - w \,\text{cof.}\, \omega \text{ et}$$
$$y = \frac{\partial v}{\partial \omega} \text{cof.}\, \omega + v \,\text{fin.}\, \omega + \frac{\partial u}{\partial \omega} \text{cof.}\, \omega + u \,\text{fin.}\, \omega$$
$$+ \frac{\partial w}{\partial \omega} \text{cof.}\, \omega + w \,\text{fin.}\, \omega,$$

tota res fcilicet hic perinde fe habet, ac fi loco v fcripfiffemus $v + u + w$. In fequentibus autem loco u et w eiusmodi formulas a v pendentes affumi conueniet, vt fit $u = \frac{A \partial \partial v}{\partial \omega^2}$ et $w = \frac{B \partial^4 v}{\partial \omega^4}$, eritque,

$$\partial s = v \partial \omega + \frac{\partial \partial v}{\partial \omega} + \frac{A \partial \partial v}{\partial \omega} + \frac{A \partial^4 v}{\partial \omega^3} + \frac{B \partial^4 v}{\partial \omega^3} + \frac{B \partial^6 v}{\partial \omega^5},$$

indeque porro

$$x = \frac{\partial v}{\partial \omega} \text{fin.}\, \omega - v \,\text{cof.}\, \omega + \frac{A \partial^3 v}{\partial \omega^3} \text{fin.}\, \omega - \frac{A \partial \partial v}{\partial \omega^2} \text{cof.}\, \omega$$
$$+ \frac{B \partial^5 v}{\partial \omega^5} \text{fin.}\, \omega - \frac{B \partial^4 v}{\partial \omega^4} \text{cof.}\, \omega,$$
$$y = \frac{\partial v}{\partial \omega} \text{cof.}\, \omega + v \,\text{fin.}\, \omega + \frac{A \partial^3 v}{\partial \omega^3} \text{cof.}\, \omega + \frac{A \partial \partial v}{\partial \omega^2} \text{fin.}\, \omega$$
$$+ \frac{B \partial^5 v}{\partial \omega^5} \text{cof.}\, \omega + \frac{B \partial^4 v}{\partial \omega^4} \text{fin.}\, \omega.$$

§. 4. His praemiffis confideremus infuper aliam curuam B Y′, pariter in B fuo axi B Z′ normaliter infiftentem, in qua abfcindatur arcus B Y′, illi arcui A Y $= s$ aequalis, cui refpondeant coordinatae B X′ $= x'$, X′ Y′ $= y'$. Nunc autem confideremus conditionem praefcriptam, vt fcilicet amplitudines horum duorum arcuum aequalium datam inter fe teneant rationem, quae fit vt $\alpha : \beta$. Hunc igitur in finem ftatuamus amplitudinem prioris curuae A Z Y $= \alpha \Phi$, pofterioris vero B Z′ Y′ $= \beta \Phi$, ita vt quod ante fuerat ω, nunc pro priore curua fit $\alpha \Phi$, pro pofteriore vero $\beta \Phi$. Quare quo vtraque curua prodeat algebraica, eiusmodi formulam pro vtriusque elemento ∂s inueftigari oportebit, quae non folum per fin. $\alpha \Phi$ et cof. $\alpha \Phi$, fed etiam per fin. $\beta \Phi$ et cof. $\beta \Phi$ multipli- Tab. I. Fig. 1 & 2.

 cata

cata euadat integrabilis, ita vt quatuor multiplicatores hic praescribantur, id quod solutionem maxime perplexam requireret; verum eo modo, quo hic sumus vsuri, negotium haud difficulter conficietur.

§. 5. Quoniam formula $v\partial\omega + \frac{\partial\partial v}{\partial\omega}$, ducta tam in sin. ω quam in cos. ω fit integrabilis, si loco ω scribamus $\alpha\Phi$, prodibit formula tam per sin. $\alpha\Phi$, quam per cos. $\alpha\Phi$ multiplicata integrabilis, quae ergo erit $\alpha v\partial\Phi + \frac{\partial\partial v}{\alpha\partial\Phi}$, siue per α diuidendo, $v\partial\Phi + \frac{\partial\partial v}{\alpha\alpha\partial\Phi}$. Haec autem formula ducta in sin. $\alpha\Phi$ dabit integrale $\frac{\partial v}{\alpha\alpha\partial\Phi}$ sin. $\alpha\Phi - \frac{v}{\alpha}$ cos. $\alpha\Phi$; at vero per cos. $\alpha\Phi$ multiplicata integrale dabit $\frac{\partial v}{\alpha\alpha\partial\Phi}$ cos. $\alpha\Phi + \frac{v}{\alpha}$ sin. $\alpha\Phi$. Simili modo haec formula: $v\partial\Phi + \frac{\partial\partial v}{\beta\beta\partial\Phi}$, euadet integrabilis, primo ducta in sin. $\beta\Phi$, deinde etiam in cos. $\beta\Phi$; priore enim casu integrale erit $\frac{\partial v}{\beta\beta\partial\Phi}$ sin. $\beta\Phi - \frac{v}{\beta}$ cos. $\beta\Phi$, posteriore vero casu integrale erit $\frac{\partial v}{\beta\beta\partial\Phi}$ cos. $\beta\Phi + \frac{v}{\beta}$ sin. $\beta\Phi$.

