1789

De motu quodam maxime memorabili, satis quidem simplici, at solutu difficillimo

Leonhard Euler

DE MOTV QVODAM

MAXIME MEMORABILI, SATIS QVIDEM SIMPLICI, AT SOLVTV DIFFICILLIMO.

Auctore

L. EVLERO.

Conuent. exhib. die 8 *Avril* 1779.

§. 1.

Concipio hic cylindrum basi sua circulari A b B A plano horizontali verticaliter insistentem, cui per plurimas spiras circumvolutum sit filum, altero termino in puncto O fixum. Quod si iam huic cylindro motus quicunque fuerit impressus, quaeritur huius motus continuatio, vnde ad quoduis tempus tam positio ipsius cylindri quam eius motus definiri queat. Hic autem assumo, motum istum sine vlla frictione aliaue resistentia super plano horizontali peragi posse; siquidem, admissa quapiam resistentia, motus determinatio vires Analyseos penitus esset superatura, quemadmodum ex sequenti solutione, qua mentem ab omni motus impedimento abstrahimus, facile colligere licebit, propterea quod ea iam calculos maxime difficiles postulat. Tab. II. Fig. 1.

§. 2. Ponamus igitur elapſo tempore $= t$, quod more ſolito in minutis ſecundis dari ſumimus, cylindrum tenere ſitum in figura repraeſentatum, cuius baſin filum ex O porrectum tangat in puncto T, a quo porro per plures ſpiras ei circumvoluatur, cuius ſitus ipſo motus initio inciderit in rectam OD, quam hic inſtar axis accipiamus, ad quam ex centro cylindri baſeos C demittamus perpendiculum CP, vt nanciſcamur binas coordinatas $OP = x$ et $PC = y$, locum puncti C determinantes. Praeterea vero ſtatuamus radium $CA = CT = a$, ipſam vero fili portionem $OT = z$, atque inſuper angulum $DOT = \theta$; ex quibus binis elementis z et θ binae coordinatae x et y ita determinabuntur, vt ſit

$$x = z \operatorname{cof.} \theta - a \operatorname{fin.} \theta \text{ et}$$
$$y = z \operatorname{fin.} \theta + a \operatorname{cof.} \theta.$$

§. 3. Quoniam vero cylindrus etiam motum habebit gyratorium circa ſuum axem verticalem, ponamus eius punctum, quod motus initio fuerat in puncto ſummo B, per tempus $= t$ proceſſiſſe in punctum b, ita vt gyratio interea facta ſit per angulum $BCb = \Phi$, cuius arcus $Bb = a\Phi$. Cum nunc ſit arcus $AT = a\theta$, erit arcus $Tb = 180 - a(\Phi + \theta)$, cui ſi addatur portio fili $OT = z$, ſumma $z + \pi - a(\Phi + \theta)$ aequalis erit longitudini fili, quod initio a puncto O vsque ad punctum B porrigebatur, quae longitudo cum ſit conſtans, eius differentiale nihilo aequabitur, vnde fiet $\partial z - a\,\partial\Phi - a\,\partial\theta = 0$ hincque $\partial\Phi = \frac{\partial z}{a} - \partial\theta$.

§. 4. Statuamus autem totius cylindri centrum grauitatis in ipſum punctum C incidere, vel potius ei verticaliter imminere, tum vero maſſam totius cylindri eiusque pondus ſta-

ſtatuamus $= M$, momentum vero inertiae reſpectu puncti C, ſeu potius reſpectu axis puncto C verticaliter inſiſtentis $= M.ac$; vbi obſeruaſſe iuuabit, ſi cylindrus ex materia vniformi conſtet, fore $c = \frac{1}{2}a$. Vt autem inueſtigatio noſtra ad omnes caſus pateat, quibus cylindrus vel non ex materia homogenea eſt conflatus, vel adeo eius loco corpus quodcunque rotundum ſubſtituatur, littera c quoſcunque alios valores recipere poterit, dummodo eius centrum grauitatis puncto C verticaliter immineat, atque inſuper in regione, vbi filum eſt circumuolutum, eius radius ſit, vti poſuimus, $CA = a$.

