1790

De curvis algebraicis, quarum longitudo exprimitur hac formula integrali $\int(v^{m-1}\,\partial v)/\surd(1-v^{2n})$

Leonhard Euler

DE CVRVIS ALGEBRAICIS, QVARVM LONGITVDO EXPRIMITVR HAC FORMVLA INTEGRALI

$$\int \frac{v^{m-1}\partial v}{\sqrt{(1-v^{2n})}}.$$

Auctore
L. EVLERO.

Conuent. exhib. d. 17 Iun. 1776.

§. 1.

Cum methodus certa huiusmodi problemata soluendi, quibus curuae algebraicae requiruntur, quarum longitudo per datam formulam integralem exprimatur, etiamnunc densissimis tenebris sit inuoluta, plurimum ad fines Analyseos amplificandos sine dubio conferet, si plura huius generis problemata particularia omni studio euoluantur, siquidem tum demum sperare licebit, fore vt tandem haec mysteria Analyseos vlterius penetremus. Hunc in finem constitui formulam propositam accuratius perscrutari, cuius quidem duo casus nulla prorsus laborant difficultate: alter scilicet, quando $m = 2n$, vel etiam $m = 4n$, vel $m = 6n$, etc. quia tum formula integrationem admittit, ideoque omnes plane curuae algebraicae rectificabiles satisfacere sunt censendae; alter vero est $m = n$, tum enim nostra formula, posito $v^n = z$, abit in in hanc: $\frac{\partial z}{n\sqrt{(1-zz)}}$, ideoque arcum circularem refert. Conſtat autem iam ſatis praeter circulum nullas alias lineas curuas algebraicas ſatisfacere poſſe.

§. 2. Vt autem noſtram quaeſtionem in genere ſoluamus, deſignemus coordinatas curuarum quaeſitarum litteris x et y, ipſos autem earum arcus littera s, ita vt ſit $\partial s = \sqrt{(\partial x^2 + \partial y^2)}$; et quaeſtio huc redit, vt pro x et y eiusmodi functiones algebraicae quantitatis v inueſtigentur, vt inde fiat $\partial s = \int \frac{v^{m-1}\partial v}{\sqrt{(1-v^{2n})}}$, cui quidem quaeſtioni ſatisfieri poſſet, ſi eiusmodi angulos ω aſſignare liceret, vt ambae iſtae formulae: $\partial x = \frac{v^{m-1}\partial v \operatorname{cof.}\omega}{\sqrt{(1-v^{2n})}}$ et $\partial y = \frac{v^{m-1}\partial v \operatorname{fin.}\omega}{\sqrt{(1-v^{2n})}}$, euaderent integrabiles. Verum nulla via patet in huiusmodi angulos inquirendi, niſi ipſa formula propoſita ante in aliam formam ad calculum angulorum magis accommodatam transformetur.

§. 3. Hunc in finem ſtatuamus $v^n = \operatorname{fin.}\Phi$, vt fiat $\sqrt{(1-v^{2n})} = \operatorname{cof.}\Phi$, tum vero erit $v^m = \operatorname{fin.}\Phi^{\frac{m}{n}}$, vbi breuitatis gratia faciamus $\frac{m}{n} = \alpha + 1$, vt ſit $v^m = \operatorname{fin.}\Phi^{\alpha+1}$, vnde differentiando erit

$$m v^{m-1}\partial v = (\alpha+1)\partial\Phi \operatorname{cof.}\Phi \operatorname{fin.}\Phi^{\alpha},$$

ita vt nunc formula reſoluenda proditura ſit

$$\partial s = \frac{\alpha+1}{m}\partial\Phi \operatorname{fin.}\Phi^{\alpha} = \frac{1}{n}\partial\Phi \operatorname{fin.}\Phi^{\alpha}.$$

Quo autem hoc negotium facilius expediamus, duas ſequentes formulas:

$$z = \operatorname{fin.}\lambda\Phi \operatorname{fin.}\Phi^{\alpha+1} \text{ et } z = \operatorname{cof.}\lambda\Phi \operatorname{fin.}\Phi^{\alpha+1}$$

ſtudio euoluamus.

Euolutio formulae prioris

$$z = \sin.\lambda\phi\,(\sin.\phi)^{\alpha+1}.$$

§. 4. Quodsi istam formulam differentiemus, prodibit

$$\frac{\partial z}{\partial\phi} = \lambda\cos.\lambda\phi\,(\sin.\phi)^{\alpha+1} + (\alpha+1)\sin.\lambda\phi\cos.\phi\,(\sin.\phi)^{\alpha},$$

siue

$$\frac{\partial z}{\partial\phi} = \sin.\phi^{\alpha}\,(\lambda\cos.\lambda\phi\sin.\phi + (\alpha+1)\sin.\lambda\phi\cos.\phi).$$

Iam in subsidium vocentur reductiones notissimae:

$$\sin.\lambda\phi\cos.\phi = \tfrac{1}{2}\sin.(\lambda+1)\phi + \tfrac{1}{2}\sin.(\lambda-1)\phi \text{ et}$$
$$\cos.\lambda\phi\sin.\phi = \tfrac{1}{2}\sin.(\lambda+1)\phi - \tfrac{1}{2}\sin.(\lambda-1)\phi,$$

quibus valoribus substitutis reperiemus:

$$\frac{2\partial z}{\partial\phi} = \sin.\phi^{\alpha}\,[(\alpha+1+\lambda)\sin.(\lambda+1)\phi + (\alpha+1-\lambda)\sin.(\lambda-1)\phi],$$

vnde colligimus hanc integrationem:

$$2\sin.\lambda\phi\sin.\phi^{\alpha+1} = (\alpha+1+\lambda)\int\partial\phi\sin.\phi^{\alpha}\sin.(\lambda+1)\phi$$
$$+ (\alpha+1-\lambda)\int\partial\phi\sin.\phi^{\alpha}\sin.(\lambda-1)\phi),$$

vbi notetur esse $\partial\phi\sin.\phi^{\alpha} = n\,\partial s$.

§. 5. Ponamus nunc statim $\lambda = \alpha + 1 = \frac{m}{n}$, atque integratio inuenta praebebit

$$\sin.\tfrac{m}{n}\phi\sin.\phi^{\frac{m}{n}} = m\int\partial s\sin.\left(\tfrac{m+n}{n}\right)\phi.$$

vnde viciffim conficitur

$$\int\partial s\sin.\tfrac{m+n}{n}\phi = \tfrac{1}{m}\sin.\tfrac{m}{n}\phi\sin.\phi^{\frac{m}{n}}.$$

Hinc si fuerit $\partial y = \partial s\sin.\left(\frac{m+n}{n}\right)\phi$, valor ipsius y erit algebraicus.

§. 6. Sumamus nunc in nostra integratione generali $\lambda = 1 + \frac{m+n}{n} = \frac{m+2n}{n}$, atque habebimus

$$\sin.\left(\tfrac{m+2n}{n}\right)\Phi \sin.\Phi^{\frac{m}{n}} = (m+n)\int \partial s \sin.\left(\tfrac{m+3n}{n}\right)\Phi - n\int \partial s \sin.\left(\tfrac{m+n}{n}\right)\Phi;$$

vbi valorem integralis posterioris iam ante definiuimus, quare integrale prius sequenti modo exprimetur:

$$\int \partial s \sin.\left(\tfrac{m+3n}{n}\right)\Phi = \tfrac{1}{m+n}\sin.\left(\tfrac{m+2n}{n}\right)\Phi \sin.\Phi^{\frac{m}{n}} + \tfrac{n}{m(m+n)}\sin.\tfrac{m}{n}\Phi \sin.\Phi^{\frac{m}{n}},\ \text{siue}$$

$$\int \partial s \sin.\left(\tfrac{m+3n}{n}\right)\Phi = \tfrac{1}{m+n}\sin.\Phi^{\frac{m}{n}}\left[\sin.\left(\tfrac{m+2n}{n}\right)\Phi + \tfrac{n}{m}\sin.\tfrac{m}{n}\Phi\right].$$

§. 7. Ponamus porro in forma generali $\lambda - 1 = \frac{m+3n}{n}$, siue $\lambda = \frac{m+4n}{n}$, ac reperiemus

$$\sin.\left(\tfrac{m+4n}{n}\right)\Phi \sin.\Phi^{\frac{m}{n}} = (m+2n)\int \partial s \sin.\left(\tfrac{m+5n}{n}\right)\Phi - 2n\int \partial s \sin.\left(\tfrac{m+3n}{n}\right)\Phi,$$

vbi cum posterius integrale modo inuenerimus, prius sequenti modo exprimetur:

$$\int \partial s \sin.\left(\tfrac{m+5n}{n}\right)\Phi = \tfrac{1}{m+2n}\sin.\left(\tfrac{m+4n}{n}\right)\Phi \sin.\Phi^{\frac{m}{n}} + \tfrac{2n}{m+2n}\int \partial s \sin.\left(\tfrac{m+3n}{n}\right)\Phi.$$