§. 6. Iam ad quaestionem propositam resoluendam, qua duae requiruntur curuae algebraicae A Y et B Y', in quibus si a terminis fixis A et B bini arcus aequales $AY = s$ et $BY' = s$ abscindantur, eorum amplitudines, seu anguli $AZY = \alpha\Phi$ et $BZ'Y' = \beta\Phi$, datam inter se teneant rationem, scilicet vt $\alpha : \beta$, accipiamus pro priore curua hanc formulam:

$$\partial s = v\partial\Phi + \frac{\partial\partial v}{\alpha\alpha\partial\Phi} + A\left(\frac{\partial\partial v}{\partial\Phi} + \frac{\partial^4 v}{\alpha\alpha\partial\Phi^3}\right),$$

hinc enim fiet

$$x = \int\partial s \text{ sin. } \alpha\Phi = \frac{\partial v}{\alpha\alpha\partial\Phi} \text{ sin. } \alpha\Phi - \frac{1}{\alpha} v \text{ cos. } \alpha\Phi$$
$$+ \frac{A\partial^3 v}{\alpha\alpha\partial\Phi^3} \text{ sin. } \alpha\Phi - \frac{A\partial\partial v \text{ cos. } \alpha\Phi}{\alpha\partial\Phi^2},$$

$$y = \int \partial s \cos.\alpha\phi = \frac{\partial v}{\alpha\alpha\partial\phi}\cos.\alpha\phi + \frac{v \sin.\alpha\phi}{\alpha}$$
$$+ \frac{A\partial^3 v}{\alpha\alpha\partial\phi^3}\cos.\alpha\phi + \frac{A\partial\partial v}{\alpha\partial\phi^2}\sin.\alpha\phi.$$

Simili modo si pro altera curua assumamus:

$$\partial s = v\partial\phi + \frac{\partial\partial v}{\beta\beta\partial\phi} + B\left(\frac{\partial\partial v}{\partial\phi} + \frac{\partial^4 v}{\beta\beta\partial\phi^3}\right),$$

eiusque coordinatas statuamus $BX' = x'$ et $X'Y' = y'$, reperiemus:

$$x' = \int \partial s \sin.\beta\phi = \frac{\partial v}{\beta\beta\partial\phi}\sin.\beta\phi - \frac{v \cos.\beta\phi}{\beta}$$
$$+ \frac{B\partial^3 v}{\beta\beta\partial\phi^3}\sin.\beta\phi - \frac{B\partial\partial v}{\beta\partial\phi^2}\cos.\beta\phi \text{ et}$$
$$y' = \int \partial s \cos.\beta\phi = \frac{\partial v}{\beta\beta\partial\phi}\cos.\beta\phi + \frac{v \sin.\beta\phi}{\beta}$$
$$+ \frac{B\partial^3 v}{\beta\beta\partial\phi^3}\cos.\beta\phi + \frac{B\partial\partial v}{\beta\partial\phi^2}\sin.\beta\phi.$$

§. 7. Nunc igitur quoniam ambo arcus AY et BY' inter se debent esse aequales, quantitates constantes A et B ita definiri oportet, vt ambae formulae pro ∂s assumtae inter se fiant aequales. At quia primae partes, littera v affectae, iam vtrinque sunt eaedem, reddantur secundae partes per $\partial\partial v$ affectae etiam inter se aequales, vnde fiet $\frac{1}{\alpha\alpha} + A = \frac{1}{\beta\beta} + B$; vltimae autem partes per $\partial^4 v$ affectae reddentur aequales, sumendo $\frac{A}{\alpha\alpha} = \frac{B}{\beta\beta}$, vnde fit $B = \frac{A\beta\beta}{\alpha\alpha}$, qui valor in praecedente aequatione substitutus dat $\frac{1}{\alpha\alpha} + A = \frac{1}{\beta\beta} + \frac{A\beta\beta}{\alpha\alpha}$, vnde colligitur $A = \frac{1}{\beta\beta}$, ergo $B = \frac{1}{\alpha\alpha}$, ita vt pro vtraque curua sit

$$\partial s = v\partial\phi + \frac{\partial\partial v}{\alpha\alpha\partial\phi} + \frac{\partial\partial v}{\beta\beta\partial\phi} + \frac{\partial^4 v}{\alpha\alpha\beta\beta\partial\phi^3}.$$

Tum autem coordinatae prioris curuae AY erunt

$$x = -\frac{v \cos.\alpha\phi}{\alpha} + \frac{\partial v \sin.\alpha\phi}{\alpha\alpha\partial\phi} - \frac{\partial\partial v \cos.\alpha\phi}{\alpha\beta\beta\partial\phi^2} + \frac{\partial^3 v \sin.\alpha\phi}{\alpha\alpha\beta\beta\partial\phi^3} \text{ et}$$
$$y = \frac{v \sin.\alpha\phi}{\alpha} + \frac{\partial v \cos.\alpha\phi}{\alpha\alpha\partial\phi} + \frac{\partial\partial v \sin.\alpha\phi}{\alpha\beta\beta\partial\phi^2} + \frac{\partial^3 v \cos.\alpha\phi}{\alpha\alpha\beta\beta\partial\phi^3};$$

pro altera autem curua BY' habebimus:

 $x' =$

$$x' = -\frac{v\ \mathrm{cos}.\ \beta\Phi}{\beta} + \frac{\partial v\ \mathrm{cos}.\ \beta\Phi}{\beta\beta\,\partial\Phi} - \frac{\partial\partial v\ \mathrm{cos}.\ \beta\Phi}{\beta\alpha\alpha\,\partial\Phi^2} + \frac{\partial^3 v\ \mathrm{sin}.\ \beta\Phi}{\alpha\alpha\beta\beta\,\partial\Phi^3} \text{ et}$$

$$y' = \frac{v\ \mathrm{sin}.\ \beta\Phi}{\beta} + \frac{\partial v\ \mathrm{cos}.\ \beta\Phi}{\beta\beta\,\partial\Phi} + \frac{\partial\partial v\ \mathrm{sin}.\ \beta\Phi}{\beta\alpha\alpha\,\partial\Phi^2} + \frac{\partial^3 v\ \mathrm{cos}.\ \beta\Phi}{\alpha\alpha\beta\beta\,\partial\Phi^3}.$$

Operae igitur pretium erit iſtum caſum ſingulari problemate complecti.