§. 5. Conſideremus nunc vires, quibus noſter cylindrus in motu ſuo ſollicitabitur, et quoniam grauitas hic non in computum venit, aliam vis actionem non ſentiet, praeter eam qua filum O T eſt tenſum, quae vis, etiamſi adhuc ſit incognita, deſignetur tantiſper littera Θ, eaque reſoluta praebet pro directione abſciſſae $OP = x$ vim Θ coſ. θ, at pro directione applicatae $PC = y$ vim $= \Theta$ ſin. θ; pro motu autem gyratorio momentum iſtius tenſionis Θ erit $= a\Theta$, quod tendet in ſenſum Bb, ideoque motum gyratorium augebit, dum contra binae vires praecedentes motibus ſecundum coordinatas ſunt contrariae. Denique ſit littera g altitudo, per quam graue libere cadendo tempore vnius minuti ſecundi deſcendit, vt celeritates per ſpatium vno minuto ſecundo percurſum exprimantur, ſiquidem maſſae per pondera definiantur.

§. 6. His praeparatis principia motus nobis ſuppeditant tres ſequentes aequationes:

$$\text{I. } \frac{\partial \partial x}{\partial t^2} = -\frac{2g\Theta \, coſ.\, \theta}{M};$$

$$\text{II. } \frac{\partial \partial y}{\partial t^2} = -\frac{2g\Theta \, ſin.\, \theta}{M};$$

$$\text{III. } \frac{\partial \partial \Phi}{\partial t^2} = +\frac{2ga\Theta}{Mac} = \frac{2g\Theta}{Mc};$$

qua-

quarum binae priores motum progreſſiuum puncti C determinant, tertia vero motum gyratorium. Quare cum ipſa tenſio Θ etiamnunc ſit incognita, eam ante omnia e calculo elidi oportet, id quod commodiſſime fieri poteſt. Quia enim ex tertia eſt $\frac{2g\Theta}{M} = \frac{c\,\partial\partial\Phi}{\partial t^2}$, hic valor in prioribus ſubſtitutus nobis praebebit has duas aequationes ſimpliciſſimas:

$$\text{I.}\quad \partial\partial x + c\,\partial\partial\Phi \operatorname{cof.}\theta = 0$$
$$\text{II.}\quad \partial\partial y + c\,\partial\partial\Phi \operatorname{fin.}\theta = 0$$

ex quibus iam totam ſolutionem deriuari oportet.

§. 7. Commode autem has duas aequationes ad duas quantitates variabiles z et θ reducere licebit; cum enim inuenerimus $\partial\Phi = \frac{\partial z}{a} - \partial\theta$, erit $\partial\partial\Phi = \frac{\partial\partial z}{a} - \partial\partial\theta$; tum vero, cum ſit

$$x = z \operatorname{cof.}\theta - a \operatorname{fin.}\theta \quad\text{et}$$
$$y = z \operatorname{fin.}\theta + a \operatorname{cof.}\theta\text{, erit}$$
$$\partial x = \partial z \operatorname{cof.}\theta - z\,\partial\theta \operatorname{fin.}\theta - a\,\partial\theta \operatorname{cof.}\theta;$$
$$\partial y = \partial z \operatorname{fin.}\theta + z\,\partial\theta \operatorname{cof.}\theta - a\,\partial\theta \operatorname{fin.}\theta;$$