§. 8. Simili modo statuamus nunc $\lambda - 1 = \frac{m+5n}{n}$, siue $\lambda = \frac{m+6n}{n}$, atque nanciscimur sequentem integrationem:

$$\sin.\left(\tfrac{m+6n}{n}\right)\Phi \sin.\Phi^{\frac{m}{n}} = (m+3n)\int \partial s \sin.\left(\tfrac{m+7n}{n}\right)\Phi - 3n\int \partial s \sin.\left(\tfrac{m+5n}{n}\right)\Phi,$$

vnde concludimus fore:

$$\int \partial s \sin.\left(\tfrac{m+7n}{n}\right)\Phi = \tfrac{1}{m+3n}\sin.\left(\tfrac{m+6n}{n}\right)\Phi \sin.\Phi^{\frac{m}{n}} + \tfrac{3n}{m+3n}\int \partial s \sin.\left(\tfrac{m+5n}{n}\right)\Phi.$$

§. 9. Lex, qua hae formulae continuo vlterius procedunt, ſatis eſt manifeſta, ita vt non opus ſit calculum vltra proſequi. At quo eas diſtinctius obtutui exponamus, ſit breuitatis gratia $\sin.\phi^{\frac{m}{n}} = \Phi$, et ſingulae formulae integrales hinc oriundae ita ſe habebunt:

$$\text{I. } \int \partial s \sin.\left(\tfrac{m+n}{n}\right)\phi = \tfrac{1}{m}\,\Phi \sin.\tfrac{m}{n}\phi,$$

$$\text{II. } \int \partial s \sin.\left(\tfrac{m+3n}{n}\right)\phi = \tfrac{1}{m+n}\,\Phi \sin.\left(\tfrac{m+2n}{n}\right)\phi + \tfrac{n}{m+n}\int \partial s \sin.\left(\tfrac{m+n}{n}\right)\phi,$$

$$\text{III. } \int \partial s \sin.\left(\tfrac{m+5n}{n}\right)\phi = \tfrac{1}{m+2n}\,\Phi \sin.\left(\tfrac{m+4n}{n}\right)\phi + \tfrac{2n}{m+2n}\int \partial s \sin.\left(\tfrac{m+3n}{n}\right)\phi,$$

$$\text{IV. } \int \partial s \sin.\left(\tfrac{m+7n}{n}\right)\phi = \tfrac{1}{m+3n}\,\Phi \sin.\left(\tfrac{m+6n}{n}\right)\phi + \tfrac{3n}{m+3n}\int \partial s \sin.\left(\tfrac{m+5n}{n}\right)\phi,$$

$$\text{V. } \int \partial s \sin.\left(\tfrac{m+9n}{n}\right)\phi = \tfrac{1}{m+4n}\,\Phi \sin.\left(\tfrac{m+8n}{n}\right)\phi + \tfrac{4n}{m+4n}\int \partial s \sin.\left(\tfrac{m+7n}{n}\right)\phi,$$

$$\text{VI. } \int \partial s \sin.\left(\tfrac{m+11n}{n}\right)\phi = \tfrac{1}{m+5n}\,\Phi \sin.\left(\tfrac{m+10n}{n}\right)\phi + \tfrac{5n}{m+5n}\int \partial s \sin.\left(\tfrac{m+9n}{n}\right)\phi,$$

etc. etc.

§. 10. Quodſi iam in ſingulis his formulis valores integralis praecedentis ſubſtituamus, adipiſcemur ſequentes integrationes ad noſtrum vſum accommodatas:

$$\text{I. } \int \partial s \sin.\left(\tfrac{m+n}{n}\right)\phi = \tfrac{\Phi}{m}\sin.\tfrac{m}{n}\phi,$$

$$\text{II. } \int \partial s \sin.\left(\tfrac{m+3n}{n}\right)\phi = \tfrac{\Phi}{m+n}\left[\sin.\left(\tfrac{m+2n}{n}\right)\phi + \tfrac{n}{m}\sin.\tfrac{m}{n}\phi\right],$$

$$\text{III. } \int \partial s \sin.\left(\tfrac{m+5n}{n}\right)\phi = \tfrac{\Phi}{m+2n}\left[\sin.\left(\tfrac{m+4n}{n}\right)\phi + \tfrac{2n}{m+n}\sin.\tfrac{m+2n}{n}\phi + \tfrac{n.2n}{m(m+n)}\sin.\tfrac{m}{n}\phi\right],$$

IV.

IV. $\int \partial s \sin. \left(\frac{m+7n}{n}\right)\Phi$

$$= \frac{\Phi}{m+3n}\left[\sin.\left(\frac{m+6n}{n}\right)\Phi + \frac{3n}{m+2n}\sin.\left(\frac{m+4n}{n}\right)\Phi + \frac{2n.3n}{(m+n)(m+2n)}\sin.\frac{(m+2n)}{n}\Phi + \frac{n.2n.3n}{m(m+n)(m+2n)}\sin.\frac{m}{n}\Phi\right],$$

V. $\int \partial s \sin. \left(\frac{m+9n}{n}\right)\Phi$

$$= \frac{\Phi}{m+4n}\left[\sin.\left(\frac{m+8n}{n}\right)\Phi + \frac{4n}{m+3n}\sin.\left(\frac{m+6n}{n}\right)\Phi + \frac{3n.4n}{(m+2n)(m+3n)}\sin.\left(\frac{m+4n}{n}\right)\Phi + \frac{2n.3n.4n}{(m+n)(m+2n)(m+3n)}\sin.\left(\frac{m+2n}{n}\right)\Phi + \frac{n.2n.3n.4n}{m(m+n)(m+2n)(m+3n)}\sin.\frac{m}{n}\Phi\right].$$

VI. $\int \partial s \sin. \left(\frac{m+11n}{n}\right)\Phi$

$$= \frac{\Phi}{m+5n}\left[\sin.\left(\frac{m+10n}{n}\right)\Phi + \frac{5n}{m+4n}\sin.\left(\frac{m+8n}{n}\right)\Phi + \frac{4n.5n}{(m+3n)(m+4n)}\sin.\left(\frac{m+6n}{n}\right)\Phi + \frac{3n.4n.5n}{(m+2n)(m+3n)(m+4n)} \times \sin.\left(\frac{m+4n}{n}\right)\Phi + \frac{2n.3n.4n.5n}{(m+n)(m+2n)(m+3n)(m+4n)}\sin.\left(\frac{m+2n}{n}\right)\Phi + \frac{n.2n.3n.4n.5n}{m(m+n)(m+2n)(m+3n)(m+4n)(m+5n)}\sin.\frac{m}{n}\right].$$

etc. etc.

vbi tantum meminisse oportet esse $\Phi = \sin.\Phi^{\frac{m}{n}}$.

§. 11. Hae formulae adhuc concinniores reddi possunt ponendo $\frac{m}{n} = k$, vt sit $\Phi = \sin.\Phi^k$, tum vero sequentes orientur formulae integrales:

I. $\int \partial s \sin.(k+1)\Phi = \dfrac{\sin.\Phi^k}{nk}\sin.k\Phi$,

II. $\int \partial s \sin.(k+3)\Phi = \dfrac{\sin.\Phi^k}{n(k+1)}\left[\sin.(k+2)\Phi + \frac{1}{k}\sin.k\Phi\right]$,

III. $\int \partial s \sin.(k+5)\Phi$

$$= \frac{\sin.\Phi^k}{n(k+2)}\left[\sin.(k+4)\Phi + \frac{2}{k+1}\sin.(k+2)\Phi + \frac{1.2}{k(k+1)}\sin.k\Phi\right],$$

IV. $\int \partial s \sin.(k+7)\Phi$

$$= \frac{\sin.\Phi^k}{n(k+3)} \left\{ \begin{array}{l} \sin.(k+6)\Phi + \frac{3}{k+2}\sin.(k+4)\Phi \\ + \frac{2.\,3}{(k+1)(k+2)}\sin.(k+2)\Phi + \frac{1.\,2.\,3}{k(k+1)(k+2)}\sin.k\Phi \end{array} \right\};$$