Problema.

§. 8. *Inuenire duas curuas algebraicas A Y et B Y′, in quibus ſi a datis terminis A et B bini arcus aequales A Y et B Y′ abſcindantur, eorum amplitudines ſeu anguli A Z Y et B Z′ Y′ datam inter ſe teneant rationem, vt* $\alpha : \beta$.

Solutio.

Statuantur amplitudines $AZY = \alpha\Phi$ et $BZ'Y' = \beta\Phi$, vt eorum ratio ſit $\alpha : \beta$, tum autem pro elemento vtriusque curuae ∂s eiusmodi formulam accipi oportet, quae in quaternos valores ſin. $\alpha\Phi$, coſ. $\alpha\Phi$; ſin. $\beta\Phi$, coſ. $\beta\Phi$, ducta euadat integrabilis, id quod eueniet, vti vidimus, ſi ſtatuatur

$$\partial s = v\,\partial\Phi + \left(\frac{1}{\alpha\alpha} + \frac{1}{\beta\beta}\right)\frac{\partial\partial v}{\partial\Phi} + \frac{1}{\alpha\alpha\beta\beta}\cdot\frac{\partial^4 . v}{\partial\Phi^3},$$

vbi loco v functionem quamcunque algebraicam anguli Φ aſſumere licet; tum enim ſi pro priore curua A Y elementum ∂s ita ordinetur, vt ſit

$$\partial s = v\,\partial\Phi + \frac{\partial\partial v}{\alpha\alpha\,\partial\Phi} + \frac{1}{\beta\beta}\left(\frac{\partial\partial v}{\partial\Phi} + \frac{\partial^4 v}{\alpha\alpha\,\partial\Phi^3}\right),$$

euidens eſt huius curuae coordinatas fore

$$AX = x = -\frac{v\ \mathrm{cos}.\ \alpha\Phi}{\alpha} + \frac{\partial v\ \mathrm{sin}.\ \alpha\Phi}{\alpha\alpha\,\partial\Phi} - \frac{\partial\partial v\ \mathrm{cos}.\ \alpha\Phi}{\alpha\beta\beta\,\partial\Phi^2} + \frac{\partial^3 v\ \mathrm{sin}.\ \alpha\Phi}{\alpha\alpha\beta\beta\,\partial\Phi^3}$$

$$AY = y = \frac{v\ \mathrm{sin}.\ \alpha\Phi}{\alpha} + \frac{\partial v\ \mathrm{cos}.\ \alpha\Phi}{\alpha\alpha\,\partial\Phi} + \frac{\partial\partial v\ \mathrm{sin}.\ \alpha\Phi}{\alpha\beta\beta\,\partial\Phi^2} + \frac{\partial^3 v\ \mathrm{cos}.\ \alpha\Phi}{\alpha\alpha\beta\beta\,\partial\Phi^3}.$$

Pro altera autem curua expreſſio elementi ∂s ita ordinetur, vt ſit

$$\partial s = v\,\partial\Phi + \frac{\partial\partial v}{\beta\beta\,\partial\Phi} + \frac{1}{\alpha\alpha}\left(\frac{\partial\partial v}{\partial\Phi} + \frac{\partial^4 v}{\beta\beta\,\partial\Phi^3}\right),$$

vnde coordinatae alterius curuae B Y′ deducuntur:

B X′

$$BX' = x' = -\frac{v\,cof.\,\beta\Phi}{\beta} + \frac{\partial v\,fin.\,\beta\Phi}{\beta\beta\,\partial\Phi} - \frac{\partial\partial v\,cof.\,\beta\Phi}{\beta\alpha\alpha\,\partial\Phi^2} - \frac{\partial^3 v\,fin.\,\beta\Phi}{\alpha\alpha\beta\beta\,\partial\Phi^3} \text{ et}$$

$$BX' = y' = \frac{v\,fin.\,\beta\Phi}{\beta} + \frac{\partial v\,cof.\,\beta\Phi}{\beta\beta\,\partial\Phi} + \frac{\partial\partial v\,fin.\,\beta\Phi}{\beta\alpha\alpha\,\partial\Phi^2} + \frac{\partial^3 v\,cof.\,\beta\Phi}{\alpha\alpha\beta\beta\,\partial\Phi^3}.$$

§. 9. Ex his formulis euidens eſt, ſi ratio amplitudinum $\alpha:\beta$ debeat eſſe aequalitatis, ſiue $\beta = \alpha$, tum ambas curuas inter ſe penitus fore easdem. Eatenus igitur hae duae curuae a ſe inuicem diſcrepabunt, quatenus numeri α et β erunt inaequales. Facile autem intelligitur, has ambas curuas pro algebraicis haberi non poſſe, niſi ratio $\alpha:\beta$ fuerit rationalis. Caeterum formula, quam hic pro ∂s inuenimus, omni attentione ideo eſt digna, quod per quadruplices multiplicatores redditur integrabilis. Totam autem hanc ſolutionem exemplo illuſtraſſe iuuabit.

Exemplum.

Quo ſtatuitur $v = cof.\,\Phi$.