hincque porro differentiando

$$\partial\partial x = \partial\partial z \operatorname{cof.}\theta - 2\,\partial z\,\partial\theta \operatorname{fin.}\theta - z\,\partial\partial\theta \operatorname{fin.}\theta$$
$$- a\,\partial\partial\theta \operatorname{cof.}\theta + a\,\partial\theta^2 \operatorname{fin.}\theta - z\,\partial\theta^2 \operatorname{cof.}\theta;$$
$$\partial\partial y = \partial\partial z \operatorname{fin.}\theta + 2\,\partial z\,\partial\theta \operatorname{cof.}\theta + z\,\partial\partial\theta \operatorname{cof.}\theta$$
$$- a\,\partial\partial\theta \operatorname{fin.}\theta - a\,\partial\theta^2 \operatorname{cof.}\theta - z\,\partial\theta^2 \operatorname{fin.}\theta;$$

quibus valoribus ſubſtitutis binae noſtrae aequationes induent ſequentem formam:

$$(-z\,\partial\theta^2 + (1 + \tfrac{c}{a})(\partial\partial z - a\,\partial\partial\theta) \operatorname{cof.}\theta$$
$$-(z\,\partial\partial\theta + 2\,\partial z\,\partial\theta - a\,\partial\theta^2) \operatorname{fin.}\theta = 0$$
$$(-z\,\partial\theta^2 + (1 + \tfrac{c}{a})(\partial\partial z - a\,\partial\partial\theta) \operatorname{fin.}\theta$$
$$+(z\,\partial\partial\theta + 2\,\partial z\,\partial\theta - a\,\partial\theta^2) \operatorname{fin.}\theta = 0.$$

§. 8.

§. 8. Hic iam commodiſſime id vſu venit, vt ſinus et coſinus anguli θ prorſus ex calculo eliminari queant, namque haec combinatio: $\mathrm{I} \times \cos.\, \theta + \mathrm{II} \times \sin.\, \theta$, praebet hanc aequationem:

$$- z \partial \theta^2 + (1 + \tfrac{c}{a})(\partial \partial z - a \partial \partial \theta) = 0;$$

at haec combinatio: $\mathrm{II.} \times \cos.\, \theta - \mathrm{I.} \times \sin.\, \theta$, dat iſtam:

$$z \partial \partial \theta + 2 \partial z . \partial \theta - a \partial \theta^2 = 0;$$

quae aequationes, ponendo brev. gr. $1 + \frac{c}{a} = n$ et $z = a v$, in ſequentes commodiores abibunt:

$$\mathrm{I.}\ n(\partial \partial v - \partial \partial \theta) - v \partial \theta^2 = 0;$$

$$\mathrm{II.}\ v \partial \partial \theta + 2 \partial v \partial \theta - \partial \theta^2 = 0;$$

in quibus adeo praeter binas variabiles v et θ vnica quantitas conſtans, ſcilicet n, reperitur, de qua notetur, eam ſemper vnitate eſſe maiorem. Totum ergo negotium iam huc eſt reductum, vt hae duae aequationes reſoluantur atque ad integrationem perducantur.

§. 9. Mirum hic ſtatim videbitur, quod cum vnica tantum relatio inter v et θ ſit inueſtiganda, hic ad duas aequationes peruenerimus; verum quia ambae aequationes ſunt differentiales ſecundi gradus, atque iam initio elementum temporis ∂t aſſumtum eſt conſtans; a quo ergo differentialia ſecunda determinationem ſuam accipiunt, reuera etiamnunc ratio temporis in has determinationes ingreditur, ita vt tres variabiles adeſſe ſint cenſendae. Cum autem iſtud elementum ∂t ex calculo noſtro exceſſerit, quoniam eius ratio nondum conſtat, eam ex calculo eliminari oportet, quod ſequenti modo fieri poterit, quo differentialia ſecundi gradus prorſus ex calculo excludentur.