V. $\int \partial s \sin.(k+9)\Phi$

$$= \frac{\sin.\Phi^k}{n(k+4)} \left\{ \begin{array}{l} \sin.(k+8)\Phi + \frac{4}{k+3}\sin.(k+6)\Phi \\ + \frac{3.\,4}{(k+2)(k+3)}\sin.(k+4)\Phi + \frac{2.\,3.\,4}{(k+1)(k+2)(k+3)}\sin.(k+2)\Phi \\ + \frac{1.\,2.\,3.\,4}{k(k+1)(k+2)(k+3)}\sin.k\Phi, \end{array} \right\},$$

VI. $\int \partial s \sin.(k+11)\Phi$

$$= \frac{\sin.\Phi^k}{n(k+5)} \left\{ \begin{array}{l} \sin.(k+10)\Phi + \frac{5}{k+4}\sin.(k+8)\Phi \\ + \frac{4.\,5}{(k+3)(k+4)}\sin.(k+6)\Phi + \frac{3.\,4.\,5}{(k+2)(k+3)(k+4)}\sin.(k+4)\Phi \\ + \frac{2.\,3.\,4.\,5}{(k+1)(k+2)(k+3)(k+4)}\sin.(k+2)\Phi \\ + \frac{1.\,2.\,3.\,4.\,5}{k(k+1)(k+2)(k+3)(k+4)}\sin.k\Phi. \end{array} \right\}.$$

§. 12. Hinc igitur patet, si i denotet numerum positiuum quemcunque, generatim integrale huius formae: $\int \partial s \sin.(k+2i+1)\Phi$, actu exhiberi posse: lege enim progressionis probe obseruata erit

$\int \partial s \sin.(k+2i+1)\Phi$

$$= \frac{\sin.\Phi^k}{n(k+i)} \left\{ \begin{array}{l} \sin.(k+2i)\Phi + \frac{i}{k+i-1}\sin.(k+2i-2)\Phi \\ + \frac{(i-1)\,i}{(k+i-2)(k+i-1)}\sin.(k+2i-4)\Phi \\ + \frac{(i-2)(i-1)\,i}{(k+i-3)(k+i-2)(k+i-1)}\sin.(k+2i-6)\Phi + \text{etc.} \end{array} \right\}.$$

Vbi tantum obseruetur haec integralia quandoque incongrua fieri posse, quod euenit, quoties in denominatoribus harum fractionum factor quispiam nihilo fit aequalis, siquidem his casibus integrale non amplius erit algebraicum. Hoc autem con-

contingere poterit, quoties k, hoc est $\frac{m}{n}$, fuerit vel $= 0$, vel numerus integer negatiuus, ipsi i aequalis vel minor; sin autem iste valor negatiuus ipsius k superet i, memoratum incommodum non amplius erit metuendum.

Euolutio formulae posterioris
$$z = \cos.\lambda\phi \sin.\phi^{\alpha+1}.$$

§. 13. Quodsi haec formula differentietur, prodibit
$$\frac{\partial z}{\partial \phi} = \sin.\phi^{\alpha}[(\alpha+1)\cos.\phi\cos.\lambda\phi - \lambda\sin.\phi\sin.\lambda\phi].$$
Cum nunc per notas reductiones sit
$$\cos.\phi\cos.\lambda\phi = \tfrac{1}{2}\cos.(\lambda-1)\phi + \tfrac{1}{2}\cos.(\lambda+1)\phi \text{ et}$$
$$\sin.\phi\sin.\lambda\phi = \tfrac{1}{2}\cos.(\lambda-1)\phi - \tfrac{1}{2}\cos.(\lambda+1)\phi,$$
his substitutis peruenietur ad hanc formam:
$$\frac{2\partial z}{\partial \phi} = \sin.\phi^{\alpha}[(\alpha+1-\lambda)\cos.(\lambda-1)\phi + (\alpha+1+\lambda)\cos.(\lambda+1)\phi],$$
vnde deducitur ista integratio:
$$2\cos.\lambda\phi\sin.\phi^{\alpha+1} = (\alpha+1-\lambda)\int\partial\phi\cos.(\lambda-1)\phi\sin.\phi^{\alpha}$$
$$+ (\alpha+1+\lambda)\int\partial\phi\cos.(\lambda+1)\phi\sin.\phi^{\alpha}.$$

§. 14. Quoniam igitur supra vidimus esse $\partial\phi\sin.\phi^{\alpha} = n\partial s$, ob $\alpha+1 = \frac{m}{n} = k$, ista integratio ad hanc formam redibit:
$$2\cos.\lambda\phi\sin.\phi^{k} = n(k-\lambda)\int\partial s\cos.(\lambda-1)\phi$$
$$+ n(k+\lambda)\int\partial s\cos.(\lambda+1)\phi,$$
ex qua deducimus
$$\int\partial s\cos.(\lambda+1)\phi = \frac{2}{n(k+\lambda)}\cos.\lambda\phi\sin.\phi^{k}$$
$$- \frac{(k-\lambda)}{k+\lambda}\int\partial s\cos.(\lambda-1)\phi.$$

§. 15. Ex hac forma generali iam deriuemus casus speciales, vt supra fecimus, ac primo quidem sumamus $\lambda = k$, vt obtineamus istud quasi principium sequentium integrationum, scilicet:

$$\text{I. } \int \partial s \operatorname{cos.} (k+1)\phi = \frac{\operatorname{sin.} \phi^k}{n k} \operatorname{cos.} k\phi.$$

Sumamus nunc $\lambda - 1 = k + 1$, siue $\lambda = k + 2$ et integratio generalis dabit

$$\text{II. } \int \partial s \operatorname{cos.} (k+3)\phi$$
$$= \frac{\operatorname{sin.} \phi^k}{n(k+1)} \operatorname{cos.} (k+2)\phi + \frac{1}{k+1} \int \partial s \operatorname{cos.} (k+1)\phi.$$

Fiat nunc $\lambda - 1 = k + 3$, siue $\lambda = k + 4$, ac prodibit

$$\text{III. } \int \partial s \operatorname{cos.} (k+5)\phi$$
$$= \frac{\operatorname{sin.} \phi^k}{n(k+2)} \operatorname{cos.} (k+4)\phi + \frac{2}{k+2} \int \partial s \operatorname{cos.} (k+3)\phi.$$

Sit iam vlterius $\lambda - 1 = k + 5$, siue $\lambda = k + 6$, ac prodibit

$$\text{IV. } \int \partial s \operatorname{cos.} (k+7)\phi$$
$$= \frac{\operatorname{sin.} \phi^k}{n(k+3)} \operatorname{cos.} (k+6)\phi + \frac{3}{k+3} \int \partial s \operatorname{cos.} (k+5)\phi.$$

Sit porro $\lambda - 1 = k + 7$, siue $\lambda = k + 8$, ac fiet

$$\text{V. } \int \partial s \operatorname{cos.} (k+9)\phi$$
$$= \frac{\operatorname{sin.} \phi^k}{n(k+4)} \operatorname{cos.} (k+8)\phi + \frac{4}{k+4} \int \partial s \operatorname{cos.} (k+7)\phi,$$

etc. etc.

§. 16. Quodsi iam in singulis formulis integralia praecedentia substituamus, nanciscemur sequentes integrationes:

$$\text{I. } \int \partial s \operatorname{cos.} (k+1)\phi = \frac{\operatorname{sin.} \phi^k}{n k} \operatorname{cos.} k\phi,$$

II.