§. 10. Cum igitur hinc ſit $\int v\,\partial\Phi = fin.\,\Phi$, ambae noſtrae curuae ſimul erunt rectificabiles; deinde ergo erit differentiando $\frac{\partial v}{\partial\Phi} = -fin.\,\Phi$; $\frac{\partial\partial v}{\partial\Phi^2} = -cof.\,\Phi$; $\frac{\partial^3 v}{\partial\Phi^3} = fin.\,\Phi$; et $\frac{\partial^4 v}{\partial\Phi^4} = cof.\,\Phi$; ex his conficitur vterque arcus

$$AY = BY' = s = fin.\,\Phi - \left(\frac{1}{\alpha\alpha} + \frac{1}{\beta\beta}\right) fin.\,\Phi + \frac{1}{\alpha\alpha\beta\beta}\, fin.\,\Phi, \text{ ſiue}$$

$$s = fin.\,\Phi \left(1 - \frac{1}{\alpha\alpha} - \frac{1}{\beta\beta} + \frac{1}{\alpha\alpha\beta\beta}\right) = fin.\,\Phi \left(1 - \frac{1}{\alpha\alpha}\right)\left(1 - \frac{1}{\beta\beta}\right),$$

vnde patet, ſi eſſet $\alpha = 1$, vel $\beta = 1$, tum iſtos arcus perpetuo fore $= 0$. Quoniam autem non tam ipſi numeri α et β, quam eorum ratio $\alpha:\beta$ praeſcribitur, nihil impedit, quo minus numeri α et β quantumuis magni accipiantur, ita vt ipſa horum numerorum magnitudo arbitrio noſtro relinquatur.

§. 11. Hinc igitur patet longitudinem vtriusque arcus perpetuo ſinui anguli Φ eſſe proportionalem, deinde vero pro priore curua AY coordinatae ita erunt determinatae:

$$AX = x = -\frac{cos.\Phi\, cos.\alpha\Phi}{\alpha} - \frac{sin.\Phi\, sin.\alpha\Phi}{\alpha\alpha}$$

$$+\frac{cos.\Phi\, cos.\alpha\Phi}{\alpha\beta\beta} + \frac{sin.\Phi\, sin.\alpha\Phi}{\alpha\alpha\beta\beta}, \text{ siue}$$

$$x = -\frac{1}{\alpha\alpha}(1 - \frac{1}{\beta\beta})(\alpha\, cos.\Phi\, cos.\alpha\Phi + sin.\Phi\, sin.\alpha\Phi);$$

$$XY = y = \frac{cos.\Phi\, sin.\alpha\Phi}{\alpha} - \frac{sin.\Phi\, cos.\alpha\Phi}{\alpha\alpha}$$

$$-\frac{cos.\Phi\, sin.\alpha\Phi}{\alpha\beta\beta} + \frac{sin.\Phi\, cos.\alpha\Phi}{\alpha\alpha\beta\beta}, \text{ siue}$$

$$y = \frac{1}{\alpha\alpha}(1 - \frac{1}{\beta\beta})(\alpha\, cos.\Phi\, sin.\alpha\Phi - sin.\Phi\, cos.\alpha\Phi).$$

Pro altera autem curua coordinatae erunt:

$$BX' = x' = -\frac{cos.\Phi\, cos.\beta\Phi}{\beta} - \frac{sin.\Phi\, cos.\beta\Phi}{\beta\beta}$$

$$+\frac{cos.\Phi\, cos.\beta\Phi}{\beta\alpha\alpha} + \frac{sin.\Phi\, sin.\beta\Phi}{\alpha\alpha\beta\beta}, \text{ siue}$$

$$x' = -\frac{1}{\beta\beta}(1 - \frac{1}{\alpha\alpha})(\beta\, cos.\Phi\, cos.\beta\Phi + sin.\Phi\, sin.\beta\Phi);$$

$$X'Y' - y' = \frac{cos.\Phi\, sin.\beta\Phi}{\beta} - \frac{sin.\Phi\, cos.\beta\Phi}{\beta\beta}$$

$$-\frac{cos.\Phi\, sin.\beta\Phi}{\beta\alpha\alpha} + \frac{sin.\Phi\, cos.\beta\Phi}{\alpha\alpha\beta\beta}, \text{ siue}$$

$$y' = \frac{1}{\beta\beta}(1 - \frac{1}{\alpha\alpha})(\beta\, cos.\Phi\, sin.\beta\Phi - sin.\Phi\, cos.\beta\Phi).$$

His autem valoribus eiusmodi constantes adiungi oportet, vt euanescant facto $\Phi = 0$.

§. 12. Quo haec clarius appareant, stabiliatur ratio illa $\alpha:\beta$ vt $1:2$, sumaturque $\alpha = 2$ et $\beta = 4$, hincque vterque arcus erit $s = \frac{45}{64} sin.\Phi$, tum vero coordinatae pro priore curua colliguntur:

$$x = \frac{15}{32} - \frac{15}{64}(2\, cos.\Phi\, cos.2\Phi + sin.\Phi\, sin.2\Phi) \text{ et}$$

$$y = \frac{15}{64}(2\, cos.\Phi\, sin.2\Phi - sin.\Phi\, cos.2\Phi),$$

pro altera autem curua erit

$$x' = \frac{3}{16} - \frac{3}{64}(4\, cos.\Phi\, cos.4\Phi + sin.\Phi\, sin.4\Phi) \text{ et}$$

$$y' = \frac{3}{64}(4\, cos.\Phi\, sin.4\Phi - sin.\Phi\, cos.4\Phi).$$

Hae

Hae autem expreſſiones ad ſimplices ſinus et coſinus reduci poſſunt; erit enim pro priore curua:

$$x = \frac{15}{32} - \frac{45}{128} \text{cof.} \, \Phi - \frac{15}{128} \text{cof.} \, 3 \Phi \text{ et}$$

$$y = \frac{15}{128} \text{fin.} \, 3 \Phi + \frac{45}{128} \text{fin.} \, \Phi,$$

pro altera autem curua ſimpliciter habebimus:

$$x' = \frac{3}{16} - \frac{15}{128} \text{cof.} \, 3 \Phi - \frac{9}{128} \text{cof.} \, 5 \Phi \text{ et}$$

$$y' = + \frac{9}{128} \text{fin.} \, 3 \Phi + \frac{15}{128} \text{fin.} \, 5 \Phi.$$

Vnde patet curuam poſteriorem ad altiorem ordinem aſſurgere quam priorem. Caeterum haud difficulter perſpicietur, omnes has curuas ex poſitione: $v = \text{cof.} \, \Phi$ oriundas, eſſe Epicycloides.