§. 10. Hunc in finem ſtatuatur $\partial\theta = p\,\partial v$, vt ſit $\partial\partial\theta = p\,\partial\partial v + \partial p\,\partial v$, quibus valoribus ſubſtitutis noſtrae aequationes induent has formas:

$$\text{I.)}\ n(1-p)\,\partial\partial v - n\,\partial p\,\partial v - v\,p\,p\,\partial v^2 = 0;$$

$$\text{II.}\ v\,p\,\partial\partial v + v\,\partial p\,\partial v + p(2-p)\,\partial v^2 = 0;$$

vnde duplici modo valor ipſius $\frac{\partial\partial v}{\partial v}$ definiri poterit; prodibit ſcilicet $1^o.)\ \frac{\partial\partial v}{\partial v} = \frac{n\,\partial p - v\,p\,p\,\partial v}{n(1-p)}$;

$$2^o.)\ \frac{\partial\partial v}{\partial v} = -\frac{v\,\partial p - p(2-p)\,\partial v}{p\,u};$$

qui valores inter ſe coaequati producunt ſequentem aequationem differentialem primi gradus:

$$p^3\,v\,v\,\partial v = n\,v\,\partial p + n\,p\,\partial v\,(1-p)(2-p),$$

cuius autem forma ita eſt comparata, vt nulla via pateat eius integrale inueſtigandi, niſi forte caſu in eiusmodi multiplicatorem incidamus, qui eam integrabilem reddat.

§. 11. Interim tamen iſtam aequationem adhuc ſimpliciorem reddere licebit, dum etiam litera n ex calculo excludi atque adeo ad ſimpliciorem poteſtatem redigi poteſt, quod fiet ponendo $v = \sqrt{n\,u}$, vnde fit $\partial v = \frac{\partial u \sqrt{n}}{2\sqrt{u}}$, et aequatio noſtra fiet

$$p^3\,u\,\partial u = 2u\,\partial p + p\,\partial u\,(1-p)(2-p).$$

Neque vero hinc quicquam vlterius concludere licet, vnde iſtum laborem alio modo aggrediamur.

Analyſis ad perfectam quaeſtionis ſolutionem perducens.

§. 12. Quoniam poſtremam aequationem differentialem primi gradus immediate ex aequationibus differentialibus ſecundi gradus deriuauimus, neque vlla adhuc integratione ſumus

mus vſi: facile intelligitur, hunc laborem plurimum ſubleuatum iri, ſi ante quasdam integrationes in ſubſidium vocemus, quam ad aequationem finalem deueniamus. Huiusmodi autem integrationes commodiſſime ex ipſis aequationibus primordialibus, quae erant

$$\text{I. } \partial\partial x + c\,\partial\partial\Phi \operatorname{cof.}\theta = 0;$$

$$\text{II. } \partial\partial y + c\,\partial\partial\Phi \operatorname{fin.}\theta = 0;$$

repetere licebit, vbi notaſſe iuuabit eſſe

$$\frac{\partial x}{a} = (\partial v - \partial\theta)\operatorname{cof.}\theta - v\,\partial\theta \operatorname{fin.}\theta$$

(ſcilicet poſito $z = a\,v$) et

$$\frac{\partial y}{a} = (\partial v - \partial\theta)\operatorname{fin.}\theta + v\,\partial\theta \operatorname{cof.}\theta.$$

Praeterea vero habebimus $\partial\Phi = \partial v - \partial\theta$.

§. 13. Nunc fiat iſta combinatio: I. $\frac{\partial x}{a}$ + II. $\frac{\partial y}{a}$, quae praebet hanc aequationem:

$$\frac{\partial x\,\partial\partial x + \partial y\,\partial\partial y}{a} + c\,\partial\partial\Phi\left(\frac{\partial x}{a}\operatorname{cof.}\theta + \frac{\partial y}{a}\operatorname{fin.}\theta\right) = 0.$$

Eſt vero

$$\frac{\partial x}{a}\operatorname{cof.}\theta + \frac{\partial y}{a}\operatorname{fin.}\theta = \partial v - \partial\theta = \partial\Phi,$$

vnde noſtra aequatio erit

$$\partial x\,\partial\partial x + \partial y\,\partial\partial y + a\,c\,\partial\Phi\,\partial\partial\Phi = 0,$$

cuius integratio manifeſto dat

$$\partial x^2 + \partial y^2 + a\,c\,\partial\Phi^2 = \text{Conſt.}$$

vbi ergo, quia elementum temporis ∂t ſumtum eſt conſtans, ſtatui poterit homogoneitate introducta,

$$\partial x^2 + \partial y^2 + a\,c\,\partial\Phi^2 = \frac{\Gamma}{a\,a}\,\partial t^2,$$