II. $\int \partial s \cos.(k+3)\Phi = \frac{\sin.\Phi^k}{n(k+1)}\left[\cos.(k+2)\Phi + \frac{1}{k}\cos.k\Phi\right]$,

III. $\int \partial s \cos.(k+5)\Phi$

$$= \frac{\sin.\Phi^k}{n(k+2)}\left\{\begin{array}{l}\cos.(k+4)\Phi + \frac{2}{k+1}\cos.(k+2)\Phi \\ + \frac{1.2}{k(k+1)}\cos.k\Phi\end{array}\right\},$$

IV. $\int \partial s \cos.(k+7)\Phi$

$$= \frac{\sin.\Phi^k}{n(k+3)}\left\{\begin{array}{l}\cos.(k+6)\Phi + \frac{3}{k+2}\cos.(k+4)\Phi \\ + \frac{2.3}{(k+1)(k+2)}\cos.(k+2)\Phi \\ + \frac{1.2.3}{k(k+1)(k+2)}\cos.k\Phi\end{array}\right\},$$

V. $\int \partial s \cos.(k+9)\Phi$

$$= \frac{\sin.\Phi^k}{n(k+4)}\left\{\begin{array}{l}\cos.(k+8)\Phi + \frac{4}{k+3}\cos.(k+6)\Phi \\ + \frac{3.4}{(k+2)(k+3)}\cos.(k+4)\Phi \\ + \frac{2.3.4}{(k+1)(k+2)(k+3)}\cos.(k+2)\Phi \\ + \frac{1.2.3.4}{k(k+1)(k+2)(k+3)}\cos.k\Phi\end{array}\right\},$$

VI. $\int \partial s \cos.(k+11)\Phi$

$$= \frac{\sin.\Phi^k}{n(k+5)}\left\{\begin{array}{l}\cos.(k+10)\Phi + \frac{5}{k+4}\cos.(k+8)\Phi \\ + \frac{4.5}{(k+3)(k+4)}\cos.(k+6)\Phi \\ + \frac{3.4.5}{(k+2)(k+3)(k+4)}\cos.(k+4)\Phi \\ + \frac{2.3.4.5}{(k+1)(k+2)(k+3)(k+4)}\cos.(k+2)\Phi \\ + \frac{1.2.3.4.5}{k(k+1)(k+2)(k+3)(k+4)}\cos.k\Phi\end{array}\right\}.$$

etc. etc.

quae formulae a praecedentibus hoc tantum discrepant, vt sinus angulorum hic in cosinus sint transmutati.

§. 17. Ex his igitur casibus facile deducimus sequentem formulam integralem:

$$\int \partial s \operatorname{cos.} (k + 2i + 1) \phi$$

$$= \frac{\operatorname{fin.} \phi^k}{n(k+i)} \left\{ \begin{array}{l} \operatorname{cos.} (k + 2i) \phi + \frac{i}{(k+i-1)} \operatorname{cos.} (k + 2i - 2) \phi \\ + \frac{i(i-1)}{(k+i-2)(k+i-1)} \operatorname{cos.} (k + 2i - 4) \phi \\ + \frac{(i-2)(i-1)i}{(k+i-3)(k+i-2)(k+i-1)} \operatorname{cos.} (k + 2i - 6) \phi \\ \text{etc.} \end{array} \right\},$$

His igitur duabus formulis generalibus euolutis quaestionem propositam sequenti modo facile resoluere licebit.

Problema.

Inuenire curuas algebraicas, quarum longitudo ita exprimatur, vt eius arcus quicunque indefinitus sit

$$s = \int \frac{v^{m-1} \partial v}{\sqrt{(1 - v^{2n})}}.$$

Solutio.

§. 18. Quaeratur primo angulus ϕ, vt sit $\operatorname{fin.} \phi = v^n$, ideoque $\operatorname{cos.} \phi = \sqrt{(1 - v^{2n})}$, tum vero, posito breuitatis gratia $\frac{m}{n} = k$, fiet $\partial s = \frac{1}{n} \partial \phi \operatorname{fin.} \phi^{k-1}$, quod cum sit elementum curuae, si coordinatae orthogonales vocentur x et y, in genere habebimus $\partial x = \partial s \operatorname{cos.} \omega$ et $\partial y = \partial s \operatorname{fin.} \omega$, quandoquidem hinc prodit $\partial x^2 + \partial y^2 = \partial s^2$.

§. 19. Totum negotium ergo huc redit, cuiusmodi angulos pro ω accipi oporteat, vt binae istae formulae differentiales euadant integrabiles, id quod ostendimus semper fieri sumendo $\omega = (k + 2i + 1) \phi$, ita vt sit

$$x = \int \partial s \operatorname{cos.} (k + 2i + 1) \phi \text{ et}$$
$$y = \int \partial s \operatorname{fin.} (k + 2i + 1) \phi;$$

tum

tum enim habebimus fequentes formulas algebraicas:

$$x = \frac{\text{fin.}\,\Phi^k}{n(k+i)} \left\{ \begin{array}{l} \text{cof.}\,(k+2i)\,\Phi + \frac{i}{(k+i-1)}\,\text{cof.}\,(k+2i-2)\,\Phi \\ + \frac{i(i-1)}{(k+i-1)(k+i-2)}\,\text{cof.}\,(k+2i-4)\,\Phi \\ + \frac{i(i-1)(i-2)}{(k+i-1)(k+i-2)(k+i-3)}\,\text{cof.}\,(k+2i-6)\,\Phi \\ + \frac{i(i-1)(i-2)(i-3)}{(k+i-1)(k+i-2)(k+i-3)(k+i-4)}\,\text{cof.}\,(k+2i-8)\,\Phi \\ \text{etc.} \end{array} \right\},$$

et

$$y = \frac{\text{fin.}\,\Phi^k}{n(k+i)} \left\{ \begin{array}{l} \text{fin.}\,(k+2i)\,\Phi + \frac{i}{(k+i-1)}\,\text{fin.}\,(k+2i-2)\,\Phi \\ + \frac{i(ii-1)}{(k+i-1)(k+i-2)}\,\text{fin.}\,(k+2i-4)\,\Phi \\ + \frac{i(i-1)(i-2)}{(k+i-1)(k+i-2)(k+i-3)}\,\text{fin.}\,(k+2i-6)\,\Phi \\ + \frac{i(i-1)(i-2)(i+3)}{(k+i-1)(k+i-2)(k+i-3)(k+i-4)}\,\text{fin.}\,(k+2i-8)\,\Phi \\ \text{etc.} \end{array} \right\},$$

vbi loco i omnes numeros integros pofitiuos, a o in infinitum vsque accipere licet; vnde fequentes folutiones fpeciales euoluiffe iuuabit.

I. Solutio fpecialis, qua $i = 0$.

§. 20. Hinc igitur refultabit folutio fimpliciffima, ambae enim coordinatae x et y ita exprimentur vt fit

$$x = \frac{\text{fin.}\,\Phi^k\ \text{cof.}\,k\,\Phi}{n\,k} \quad \text{et} \quad y = \frac{\text{fin.}\,\Phi^k\ \text{fin.}\,k\,\Phi}{n\,k},$$

quae folutio femper eft realis, nifi fuerit $k = 0$, tum autem foret quoque $m = 0$ et $\partial s = \frac{\partial v}{v\sqrt{(1 - v^{2n})}} = \frac{1}{n}\,\frac{\partial \Phi}{\text{fin.}\,\Phi}$, vnde fit $s = \frac{1}{n}\,l\,\text{tang.}\,\frac{1}{2}\Phi$, ficque arcus per fimplicem logarithmum exprimeretur; tales autem curuas algebraicas nullo modo exhiberi

hiberi posse satis est euictum. Caeterum pro omnibus reliquis casibus, quemcunque valorem rationalem habuerit k, semper erit

$$x\,x + y\,y = \frac{\text{sin.}\ \Phi^{2k}}{n\,n\,k\,k} = \frac{v^{2nk}}{n\,n\,k\,k} = \frac{v^{2m}}{m\,m},$$

ideoque chorda $\sqrt{(x\,x + y\,y)} = \frac{v^m}{m}$.

II. Solutio specialis, qua $i = 1$.

§. 21. Hoc igitur casu ambae coordinatae ita erunt expressae:

$$x = \frac{\text{sin.}\ \Phi^k}{n\,(k+1)}\left[\text{cos.}\,(k+2)\,\Phi + \frac{1}{k}\,\text{cos.}\ k\,\Phi\right] \text{ et}$$

$$y = \frac{\text{sin.}\ \Phi^k}{n\,(k+1)}\left[\text{sin.}\,(k+2)\,\Phi + \frac{1}{k}\,\text{sin.}\ k\,\Phi\right],$$

vnde conficitur chorda

$$\sqrt{(x\,x + y\,y)} = \frac{\text{sin.}\ \Phi^k}{n\,(k+1)}\sqrt{\left(1 + \frac{1}{k\,k} + \frac{2}{k}\,\text{cos.}\ 2\,\Phi\right)} \text{ siue}$$

$$\sqrt{(x\,x + y\,y)} = \frac{v^m}{m\,(m+n)}\sqrt{[(m+n)^2 - 4\,m\,n\,v^{2n}]},$$

haecque solutio semper valebit, praeter duos casus excipiendos, qui sunt vel $k = 0$, vel $k = -1$.