Problema generalius.

§. 13. *Inuenire tres curuas algebraicas* $A\Upsilon$, $B\Upsilon'$ *et* $C\Upsilon''$, *in quibus ſi a punctis fixis* A, B, C *arcus aequales* $A\Upsilon$, $B\Upsilon'$ *et* $C\Upsilon''$ *abſcindantur, eorum amplitudines, ſeu anguli* $AZ\Upsilon$, $BZ'\Upsilon'$ $CZ''\Upsilon''$, *datam inter ſe teneant rationem, vt* $\alpha : \beta : \gamma$. Tab. I. Fig. 1. 2. 3.

Solutio.

Ponantur iſtae amplitudines, ſeu anguli $AZY = \alpha\Phi$; $BZ'Y' = \beta\Phi$ et $CZ''Y'' = \gamma\Phi$; ac pro elemento cuiusque curuae ∂s eiusmodi formula requiritur, quae tam per ſinus quam coſinus ſingularum amplitudinum multiplicata euadat integrabilis; huiusmodi autem formula, vti mox patebit, eſt haec:

$$\partial s = v \partial \Phi + \left(\frac{1}{\alpha\alpha} + \frac{1}{\beta\beta} + \frac{1}{\gamma\gamma}\right) \frac{\partial \partial v}{\partial \Phi}$$
$$+ \left(\frac{1}{\alpha^2 \beta^2} + \frac{1}{\alpha^2 \gamma^2} + \frac{1}{\beta^2 \gamma^2}\right) \frac{\partial^4 v}{\partial \Phi^3} + \frac{1}{\alpha\alpha\beta\beta\gamma\gamma} \frac{\partial^6 v}{\partial \Phi^5},$$

vbi pro v functionem quamcunque algebraicam anguli Φ accipere licet, id quod ita eſt intelligendum, vt v ſit functio quaecunque algebraica ſiue ſinus, ſiue coſinus, ſiue tangentis anguli

anguli Φ. Haec enim expreſſio pro ∂s aſſumta ita eſt comparata, vt ſi multiplicetur per ſingulos ſex hos factores: 1°. ſin. $\alpha\Phi$; 2°. coſ. $\alpha\Phi$; 3°. ſin. $\beta\Phi$; 4°. coſ. $\beta\Phi$; 5°. ſin. $\gamma\Phi$ et 6°. coſ. $\gamma\Phi$, euadat integrabilis, quemadmodum ex ſequentibus perſpicietur.

§. 14. Pro prima curua AY, cuius amplitudo eſt $=\alpha\Phi$, elementum curuae ita repraeſentetur:

$$\partial s = v\,\partial\Phi + \frac{\partial\partial v}{\alpha\alpha\,\partial\Phi} + \left(\frac{1}{\beta\beta} + \frac{1}{\gamma\gamma}\right)\left(\frac{\partial\partial v}{\partial\Phi} + \frac{\partial^4 v}{\alpha\alpha\,\partial\Phi^3}\right) + \frac{1}{\beta\beta\gamma\gamma}\left(\frac{\partial^4 v}{\partial\Phi^3} + \frac{\partial^6 v}{\alpha\alpha\,\partial\Phi^5}\right),$$

quae expreſſio manifeſto ab ante propoſita non diſcrepat. Haec autem terna membra integrabilia euadunt, ſiue ducantur in ſin. $\alpha\Phi$ ſiue in coſ. $\alpha\Phi$, vnde colligentur coordinatae huius curuae, ſcilicet:

$$AX = x = -\frac{v\,\mathrm{cof.}\,\alpha\Phi}{\alpha} + \frac{\partial v\,\mathrm{fin.}\,\alpha\Phi}{\alpha\alpha\,\partial\Phi} + \left(\frac{1}{\beta\beta} + \frac{1}{\gamma\gamma}\right)\left(-\frac{\partial\partial v\,\mathrm{cof.}\,\alpha\Phi}{\alpha\,\partial\Phi^2} + \frac{\partial^3 v\,\mathrm{fin.}\,\alpha\Phi}{\alpha\alpha\,\partial\Phi^3}\right) + \frac{1}{\beta\beta\gamma\gamma}\left(-\frac{\partial^4 v\,\mathrm{cof.}\,\alpha\Phi}{\alpha\,\partial\Phi^4} + \frac{\partial^5 v\,\mathrm{fin.}\,\alpha\Phi}{\alpha\alpha\,\partial\Phi^5}\right),$$

$$XY = y = \frac{v\,\mathrm{fin.}\,\alpha\Phi}{\alpha} + \frac{\partial v\,\mathrm{cof.}\,\alpha\Phi}{\alpha\alpha\,\partial\Phi} + \left(\frac{1}{\beta\beta} + \frac{1}{\gamma\gamma}\right)\left(\frac{\partial\partial v\,\mathrm{fin.}\,\alpha\Phi}{\alpha\,\partial\Phi^2} + \frac{\partial^3 v\,\mathrm{cof.}\,\alpha\Phi}{\alpha\alpha\,\partial\Phi^3}\right) + \frac{1}{\beta\beta\gamma\gamma}\left(\frac{\partial^4 v\,\mathrm{fin.}\,\alpha\Phi}{\alpha\,\partial\Phi^4} + \frac{\partial^5 v\,\mathrm{cof.}\,\alpha\Phi}{\alpha\alpha\,\partial\Phi^5}\right).$$