ſiue $\frac{\partial x^2 + \partial y^2 + a\,c\,\partial\Phi^2}{a\,t^2} = \frac{\Gamma}{a\,a}$, quae aequatio per maſſam corporis M multiplicata (ob $\frac{\partial x^2 + \partial y^2}{\partial t^2}$ = quadrato celeritatis centri gra-

 vitatis

vitatis C) inuoluit primo vim viuam motus progreſſiui; praeterea pars $\frac{M a c \partial \Phi^2}{\partial t^2}$ (ob celeritatem angularem $= \frac{\partial \Phi}{\partial t}$, et momentum inertiae $= M a c$) exprimit vim viuam motus gyratorii. Sicque haec aequatio inuenta nobis declarat totam vim viuam noſtri corporis perpetuo manere eiusdem quantitatis, quippe quae ſemper aequalis erit vi viuae initio impreſſae.

§. 14. Nunc iam hanc aequationem ad binas variabiles v et θ transferamus, et cum ſit

$$\frac{\partial x^2 + \partial y^2}{a a} = (\partial v - \partial \theta)^2 + v v \partial \theta^2,$$

tum vero $\frac{c \partial \Phi^2}{a} = \frac{c}{a} (\partial v - \partial \theta)^2$, hinc quia poſuimus $1 + \frac{c}{a} = n$, habebimus hanc aequationem ſemel integratam:

$$n (\partial v - \partial \theta)^2 + v v \partial \theta^2 = \Gamma \partial t^2.$$

At vero haec aequatio parum iuuaret, ſi non inſuper aliam elicuerimus, id quod facile ſuccedet, vtendo hac combinatione: II. x — I. y, quae nobis praebet

$$x \partial \partial y - y \partial \partial x + c \partial \partial \Phi (x \text{ ſin.} \theta - y \text{ coſ.} \theta) = 0,$$

quae aequatio, ob x ſin. $\theta - y$ coſ. $\theta = - a$, abit in ſequentem:

$$x \partial \partial y - y \partial \partial x - a c \partial \partial \Phi = 0,$$

cuius integrale manifeſto eſt

$$x \partial y - y \partial x - a c \partial \Phi = \text{Conſt.} = a^2 \Delta \partial t.$$

Vbi notaſſe iuuabit formulam $x \partial y - y \partial x$ exprimere duplum elementum areae motu centri grauitatis C circa polum O deſcriptae. Similique modo formula $a c \partial \Phi$ ſpectari poteſt tanquam duplum elementum areae motu gyratorio deſcriptae, ita vt etiam hoc caſu deſcriptio arearum ſit tempori proportionalis.

§. 15.

§. 15. Transferamus nunc etiam hanc aequationem ad bina elementa v et θ; et cum sit $\frac{y}{x} = \frac{v \sin.\theta + \cos.\theta}{v \cos.\theta - \sin.\theta}$, erit differentiando $\frac{x \partial y - y \partial x}{x x} = \frac{v v \partial \theta - \partial v + \partial \theta}{(v \cos.\theta - \sin.\theta)^2}$, vnde colligitur

$$x \partial y - y \partial x = a a (v v \partial \theta - \partial v + \partial \theta);$$

tum vero erit $a c \partial \Phi = a c (\partial v - \partial \theta)$. Hinc ergo diuidendo per $a c$ nouam aequationem integratam sumus adepti, quae erit

$$v v \partial \theta - n (\partial v - \partial \theta) = \Delta \partial t,$$

quam ergo combinari conuenit cum ante inuenta

$$n (\partial v - \partial \theta)^2 + v v \partial \theta^2 = \Gamma \partial t^2.$$

Duas igitur inuenimus aequationes differentiales primi gradus, vnde tam v quam angulum θ ad quoduis tempus t inuestigare nobis incumbit.