III. Solutio specialis, qua $i = 2$.

§. 22. Hoc igitur casu ambae coordinatae erunt ita expressae:

$x =$

$$x = \frac{\text{ſin.}\,\phi^k}{n(k+2)}\Big[\text{coſ.}\,(k+4)\,\phi$$
$$+ \frac{2}{k+1}\,\text{coſ.}\,(k+2)\,\phi + \frac{2}{(k+1)}\cdot\frac{1}{k}\,\text{coſ.}\,k\,\phi\Big] \text{ et}$$
$$y = \frac{\text{ſin.}\,\phi^k}{n(k+2)}\Big[\text{ſin.}\,(k+4)\,\phi$$
$$+ \frac{2}{k+1}\,\text{ſin.}\,(k+2)\,\phi + \frac{2}{(k+1)}\cdot\frac{1}{k}\,\text{ſin.}\,k\,\phi\Big].$$

Hic igitur tres caſus excipi oportet, quibus hae formulae ceſſant eſſe algebraicae: primo ſcilicet ſi $k = 0$; 2°. ſi $k = -1$; 3°. ſi $k = -2$.

IV. Solutio ſpecialis, qua $i = 3$.

§. 23. Hoc igitur caſu ambae coordinatae ſequenti modo reperientur expreſſae:

$$x = \frac{\text{ſin.}\,\phi^k}{n(k+3)}\Big[\text{coſ.}\,(k+6)\,\phi + \frac{3}{k+2}\,\text{coſ.}\,(k+4)\,\phi$$
$$+ \frac{3}{k+2}\cdot\frac{2}{k+1}\,\text{coſ.}\,(k+2)\,\phi + \frac{3}{k+2}\cdot\frac{2}{k+1}\cdot\frac{1}{k}\,\text{coſ.}\,k\,\phi\Big],$$
$$y = \frac{\text{ſin.}\,\phi^k}{n(k+3)}\Big[\text{ſin.}\,(k+6)\,\phi + \frac{3}{k+2}\,\text{ſin.}\,(k+4)\,\phi$$
$$+ \frac{3}{k+2}\cdot\frac{2}{k+1}\,\text{ſin.}\,(k+2)\,\phi + \frac{3}{k+2}\cdot\frac{2}{k+1}\cdot\frac{1}{k}\,\text{ſin.}\,k\,\phi\Big].$$

Hae ergo formulae quatuor caſibus erunt inutiles: 1°. $k = 0$; 2°. $k = -1$; 3°. $k = -2$; 4°. $k = -3$, quippe quibus termini in infinitum excreſcentes abirent in arcus circulares, neque igitur formulae amplius eſſent algebraicae.

V. Solutio specialis,
qua $i = 4$.

§. 24. Hoc igitur casu coordinatae sequenti modo exprimentur:

$$x = \frac{\sin.\,\Phi^k}{n(k+4)}\left\{\begin{array}{l}\sin.(k+8)\Phi + \frac{4}{k+3}\sin.(k+6)\Phi \\ + \frac{4}{k+3}\cdot\frac{3}{k+2}\cdot\cos.(k+4)\Phi \\ + \frac{4}{k+3}\cdot\frac{3}{k+2}\cdot\frac{2}{k+1}\cos.(k+2)\Phi \\ + \frac{4}{k+3}\cdot\frac{3}{k+2}\cdot\frac{2}{k+1}\cdot\frac{1}{k}\cos.k\Phi\end{array}\right\},$$

$$y = \frac{\sin.\,\Phi^k}{n(k+4)}\left\{\begin{array}{l}\sin.(k+8)\Phi + \frac{4}{k+3}\sin.(k+6)\Phi \\ + \frac{4}{k+3}\cdot\frac{3}{k+2}\sin.(k+4)\Phi \\ + \frac{4}{k+3}\cdot\frac{3}{k+2}\cdot\frac{2}{k+1}\cdot\sin.(k+2)\Phi \\ + \frac{4}{k+3}\cdot\frac{3}{k+2}\cdot\frac{2}{k+1}\cdot\sin.k\Phi\end{array}\right\},$$

vbi manifestum est has formulas, praeter quatuor casus ante notatos, insuper casu $k = -4$ fieri inutiles.

Corollarium.

§. 25. Exceptis igitur casibus quibus k aequatur numero integro negatiuo, methodus nostra semper suppeditat innumerabiles curuas algebraicas; neque tamen idcirco haec solutio pro generali est habenda, cum etiam casibus memoratis, quibus numerus solutionum nostrarum limitatur, nihilominus infinitas solutiones aliis methodis assignare liceat; vbi quidem semper excludi oportet casum $k = 0$, quippe quo certum est nullas curuas algebraicas satisfacere posse. Innumerabilitatem solutionum, pro casu $k = -1$, ostendisse operae erit pretium.

Euo-

Euolutio caſus
quo $k = -1$.

§. 26. Hoc igitur caſu noſtra methodus vnicam praebet curuam algebraicam, his coordinatis contentam: $x = -\frac{1}{n} \cdot \frac{cof.\Phi}{fin.\Phi}$ et $y = \frac{1}{n}$, quae ergo eſt linea recta axi parallela. Cum autem ſit $\partial s = \frac{\partial \Phi}{n\, fin.\Phi}$, erit $s = -\frac{1}{n} \cot.\Phi$; ſicque omnes plane curuae algebraicae rectificabiles hoc caſu ſatisfaciunt. Sumta enim quacunque tali curua, cuius arcus s per formulam algebraicam exprimatur, ſemper aſſignari poterit angulus Φ, vt fiat $-\frac{1}{n}\cot.\Phi = s$; vnde patet praeter lineam rectam quam inuenimus, omnes plane curuas rectificabiles ſatisfacere.

Corollarium.

§. 27. Cum igitur formula noſtra differentialis $\partial s = \frac{v^{m-1}\partial v}{\sqrt{(1 - v^{2n})}}$, ſemper abſolute euadat integrabilis, quoties k, ſiue $\frac{m}{n}$, fuerit vel numerus integer poſitiuus par, vel etiam numerus integer negatiuus impar; manifeſtum eſt his omnibus caſibus omnes plane curuas algebraicas rectificabiles perinde eſſe ſatisfacturas, ideoque reuera his caſibus infinities plures curvae algebraicae noſtro problemati ſatisfacient, quam noſtra methodus nobis ſuppeditauit. Verum etiam, dummodo k ſit numerus negatiuus integer, ſemper innumerabiles curuas algebraicas aſſignare licet, quod pro caſu $k = -2$ oſtendiſſe ſufficiet.

Euolutio caſus
quo $k = -2$.

§. 28. Hoc igitur caſu methodus ſuperior duas tantum nobis largitur curuas algebraicas, ſcilicet:

 1°.)

$1^{\circ}.\ x = -\frac{cof.\, 2\Phi}{2n\, fin.\, \Phi^2}$ et $y = \frac{fin.\, 2\Phi}{2n\, fin.\, \Phi^2}$,

$2^{\circ}.\ x = -\frac{1}{n\, fin.\, \Phi^2}(1 - \frac{1}{2} cof.\, 2\Phi)$ et $y = -\frac{fin.\, 2\Phi}{2n\, fin.\, \Phi}$.

Cum autem hoc casu sit $\partial s = \frac{\partial\Phi}{n\, fin.\, \Phi^3}$, statuatur $cot.\, \Phi = t$, eritque $\frac{\partial\Phi}{fin.\, \Phi^2} = -\partial t$, ideoque $\partial s = -\frac{\partial t}{n\, fin.\, \Phi}$, quia vero est $fin.\, \Phi = \frac{1}{\sqrt{(1+tt)}}$, fiet $\partial s = -\frac{1}{n}\partial t \sqrt{(1+tt)}$, quod cum sit elementum arcus parabolici, nuper iam demonstraui infinitas curuas algebraicas satisfacere, atque hoc idem quoque tenendum est, si littera k cuicunque numero impari negatiuo maiori aequetur.

Scholion.

§. 29. Ex his iam facile colligere licet, etiam in genere pro omnibus valoribus ipsius k reuera infinities plures curvas algebraicas esse satisfacturas, quam methodus nostra nobis suppeditat, etiamsi adeo innumerabiles exhibeat. Interim tamen duos casus excipi necesse est: alterum quo $k = 0$, pro quo iam notauimus, nullas plane curuas algebraicas satisfacere; alterum vero quo $k = 1$; cum enim sit $\partial s = \frac{1}{n}\partial\Phi$, arcus s ipse arcui circulari aequari deberet, cui conditioni solus circulus satisfacere est monstratus, id quod etiam nostrae solutiones manifesto declarabunt.

Euolutio casus quo $k = 1$.