§. 15. Pro ſecunda curua BY′ operatio ſimili modo inſtitui poterit; verum quia litterae α, β, γ inter ſe permutari poſſunt, per analogiam eius coordinatae ex praecedentibus facillime formantur, ſiquidem erit

$$BY' = x' = -\frac{v\,\mathrm{cof.}\,\beta\Phi}{\beta} + \frac{\partial v\,\mathrm{fin.}\,\beta\Phi}{\beta\beta\,\partial\Phi} + \left(\frac{1}{\gamma\gamma} + \frac{1}{\alpha\alpha}\right)\left(-\frac{\partial\partial v\,\mathrm{cof.}\,\beta\Phi}{\beta\,\partial\Phi^2} + \frac{\partial^3 v\,\mathrm{fin.}\,\beta\Phi}{\beta\beta\,\partial\Phi^3}\right) + \frac{1}{\gamma\gamma\alpha\alpha}\left(-\frac{\partial^4 v\,\mathrm{cof.}\,\beta\Phi}{\beta\,\partial\Phi^4} + \frac{\partial^5 v\,\mathrm{fin.}\,\beta\Phi}{\beta\beta\,\partial\Phi^5}\right)$$ et

$X'Y' =$

$$X'Y' = y' = \frac{v \sin. \beta \Phi}{\beta} + \frac{\partial v \cos. \beta \Phi}{\beta \beta \partial \Phi}$$
$$+ \left(\frac{1}{\gamma\gamma} + \frac{1}{\alpha\alpha}\right)\left(\frac{\partial \partial v \sin. \beta \Phi}{\beta \partial \Phi^2} + \frac{\partial^3 v \cos. \beta \Phi}{\beta \beta \partial \Phi^3}\right)$$
$$+ \frac{1}{\gamma\gamma\alpha\alpha}\left(\frac{\partial^4 v \sin. \beta \Phi}{\beta \partial \Phi^4} + \frac{\partial^5 v \cos. \beta \Phi}{\beta \beta \partial \Phi^5}\right).$$

§. 16. Eodem modo pro tertia curua C Y″ coordinatae sequenti modo exprimentur:

$$C Y'' = x'' = -\frac{v \cos. \gamma \Phi}{\gamma} + \frac{\partial v \sin. \gamma \Phi}{\gamma \gamma \partial \Phi}$$
$$+ \left(\frac{1}{\alpha\alpha} + \frac{1}{\beta\beta}\right)\left(-\frac{\partial \partial v \cos. \gamma \Phi}{\gamma \partial \Phi^2} + \frac{\partial^3 v \sin. \gamma \Phi}{\gamma \gamma \partial \Phi^3}\right)$$
$$+ \frac{1}{\alpha\alpha\beta\beta}\left(-\frac{\partial^4 v \cos. \gamma \Phi}{\gamma \partial \Phi^4} + \frac{\partial^5 v \sin. \gamma \Phi}{\gamma \gamma \partial \Phi^5}\right)$$

$$X'' Y'' = y'' = \frac{v \sin. \gamma \Phi}{\gamma} + \frac{\partial v \cos. \gamma \Phi}{\gamma \gamma \partial \Phi}$$
$$+ \left(\frac{1}{\alpha\alpha} + \frac{1}{\beta\beta}\right)\left(\frac{\partial \partial v \sin. \gamma \Phi}{\gamma \partial \Phi^2} + \frac{\partial^3 v \cos. \gamma \Phi}{\gamma \gamma \partial \Phi^3}\right)$$
$$+ \frac{1}{\alpha\alpha\beta\beta}\left(\frac{\partial^4 v \sin. \gamma \Phi}{\gamma \partial \Phi^4} + \frac{\partial^5 v \cos. \gamma \Phi}{\gamma \gamma \partial \Phi^5}\right).$$

§. 17. Quoniam in his tribus curuis arcus abſciſſi A Y, B Y′, C Y″, ſunt inter ſe aequales, eorum longitudo communis per integrationem colligitur fore

$$s = \int v \partial \Phi + \left(\frac{1}{\alpha\alpha} + \frac{1}{\beta\beta} + \frac{1}{\gamma\gamma}\right)\frac{\partial v}{\partial \Phi}$$
$$+ \left(\frac{1}{\alpha\alpha\beta\beta} + \frac{1}{\alpha\alpha\gamma\gamma} + \frac{1}{\beta\beta\gamma\gamma}\right)\frac{\partial^3 v}{\partial \Phi^3} + \frac{1}{\alpha\alpha\beta\beta\gamma\gamma}\frac{\partial^5 v}{\partial \Phi^5}.$$

Vnde patet, ſi modo prima formula $\int v \partial \Phi$ integrationem admittat, tum has curuas ſimul fore rectificabiles; contra autem functionem v facile ita aſſumere licebit, vt rectificatio harum curuarum a data quadratura pendeat.

§. 18. Hae ergo tres curuae etiam ita ſunt comparatae, vt ſi in una earum A Y a termino fixo A arcus quicunque A Y $= s$ abſcindatur, in binis reliquis a terminis itidem fixis B et C arcus B Y′ et C Y″ illi aequales facile abſcindi queant. Quaeratur enim primo amplitudo arcus abſciſſi A Y, quae ſit $= \omega$, ac ponatur $\omega = \alpha \Phi$, vt ſit $\Phi = \frac{\omega}{\alpha}$; tum in fecunda curua quaeratur arcus B Y′, cuius amplitudo ſit $=\beta\Phi = \frac{\beta\omega}{\alpha}$, id quod praeſtabitur, quaerendo punctum Y′, vbi tangens ad axem inclinetur ſub angulo $= 90 - \beta\Phi$, quo facto arcus B Y′ aequalis erit arcui A Y; ſimilique modo in tertia curua arcus C Y″ reperietur.

Problema.

Tab. I. *Inuenire quatuor curuas algebraicas A Υ, B Υ′, C Υ″ et*
Fig. 1..4. *D Υ‴, ſuis axibus in punctis A, B, C, D normaliter inſiſtentes, in quibus ſi ab his terminis arcus aequales A Υ, B Υ′, C Υ″, D Υ‴ abſcindantur, eorum amplitudines eandem inter ſe teneant rationem, quam habent quatuor numeri α, β, γ et δ.*

Solutio.