§. 16. Nunc facile patet easdem has aequationes integratas ex binis differentialibus secundi gradus §. 8. deriuari potuisse, inde enim haec combinatio: I. $(\partial v - \partial \theta) +$ II. $v \partial \theta$ praebet

$$n (\partial v - \partial \theta)(\partial \partial v - \partial \partial \theta) - (\partial v - \partial \theta) v \partial \theta^2$$
$$+ v v \partial \theta \partial \partial \theta + 2 v \partial v \partial \theta^2 - v \partial \theta^3 = 0,$$

quae reducta ad hanc aequationem:

$$n (\partial v - \partial \theta)(\partial \partial v - \partial \partial \theta) + v \partial v \partial \theta^2 + v v \partial \theta \partial \partial \theta = 0,$$

manifesto praebet hoc integrale:

$$\tfrac{1}{2} n (\partial v - \partial \theta)^2 + \tfrac{1}{2} v v \partial \theta^2 = C = \tfrac{1}{2} \Gamma \partial t^2.$$

Simili modo haec combinatio: II. v — I. dat

$$v v \partial \partial \theta + 2 v \partial v \partial \theta - v \partial \theta^2 - n (\partial \partial v - \partial \partial \theta) + v \partial \theta^2 = 0,$$

siue

$$v v \partial \partial \theta + n \partial \partial \theta + 2 v \partial v \partial \theta - n \partial \partial v = 0,$$

 cuius

cuius integrale eſt

$$v\,v\,\partial\theta - n(\partial v - \partial\theta) = C = \Delta\,\partial t.$$

§. 17. Diuidamus nunc harum aequationum integratarum priorem per quadratum poſterioris, vt elementum temporis ∂t elidamus, ſicque nanciſcemur hanc aequationem:

$$\frac{n(\partial v - \partial\theta)^2 + v\,v\,\partial\theta^2}{[v\,v\,\partial\theta - n(\partial v - \partial\theta)]^2} = \frac{\Gamma}{\Delta\,\Delta},$$

in qua ſi ſtatuamus vt ſupra $\partial\theta = p\,\partial v$, prodibit haec aequatio:

$$\frac{n(1-p)^2 + v\,v\,p\,p}{[v\,v\,p - n(1-p)]^2} = \frac{\Gamma}{\Delta\,\Delta}.$$

Sicque adeo inter binas quantitates v et p relationem algebraicam ſumus adepti, quae ergo erit integralis aequationis illius differentialis, quae inextricabilis erat viſa, ſcilicet:

$$v\,v\,p^3\,\partial v = n\,v\,\partial p + n\,p\,\partial v\,(1-p)(2-p).$$

Quod ſi enim illa aequatio differentietur, haec ipſa prodibit, vti calculum inſtituens mox reperiet.

§. 18. Quo has formulas ſimpliciores reddamus, ſtatuamus porro, vt ſupra fecimus, $v\,v = n\,u$, atque huius aequationis differentialis:

$$p^3\,u\,\partial u = 2u\,\partial p + p\,\partial u\,(1-p)(2-p)$$

integrale erit: $\frac{(1-p)^2 + u\,p\,p}{[p\,u - (1-p)]^2} = \frac{n\,\Gamma}{\Delta\,\Delta}$. Hinc occaſionem arripio ſequentia theoremata ſubiungendi, quae, quoties fieri licet, multo generalius integrationem talium aequationum declarant.

Theorema I.