§. 30. Pro hoc ergo casu solutio specialis prima praebet has coordinatas:

$$x = \frac{1}{n} fin.\, \Phi\, cof.\, \Phi \text{ et } y = \frac{1}{n} fin.\, \Phi^2.$$

Cum igitur sit

$$x = \frac{1}{2n} fin.\, 2\Phi \text{ et } y = \frac{1}{2n}(1 - cof.\, 2\Phi) \text{ erit},$$

$$\frac{1}{2n} cof.\, 2\Phi = \frac{1}{2n} - y,$$

additis

additis ergo quadratis erit

$$x x + (\tfrac{1}{2n} - y)^2 = \tfrac{1}{4nn},$$

quae aequatio manifesto est pro circulo.

§. 31. Secunda vero solutio specialis pro hoc casu nobis dat

$$x = \tfrac{\sin.\Phi}{2n}(\cos.3\Phi + \cos.\Phi) \text{ et}$$

$$y = \tfrac{\sin.\Phi}{2n}(\sin.3\Phi + \sin.\Phi),$$

quae formulae per reductiones notas abeunt in has:

$$4nx = \sin.4\Phi \text{ et } 4ny = 1 - \cos.4\Phi,$$

ideoque $\cos.4\Phi = 1 - 4ny$. Additis igitur quadratis orietur $16nnxx + (1 - 4ny)^2 = 1$, quae itidem est pro circulo.

§. 32. Simili modo solutio specialis tertia praebet

$$x = \tfrac{\sin.\Phi}{3n}(\cos.5\Phi + \cos.3\Phi + \cos.\Phi) \text{ et}$$

$$y = \tfrac{\sin.\Phi}{3n}(\sin.5\Phi + \sin.3\Phi + \sin.\Phi),$$

quae pariter more solito reductae dant $6nx = \sin.6\Phi$ et $6ny = 1 - \cos.6\Phi$, vnde si angulum 6Φ eliminemus, manifesto resultat aequatio ad circulum.

§. 33. Quin etiam hoc idem in genere ostendere licet, quandoquidem sumto $k = 1$ reperitur

$$x = \frac{\sin.\Phi}{n(i+1)}\left\{\begin{array}{l}\cos.(2i+1)\Phi + \cos.(2i-1)\Phi \\ + \cos.(2i-2)\Phi + \cos.(2i-3)\Phi \\ + \text{etc.} \;\ldots\ldots + \cos.\Phi\end{array}\right\},$$

$$y = \frac{\sin.\Phi}{n(i+1)}\left\{\begin{array}{l}\sin.(2i+1)\Phi + \sin.(2i-1)\Phi \\ + \sin.(2i-2)\Phi + \sin.(2i-3)\Phi \\ + \text{etc.} \;\ldots\ldots + \sin.\Phi\end{array}\right\}.$$

Reductionibus igitur adhibitis colligetur fore

$$2n(i+1)x = \text{fin.}(2i+2)\Phi \text{ et}$$
$$2n(i+1)y = 1 - \text{cof.}(2i+2)\Phi,$$

vnde patet curuam ſatisfacientem perpetuo manere circulum.

Scholion.

§. 34. Euidens autem eſt, reliquis caſibus omnibus ſolutiones methodo noſtra datas maxime a ſe inuicem eſſe diſcrepaturas, atque adeo continuo ad altiores curuarum ordines eſſe aſcenſuras. Interim tamen, etiamſi ſolutio noſtra infinitas praebeat curuas ſatisfacientes, nullum plane eſt dubium, quin praeter eas innumerabiles aliae reuera aſſignari queant, quemadmodum pro caſibus, quibus curuae debent eſſe rectificabiles, iam ſatis eſt oſtenſum. Eandem ſolutionum multiplicitatem inſuper alio caſu, quo $k = 3$, declaraſſe iuuabit.

Euolutio caſus quo $k = 3$.

§. 35. Hoc quidem caſu noſtra methodus infinitas exhibet curuas algebraicas; verum praeter illas ſequenti modo innumerabiles alias inuenire licebit. Cum enim ſit $\partial s = \frac{1}{n}\partial\Phi \,\text{fin.}\Phi^2$, erit $\partial s = \frac{\partial\Phi}{2n}(1 - \text{cof.}\, 2\Phi)$, quae formula nobis ſequentes valores pro ∂x et ∂y aſſumendos ſuggerit:

$$\partial x = \frac{\partial\Phi}{2n}(1 - \text{cof.}\, 2\Phi)\,\text{cof.}\,\lambda\Phi \text{ et}$$
$$\partial y = \frac{\partial\Phi}{2n}(1 - \text{cof.}\, 2\Phi)\,\text{fin.}\,\lambda\Phi,$$

quae formulae manifeſto ſemper integrationem admittent, ſolo caſu $\lambda = \pm 2$ excepto. Quodſi enim reductiones notae in ſubſidium vocentur, proueniet

$$\frac{4n\partial x}{\partial\Phi} = 2\,\text{cof.}\,\lambda\Phi - \text{cof.}(\lambda+2)\Phi - \text{cof.}(\lambda-2)\Phi \text{ et}$$
$$\frac{4n\partial y}{\partial\Phi} = 2\,\text{fin.}\,\lambda\Phi - \text{fin.}(\lambda+2)\Phi - \text{fin.}(\lambda-2)\Phi,$$

quae

quae ergo formulae integratae nobis praebent

$$4nx = \frac{2\,\text{fin.}\,\lambda\Phi}{\lambda} - \frac{\text{fin.}\,(\lambda+2)\Phi}{\lambda+2} - \frac{\text{fin.}\,(\lambda-2)\Phi}{\lambda-2} \text{ et}$$

$$4ny = -\frac{2\,\text{cof.}\,\lambda\Phi}{\lambda} + \frac{\text{cof.}\,(\lambda+2)\Phi}{\lambda+2} + \frac{\text{cof.}\,(\lambda-2)\Phi}{\lambda-2}.$$

§. 36. Quoniam hic pro λ non folum omnes numeros integros, verum etiam omnes fractiones affumere licet, euidens eft iftam folutionem infinities latius patere, quam fupra exhibitam. Quin etiam manifeftum eft iftas nouas folutiones omnes a fuperioribus penitus effe diuerfas.

§. 37. Eodem modo cafus tractari poterunt, quibus litterae k valor integer pofitiuus quicunque tribuitur, propterea quod poteftatem fin. Φ^k femper in finus vel cofinus fimplices refoluere licet, quae partes deinde tam in finus $\lambda\Phi$ quam in cof. $\lambda\Phi$ ductae euadent integrabiles, dummodo $\lambda\Phi$ non tale fit multiplum ipfius Φ, cuiusmodi ex illa refolutione funt natae.

§. 38. Quoniam haec maximae funt generalia atque ob hanc ipfam caufam maiori illuftratione indigeant, referamus formulas fupra inuentas ad cafum quempiam fpecialem et in curuas algebraicas inquiramus, quarum arcus fiue per arcum curuae elafticae $\int \frac{\partial v}{\sqrt{(1-v^4)}}$, fiue per applicatam eiusdem curuae, $\int \frac{vv\,\partial v}{\sqrt{(1-v^4)}}$, exprimatur.

Exemplum I.

§. 39. *Inuenire curuas algebraicas, quarum arcus fit* $s = \int \frac{\partial x}{\sqrt{(1-v^4)}}$.

Cum igitur hic fit $m = 1$, et $n = 2$, erit $k = \frac{1}{2}$, vnde folutionum fpecialium fupra datarum prima nobis praebebit $x = \text{cof.}\,\frac{1}{2}\Phi\sqrt{\text{fin.}\,\Phi}$ et $y = \text{fin.}\,\frac{1}{2}\Phi\sqrt{\text{fin.}\,\Phi}$. Quo nunc hinc angulum Φ eliminemus, quaeramus $xx + yy = \text{fin.}\,\Phi$ et $2xy = \text{fin.}\,\Phi\,\text{fin.}\,\frac{1}{2}\Phi\,\text{cof.}\,\frac{1}{2}\Phi = \text{fin.}\,\Phi^2$, eritque

$2xy$

$2xy = (xx + yy)^2$, quae ergo curua eſt ordinis quarti et ſub nomine Lemniſcatae cognita, cuius adeo omnes arcus pari modo, quo circulares, intèr ſe comparari poſſe iam dudum a Geometris eſt oſtenſum.

§. 40. Simili modo ſequentes ſolutiones ſpeciales perducent ad alias curuas algebraicas eiusdem indolis, quae autem ad multo altiores ordines aſſurgent, quas hic idcirco fuſius euoluere ſuperfluum foret.

Exemplum 2.

§. 41. *Inuenire formulam algebraicam, cuius arcus ſit* $s = \int \frac{vv\partial v}{\sqrt{(1 - v^4)}}$.