§. 19. Poſitis his quatuor amplitudinibus $AZY = \alpha\Phi$; $B Z' Y' = \beta\Phi$; $C Z'' Y'' = \gamma\Phi$ et $D Z''' Y''' = \delta\Phi$, totum negotium eo redit, vt eiusmodi formula differentialis pro elemento curuae ∂s inueſtigetur, quae tam per ſinus quam coſinus horum quatuor angulorum multiplicata euadat integrabilis, cui conditioni ſatisfacere facile perſpicietur haec formula:

$$\partial s = \begin{cases} v\,\partial\Phi + \left(\frac{1}{\alpha\alpha} + \frac{1}{\beta\beta} + \frac{1}{\gamma\gamma} + \frac{1}{\delta\delta}\right) \\ + \frac{1}{\alpha\alpha\beta\beta} + \frac{1}{\alpha\alpha\gamma\gamma} + \frac{1}{\alpha\alpha\theta\theta} + \frac{1}{\beta\beta\theta\theta} + \frac{1}{\gamma\gamma\theta\theta}\Big) \frac{\partial^4 v}{\partial\Phi^3} \\ + \frac{1}{\alpha\alpha\beta\beta\gamma\gamma} + \frac{2}{\alpha\alpha\beta\beta\theta\theta} + \frac{1}{\alpha\alpha\gamma\gamma\theta\theta} + \frac{\beta\beta\gamma\gamma\theta\theta}{\beta\beta\gamma\gamma\theta\theta}\Big) \frac{\partial^6 v}{\partial\Phi^5} \\ + \frac{1}{\alpha\alpha\beta\beta\gamma\gamma\theta\theta}\,\frac{\partial^8 v}{\partial\Phi^7}. \end{cases}$$

§. 20. Quodſi enim hinc coordinatas primae curuae AY elicere velimus, elementum curuae ∂s ſequenti modo per membra repraeſentetur:

$\partial s =$

$$\partial s = (v\,\partial\Phi + \frac{\partial\partial v}{\alpha\alpha\,\partial\Phi}) + (\frac{1}{\beta\beta} + \frac{1}{\gamma\gamma} + \frac{1}{\delta\delta})(\frac{\partial\partial v}{\partial\Phi} + \frac{\partial^4 v}{\alpha\alpha\,\partial\Phi^3})$$
$$+ (\frac{1}{\beta\beta\gamma\gamma} + \frac{1}{\beta\beta\delta\delta} + \frac{1}{\gamma\gamma\delta\delta})(\frac{\partial^4 v}{\partial\Phi^3} + \frac{\partial^6 v}{\alpha\alpha\,\partial\Phi^5})$$
$$+ \frac{1}{\beta\beta\gamma\gamma\delta\delta}(\frac{\partial^6 v}{\partial\Phi^5} + \frac{\partial^8 v}{\alpha\alpha\,\partial\Phi^7}),$$

haec enim forma primo in ſin. $\alpha\Phi$, deinde in coſ. $\alpha\Phi$, ducta et integrata dabit ambas coordinatas primae curuae, ſcilicet:

$$AX = x = -\frac{v\,\text{coſ.}\,\alpha\Phi}{\alpha} + \frac{\partial v\,\text{ſin.}\,\alpha\Phi}{\alpha\alpha\,\partial\Phi} + (\frac{1}{\beta\beta} + \frac{1}{\gamma\gamma} + \frac{1}{\delta\delta}) \times$$
$$\times (-\frac{\partial\partial v\,\text{coſ.}\,\alpha\Phi}{\alpha\,\partial\Phi^2} + \frac{\partial^3 v\,\text{ſin.}\,\alpha\Phi}{\alpha\alpha\,\partial\Phi^3})$$
$$+ (\frac{1}{\beta\beta\gamma\gamma} + \frac{1}{\beta\beta\delta\delta} + \frac{1}{\gamma\gamma\delta\delta})(-\frac{\partial^4 v\,\text{coſ.}\,\alpha\Phi}{\alpha\,\partial\Phi^4} + \frac{\partial^5 v\,\text{ſin.}\,\alpha\Phi}{\alpha\alpha\,\partial\Phi^5}$$
$$+ \frac{1}{\beta\beta\gamma\gamma\delta\delta}(-\frac{\partial^6 v\,\text{coſ.}\,\alpha\Phi}{\alpha\,\partial\Phi^6} + \frac{\partial^7 v\,\text{ſin.}\,\alpha\Phi}{\alpha\alpha\,\partial\Phi^7}) \text{ et}$$

$$XY = y = \frac{v\,\text{ſin.}\,\alpha\Phi}{\alpha} + \frac{\partial v\,\text{coſ.}\,\alpha\Phi}{\alpha\alpha\,\partial\Phi} + (\frac{1}{\beta\beta} + \frac{1}{\gamma\gamma} + \frac{1}{\delta\delta}) \times$$
$$\times (\frac{\partial\partial v\,\text{ſin.}\,\alpha\Phi}{\alpha\,\partial\Phi^2} + \frac{\partial^3 v\,\text{coſ.}\,\alpha\Phi}{\alpha\alpha\,\partial\Phi^3})$$
$$+ (\frac{1}{\beta\beta\gamma\gamma} + \frac{1}{\beta\beta\delta\delta} + \frac{1}{\gamma\gamma\delta\delta})(\frac{\partial^4 v\,\text{ſin.}\,\alpha\Phi}{\alpha\,\partial\Phi^4} + \frac{\partial^5 v\,\text{coſ.}\,\alpha\Phi}{\alpha\alpha\,\partial\Phi^5})$$
$$+ \frac{1}{\beta\beta\gamma\gamma\delta\delta}(\frac{\partial^6 v\,\text{ſin.}\,\alpha\Phi}{\alpha\,\partial\Phi^6} + \frac{\partial^7 v\,\text{coſ.}\,\alpha\Phi}{\alpha\alpha\,\partial\Phi^7}).$$