§. 19. Propoſita hac aequatione differentiali: $u\,\partial u + P\,\partial u + u\,\partial Q = 0$, in qua litterae P et Q ſint eiusmodi functiones ipſius p, vt valor huius formulae: $\frac{(\alpha P + \beta Q)^\beta}{(\beta P + \alpha Q)^\alpha}$, fiat

quan-

quantitas constans; tum integrale istius aequationis erit

$$\frac{[(\alpha+\beta)u+\alpha P+\beta Q]^\beta}{[(\alpha+\beta)u+\beta P+\alpha Q]^\alpha}=\text{Const.}$$

Theorema II.

§. 20. Si fuerit P functio quaecunque ipsius p, tum semper huius aequationis differentialis:

$$(\alpha-\beta)u\partial u+\partial u(\alpha A P^\alpha-\beta B P^\beta)+\frac{\alpha\beta u\partial P}{P}(B P^\beta-A P^\alpha)=0,$$

integrale completum erit $\frac{(u+A P^\alpha)^\beta}{(u+B P^\beta)^\alpha}=\text{Const.}$

§. 21. Quoniam igitur ad aequationem algebraicam inter quantitates v et p, vel etiam inter u et p, posito scilicet $vv=nu$ peruenimus, alteram earum per alteram definire licebit. Cum enim sit $\frac{ppu+(1-p)^2}{[pu-(1-p)]^2}=\frac{n\Gamma}{\Delta\Delta}$, hinc facile valor ipsius u per p determinari posset; verum pro instituto nostro expediet vicissim p per u exprimi. Hunc in finem statuamus $\frac{1-p}{p}=q$, vt nostra aequatio euadat $\frac{u+qq}{(u-q)^2}=\frac{n\Gamma}{\Delta\Delta}$, vnde facile q per u definitur. Posito enim breuitatis gratia $\frac{n\Gamma}{\Delta\Delta}=\frac{1}{\lambda}$, vt habeamus $\lambda u+\lambda qq=uu-2qu+qq$, extractio radicis praebet $q=\frac{u+\sqrt{\lambda u(1-\lambda+u)}}{1-\lambda}=\frac{1-p}{p}$. Hinc vnitatem vtrinque addendo fiet $\frac{1}{p}=\frac{1-\lambda+u+\sqrt{\lambda u(1-\lambda+u)}}{1-\lambda}$, quae expressio reducitur ad hanc:

$$\frac{1}{p}=\frac{\sqrt{1-\lambda+u}\,[\sqrt{1-\lambda+u}+\sqrt{\lambda u}]}{1-\lambda},$$

siue posito $1-\lambda=m$, vt sit $\lambda=1-m$, erit

$$\frac{1}{p}=\frac{\sqrt{m+u}\,[\sqrt{m+u}+\sqrt{(1-m)u}]}{m}.$$

§. 22.

§. 22. Nunc hanc fractionem supra et infra multiplicemus per $\sqrt{(m+u)} - \sqrt{(1-m)u}$, vt prodeat

$\frac{1}{p} = \frac{(1+u)\sqrt{(m+u)}}{\sqrt{(m+u)} - \sqrt{(1-m)u}}$, vnde inuertendo colligimus

$$p = \frac{\sqrt{(m+u)} - \sqrt{(1-m)u}}{(1+u)\sqrt{(m+u)}} = \frac{1}{1+u} - \frac{\sqrt{(1-m)u}}{(1+u)\sqrt{(m+u)}}.$$

Nunc igitur loco u restituto valore $\frac{vv}{n}$ erit

$$p = \frac{n}{vv+n} - \frac{nv\sqrt{(1-m)}}{(n+vv)\sqrt{(mn+vv)}},$$

vnde cum posuerimus $\partial\theta = p\,\partial v$, angulum θ ex sequenti aequatione definiri oportet:

$$\partial\theta = \frac{n\,\partial v}{n+vv} - \frac{nv\,\partial v\sqrt{(1-m)}}{(n+vv)\sqrt{(mn+vv)}}.$$

Hoc elemento inuento etiam elementum temporis ∂t definire poterimus ope aequationis $vv\,\partial\theta + n(\partial v - \partial\theta) = \Delta\,\partial t$, siue huius $\Delta\,\partial t = \partial v\,[p(vv+n) - n]$. Cum enim sit

$$p(vv+n) = n - \frac{n\sqrt{(1-m)}vv}{\sqrt{(mn+vv)}},$$

colligitur fore

$$\Delta\,\partial t = -\frac{nv\,\partial v\sqrt{(1-m)}}{\sqrt{(mn+vv)}}.$$

Sicque solutio nostri problematis perducta est ad integrationem harum duarum formularum.