Hic igitur eſt $m = 3$ et $n = 2$, ideoque $k = \frac{3}{2}$, vnde ſpecies prima praebet

$$x = \tfrac{1}{3} \text{ſin.}\, \Phi^{\frac{3}{2}} \text{coſ.}\, \tfrac{3}{2}\Phi \text{ et } y = \tfrac{1}{3} \text{ſin.}\, \Phi^{\frac{3}{2}} \text{ſin.}\, \tfrac{3}{2}\Phi,$$

vnde erit

$$9(xx + yy = \text{ſin.}\, \Phi^3 \text{ et}$$

$$18\, xy = 2 \text{ſin.}\, \Phi^3 \text{ſin.}\, \tfrac{3}{2}\Phi \text{coſ.}\, \tfrac{3}{2}\Phi = \text{ſin.}\, \Phi^3 \text{ſin.}\, 3\Phi.$$

Cum igitur ſit $\text{ſin.}\, 3\Phi = 3 \text{ſin.}\, \Phi - 4 \text{ſin.}\, \Phi^3$, erit

$$18\, xy = 3 \text{ſin.}\, \Phi^4 - 4 \text{ſin.}\, \Phi^6,$$

hinc porro

$$3 \text{ſin.}\, \Phi^4 = 18\, xy + 324\,(xx + yy)^2, \text{ ſiue}$$

$$\text{ſin.}\, \Phi^4 = 6\, xy + 108\,(xx + yy)^2.$$

Hinc igitur deducimus binos valores pro $\text{ſin.}\, \Phi^{12}$, vnde naſcitur ſequens aequatio:

$$216\,[xy + 18\,(xx + yy)^2]^3 = 9^4\,(xx + yy)^4,$$

quae aequatio aſſurgit ad ordinem duodecimum, videturque eſſe ſimpliciſſima, quae huic conditioni ſatisfaciat.

Scho-

Scholion.

§. 42. Principia autem, quae hic ſtabiliuimus, quaeſtionibus multo magis complicatis reſoluendis ſufficiunt, quemadmodum in ſequenti problemate adhuc ſumus oſtenſuri.

Problema magis generale.

Inuenire curuas algebraicas, quarum arcus indefiniti s *ita exprimantur, vt ſit*

$$s = \int \frac{v^{m-1}\partial v}{\sqrt{(1-v^{2n})}}(a + b v^{2n} + c v^{4n} + d v^{6n} + \text{etc.})$$

Solutio.

§. 43. Quotcunque terminos iſta expreſſio contineat, ſufficiet ſolutionem ad tres terminos accommodaſſe, quandoquidem hinc facile perſpicietur, quomodo calculum ad quotcunque terminos extendi oporteat. Statuamus igitur vt ante $v^n = \text{ſin.}\,\Phi$, ac poſito $\frac{m}{n} = k$, quia inde fit

$$\frac{v^{m-1}\partial v}{\sqrt{(1-v^{2n})}} = \tfrac{1}{n}\partial\Phi\,\text{ſin.}\,\Phi^{k-1},$$

pro noſtro problemate habebimus:

$$\partial s = \tfrac{1}{n}\partial\Phi\,\text{ſin.}\,\Phi^{k-1}(a + b\,\text{ſin.}\,\Phi^2 + c\,\text{ſin.}\,\Phi^4).$$

§. 44. Cum nunc hic habeamus tres partes, in quibus exponentes ipſius ſin. Φ ſunt $k-1$; $k+1$; $k+3$; qui binario aſcendunt, ponamus pro parte ſecunda $k+2=k'$, ac pro tertia $k+4=k''$, vt ternae noſtrae partes fiant

$$\partial s = \tfrac{a\partial\Phi}{n}\,\text{ſin.}\,\Phi^{k-1} + \tfrac{b\partial\Phi}{n}\,\text{ſin.}\,\Phi^{k'-1} + \tfrac{c\partial\Phi}{n}\,\text{ſin.}\,\Phi^{k''-1}.$$

Has igitur ſingulatim multiplicemus per coſinum et ſinum eiusdem anguli $(k+2i+1)\Phi$, qui pro parte ſecunda erit $(k'$

$(k'+2i-1)\Phi$, pro tertia autem $(k''+2i-3)\Phi$; vbi tantum notari oportet numerum integrum i ita accipi debere, vt vltimus numerus $2i-3$ maneat pofitiuus.

§. 45. His igitur conftitutis ex formula ∂s prorfus vt fupra determinare licebit elementa coordinatarum ∂x et ∂y, ponendo fcilicet

$$\partial x = \partial s \operatorname{cof.} (k+2i+1)\Phi \text{ et}$$
$$\partial y = \partial s \operatorname{fin.} (k+2i+1)\Phi,$$

quandoquidem hinc fiet $\partial x^2 + \partial y^2 = \partial s^2$, vnde ternis partibus pro ∂s fcribendis ipfae coordinatae ita exprimentur:

$$x = \left\{ \begin{array}{l} \frac{a}{n}\int \partial\Phi \operatorname{fin.} \Phi^{k-1} \operatorname{cof.} (k+2i+1)\Phi \\ +\frac{b}{n}\int \partial\Phi \operatorname{fin.} \Phi^{k'-1} \operatorname{cof.} (k'+2i-1)\Phi \\ +\frac{c}{n}\int \partial\Phi \operatorname{fin.} \Phi^{k''-1} \operatorname{cof.} (k''+2i-3)\Phi \end{array} \right\},$$

$$y = \left\{ \begin{array}{l} \frac{a}{n}\int \partial\Phi \operatorname{fin.} \Phi^{k-1} \operatorname{fin.} (k+2i+1)\Phi \\ +\frac{b}{n}\int \partial\Phi \operatorname{fin.} \Phi^{k'-1} \operatorname{fin.} (k'+2i-1)\Phi \\ +\frac{c}{n}\int \partial\Phi \operatorname{fin.} \Phi^{k''-1} \operatorname{fin.} (k''+2i-3)\Phi \end{array} \right\}.$$

Vbi integralia fingularum partium per formulas fupra §. 13 et §. 18. exhibitas affignare licet, fiquidem ibi dedimus integralia harum formularum:

$$\int \partial s \operatorname{fin.} (k+2i+1)\Phi \text{ et } \int \partial s \operatorname{cof.} (k+2i+1)\Phi,$$

exiftente $\partial s = \frac{\partial\Phi}{n} \operatorname{fin.} \Phi^{k-1}$.

§. 46. Cum igitur hic loco i innumerabiles numeros integros affumere liceat, manifeftum eft etiam pro hoc problemate infinitas exhiberi poffe folutiones, fi modo excipiantur cafus illi fingulares, quibus quispiam denominator euanescit, id quod euenit, quando k vel cyphrae, vel numero negatiuo

tiuo integro aequatur. Caeterum hoc problema exemplo particulari illuſtraſſe iuuabit.

Exemplum.

§. 47. *Inuenire curuas algebraicas, pro quibus ſit*
$s = \int \frac{\partial v}{\sqrt{(1 - vv)}} (a + b v^2 + c v^4)$.

Hic ergo erit $m = 1$; $n = 1$ et $k = 1$, ideoque $k' = 3$ et $k'' = 5$, quamobrem ambae coordinatae in genere ita exprimentur:

$$x = \left\{ \begin{array}{l} a \int \partial \Phi \operatorname{cof.} (2i + 2) \Phi \\ + b \int \partial \Phi \operatorname{fin.} \Phi^2 \operatorname{cof.} (2i + 2) \Phi \\ + c \int \partial \Phi \operatorname{fin.} \Phi^4 \operatorname{cof.} (2i + 2) \Phi \end{array} \right\},$$

$$y = \left\{ \begin{array}{l} a \int \partial \Phi \operatorname{fin.} (2i + 2) \Phi, \\ + b \int \partial \Phi \operatorname{fin.} \Phi^2 \operatorname{fin.} (2i + 2) \Phi \\ + c \int \partial \Phi \operatorname{fin.} \Phi^4 \operatorname{fin.} 2i + 2) \Phi, \end{array} \right\}.$$

Vbi autem notandum eſt numerum i vnitate maiorem capi debere, ne $2i - 3$ fiat negatiuum.