Eodem modo expreſſiones pro coordinatis reliquarum curuarum per ſolam analogiam formabuntur, dum quatuor litterae α, β, γ, δ, ordine promouentur, ita vt ſuperfluum foret iſtas formulas ſatis prolixas hic euoluere. Caeterum cum functio v penitus arbitrio noſtro relinquatur, dummodo fuerit algebraica, facile intelligitur, hanc ſolutionem maxime eſſe generalem, id quod eo magis eſt mirandum, quod ſine vlla difficultate ad plures quotcunque curuas, tali indole inter ſe connexas, accommodari poſſit, dum alioquin inuentio huiusmodi formularum differentialium, quae per plures factores multiplicatae euadunt integrabiles, maximis difficultatibus obuoluta deprehenditur.

§. 21. Quantuscunque enim fuerit numerus huiusmodi curuarum inueniendarum, quarum amplitudines ſeſe habere

 debeant

debeant in ratione numerorum α, β, γ, δ, ε, ζ, etc. tum primo confiderentur fractiones $\frac{1}{\alpha\alpha}$; $\frac{1}{\beta\beta}$; $\frac{1}{\gamma\gamma}$; $\frac{1}{\delta\delta}$; $\frac{1}{\varepsilon\varepsilon}$; etc. quarum fumma ponatur $= A$, tum vero fumma productorum ex binis ftatuatur $= B$, fumma productorum ex ternis $= C$, ex quaternis $= D$, etc. donec perueniatur ad productum ex omnibus, quibus conftitutis, fi elementum curuae ∂s hac formula referatur:

$$\partial s = v\,\partial\Phi + A\,\frac{\partial\partial v}{\partial\Phi} + B\,\frac{\partial^4 v}{\partial\Phi^3} + C\,\frac{\partial^6 v}{\partial\Phi^5} + D\,\frac{\partial^8 v}{\partial\Phi^7}, \text{ etc.}$$

ea hac infigni praedita erit proprietate, vt fiue multiplicetur per finus fingulorum angulorum $\alpha\Phi$, $\beta\Phi$, $\gamma\Phi$, $\delta\Phi$, etc. fiue per eorum cofinus, ea femper integrationem admittat, vnde huic Analyfeos generi, parum adhuc exculto, haud contemnendum incrementum allatum eft cenfendum.

DE

Fig. 1.

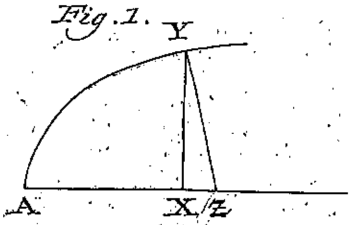

Fig. 2.

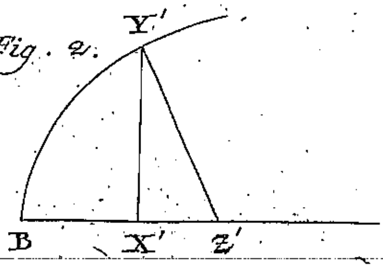

Fig. 3.

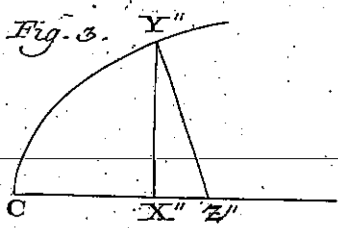

Fig. 4.

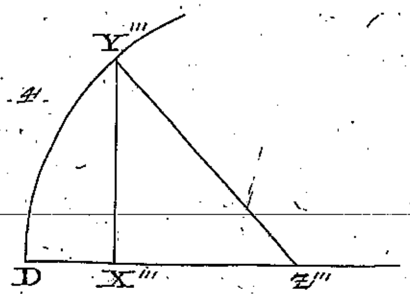

Fig. 5.

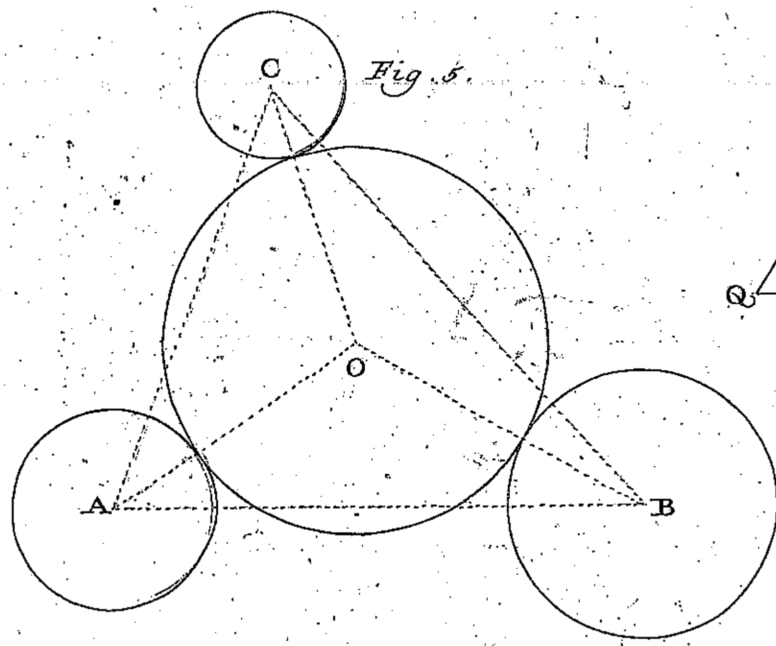

Fig. 6.