§. 48. Quo igitur curuam ſimpliciſſimam ſatisfacientem nanciſcamur, ſumamus $i = 2$, atque formulae integrales pro coordinatis erunt:

$$\begin{array}{l} x = a \int \partial \Phi \operatorname{cof.} 6 \Phi \\ \quad + b \int \partial \Phi \operatorname{fin.} \Phi^2 \operatorname{cof.} 6 \Phi \\ \quad + c \int \partial \Phi \operatorname{fin.} \Phi^4 \operatorname{cof.} 6 \Phi \text{ et} \\ y = a \int \partial \Phi \operatorname{fin.} 6 \Phi \\ \quad + b \int \partial \Phi \operatorname{fin.} \Phi^2 \operatorname{fin.} 6 \Phi \\ \quad + c \int \partial \Phi \operatorname{fin.} \Phi^4 \operatorname{fin.} 6 \Phi. \end{array}$$

Iam pro primis partibus eſt $k = 1$ et $\partial s = \partial \Phi$, vnde erit

$$\int \partial s \cos. 6\,\Phi = \int \partial s \cos. (k+5)\,\Phi$$
$$= \frac{\sin. \Phi}{3} (\cos. 5\,\Phi + \cos. 3\,\Phi + \cos. \Phi) \text{ et}$$
$$\int \partial s \sin. 6\,\Phi = \int \partial s \sin. (k+5)\,\Phi$$
$$= \frac{\sin. \Phi}{3} (\sin. 5\,\Phi + \sin. 3\,\Phi + \sin. \Phi),$$

qui valores reducti dabunt:

$$\int \partial s \cos. 6\,\Phi = \tfrac{1}{6} \sin. 6\,\Phi \text{ et}$$
$$\int \partial s \sin. 6\,\Phi = \tfrac{1}{6} (1 - \cos. 6\,\Phi),$$

quas formulas per quantitatem a multiplicari oportet.

§. 49. Pro partibus secundis habemus $\partial s = \partial \Phi \sin. \Phi^2$ et $k = 3$, vnde nanciscimur:

$$\int \partial \Phi \sin. \Phi^2 \cos. 6\,\Phi = \int \partial s \cos. (k+3)\,\Phi$$
$$= \frac{\sin. \Phi^3}{4} (\cos. 5\,\Phi + \tfrac{1}{3} \cos. 3\,\Phi) \text{ et}$$
$$\int \partial \Phi \sin. \Phi^2 \sin. 6\,\Phi = \int \partial s \sin. (k+3)\,\Phi$$
$$= \frac{\sin. \Phi^3}{4} (\sin. 5\,\Phi + \tfrac{1}{3} \sin. 3\,\Phi).$$

Prior forma ob $\sin. \Phi^3 = \tfrac{3}{4} \sin. \Phi - \tfrac{1}{4} \sin. 3\,\Phi$, transit in hanc:

$$\int \partial s \cos. 6\,\Phi = \tfrac{1}{16} [3 \sin. \Phi \cos. 5\,\Phi - \sin. 3\,\Phi \cos. 5\,\Phi$$
$$+ \sin. \Phi \cos. 3\,\Phi - \tfrac{1}{3} \sin. 3\,\Phi \cos. 3\,\Phi],$$

ideoque

$$\int \partial s \cos. 6\,\Phi = \tfrac{1}{32} (-2 \sin. 4\,\Phi + \tfrac{8}{3} \sin. 6\,\Phi - 1 \sin. 8\,\Phi)$$
$$= -\tfrac{1}{16} \sin. 4\,\Phi + \tfrac{1}{12} \sin. 6\,\Phi - \tfrac{1}{32} \sin. 8\,\Phi.$$

Simili modo habebimus

$$\int \partial s \sin. 6\,\Phi = \tfrac{1}{16} [3 \sin. \Phi \sin. 5\,\Phi - \sin. 3\,\Phi \sin. 5\,\Phi$$
$$+ \sin. \Phi \sin. 3\,\Phi - \tfrac{1}{3} \sin. 3\,\Phi^2],$$

ideoque

$$\int \partial s \sin. 6\,\Phi = \tfrac{1}{32} (2 \cos. 4\,\Phi - \tfrac{8}{3} \cos. 6\,\Phi + \cos. 8\,\Phi)$$
$$= + \tfrac{1}{16} \sin. 4\,\Phi - \tfrac{1}{12} \sin. 6\,\Phi + \tfrac{1}{32} \sin. 8\,\Phi).$$

§. 50.

§. 50. Verum in hoc negotio formulis supra datis penitus carere possumus; cum enim sit $\sin.\Phi^2 = \frac{1}{2} - \frac{1}{2}\cos.2\Phi$, erit primo pro partibus secundis littera b affectis:

$$\int\partial\Phi \sin.\Phi^2 \cos.6\Phi = \tfrac{1}{2}\int\partial\Phi \cos.6\Phi(1-\cos.2\Phi)$$
$$= \tfrac{1}{2}\int\partial\Phi(\cos.6\Phi - \tfrac{1}{2}\cos.8\Phi - \tfrac{1}{2}\cos.4\Phi),$$

cuius integrale manifesto est

$$= \tfrac{1}{12}\sin.6\Phi - \tfrac{1}{32}\sin.8\Phi - \tfrac{1}{16}\sin.4\Phi.$$

Deinde ob

$$\sin.\Phi^2 \sin.6\Phi = \tfrac{1}{2}\sin.6\Phi - \tfrac{1}{2}\cos.2\Phi \sin.6\Phi$$
$$= \tfrac{1}{2}\sin.6\Phi - \tfrac{1}{4}\sin.8\Phi - \tfrac{1}{4}\sin.4\Phi$$

habebimus

$$\int\partial s \sin.6\Phi = -\tfrac{1}{12}\cos.6\Phi + \tfrac{1}{32}\cos.8\Phi + \tfrac{1}{16}\cos.4\Phi,$$

quas formulas per litteram b multiplicari oportet.

§. 51. Denique pro tertiis partibus littera c affectis cum sit

$$\sin.\Phi^4 = \tfrac{3}{8} - \tfrac{1}{2}\cos.2\Phi + \tfrac{1}{8}\cos.4\Phi,$$

erit

$$\sin.\Phi^4 \cos.6\Phi = \tfrac{3}{8}\cos.6\Phi - \tfrac{1}{4}\cos.8\Phi - \tfrac{1}{4}\cos.4\Phi$$
$$+ \tfrac{1}{16}\cos.10\Phi + \tfrac{1}{16}\cos.2\Phi;$$

vnde integrando nanciscimur:

$$\int\partial\Phi \sin.\Phi^4 \cos.6\Phi = \tfrac{1}{16}\sin.6\Phi - \tfrac{1}{32}\sin.8\Phi$$
$$- \tfrac{1}{16}\sin.4\Phi + \tfrac{1}{160}\sin.10\Phi + \tfrac{1}{32}\sin.2\Phi.$$

Deinde vero erit

$$\int\partial\Phi \sin.\Phi^4 \sin.6\Phi = \tfrac{3}{8}\sin.6\Phi - \tfrac{1}{4}\sin.8\Phi - \tfrac{1}{4}\sin.4\Phi$$
$$+ \tfrac{1}{16}\sin.10\Phi + \tfrac{1}{16}\sin.2\Phi,$$

ideoque integrando habebimus:

$$\int\partial\Phi \sin.\Phi^4 \sin.6\Phi = -\tfrac{1}{16}\cos.6\Phi + \tfrac{1}{32}\cos.8\Phi$$
$$+ \tfrac{1}{16}\cos.4\Phi - \tfrac{1}{160}\cos.10\Phi - \tfrac{1}{32}\cos.2\Phi.$$

§. 52. His igitur colligendis ambae coordinatae x et y sequenti modo expressae reperiuntur:

$$x = \tfrac{c}{32}\,\text{sin.}\,2\,\Phi - \tfrac{(b+c)}{16}\,\text{sin.}\,4\,\Phi + \left(\tfrac{a}{6} + \tfrac{b}{12} + \tfrac{c}{18}\right)\text{sin.}\,6\,\Phi$$
$$- \tfrac{(b+c)}{32}\,\text{sin.}\,8\,\Phi + \tfrac{c}{160}\,\text{sin.}\,10\,\Phi,$$

$$y = -\tfrac{c}{32}\,\text{cos.}\,2\,\Phi + \tfrac{(b+c)}{16}\,\text{cos.}\,4\,\Phi - \left(\tfrac{a}{6} + \tfrac{b}{12} + \tfrac{c}{18}\right)\text{cos.}\,6\,\Phi$$
$$+ \tfrac{b+c}{32}\,\text{cos.}\,8\,\Phi - \tfrac{c}{160}\,\text{cos.}\,10\,\Phi,$$

vbi constantem $\frac{a}{6}$ in prima parte pro y ingressam omisimus.

DE