§. 23. Hic autem statim in oculos incurrit *integratio temporis* t, siquidem manifesto fiet

$$\Delta t = -n\sqrt{(1-m)}\sqrt{(mn+vv)} + C,$$

vbi quidem signum contrarium prodiisset, si in resolutione aequationis quadratae altera radice essemus vsi; hoc autem ipsa quaestionis natura postulat, cum distantia v continuo crescat, sicque mutato signo formulae radicalis $\sqrt{(mn+vv)}$ reuera habebimus:

$$\Delta t = +n\sqrt{(1-m)}\sqrt{(mn+vv)} + C,$$

vnde

vnde ſi motus initio ſumamus fuiſſe $v = f$, erit

$$\Delta t = n\sqrt{1-m}\,(\sqrt{mn+vv} - \sqrt{mn+ff}).$$

§. 24. *Pro inueſtigatione anguli* θ ſtatim quoque ſignum radicalis immutemus, vt habeamus

$$\partial\theta = \frac{n\,\partial v}{n+vv} + \frac{n v\,\partial v\,\sqrt{(1-m)}}{(n+vv)\sqrt{(mn-vv)}},$$

cuius expreſſionis pars prior nulla laborat difficultate, cum ſit

$$\int \frac{n\,\partial v}{n+vv} = \sqrt{n} \times \text{A tang.}\,\frac{v}{\sqrt{n}}.$$

Tantum ergo ſupereſt, vt etiam partis poſterioris integrale inveſtigetur, quem in finem ponamus breuitatis gratia

$$\frac{n v\,\partial v\,\sqrt{(1-m)}}{(n+vv)\sqrt{(mn+vv)}} = \partial V,$$

vt habeamus

$$\theta = \sqrt{n} \times \text{A tang.}\,\frac{v}{\sqrt{n}} + V,$$

et poſito $\sqrt{(mn+vv)} = s$, vnde fit $v\,\partial v = s\,\partial s$, prodibit

$$\partial V = \frac{n\,\partial s\,\sqrt{(1-m)}}{n-mn+ss} = \frac{n\,\partial s\,\sqrt{(1-m)}}{n(1-m)+ss},$$

cuius integrale pariter per angulum exprimitur, ſiquidem

$$V = \sqrt{n} \times \text{A tang.}\,\frac{s}{\sqrt{n(1-m)}} = \sqrt{n} \times \text{A tang.}\,\sqrt{\frac{mn+vv}{n(1-m)}}.$$

§. 25. His igitur partibus coniungendis adipiſcimur

$$\frac{\theta}{\sqrt{n}} = \text{A tang.}\,\frac{v}{\sqrt{n}} + \text{A tang.}\,\sqrt{\frac{mn+vv}{n(1-m)}} - C.$$

Hinc ſi ambo arcus in vnum colligantur, fiet

$$\frac{\theta}{\sqrt{n}} = \text{A tang.}\,\frac{v\sqrt{n(1-m)} + \sqrt{n(mn+vv)}}{n\sqrt{(1-m)} - v\sqrt{(mn+vv)}} - C.$$

Conſtantis adiectae C valor ex ſtatu initiali definiri debet, pro quo ſi fuerit $\theta = 0$ et $v = f$, erit iſta conſtans

$$C = \text{A tang.}\,\frac{f\sqrt{n(1-m)} + \sqrt{n(mn+ff)}}{n\sqrt{(1-m)} - f\sqrt{(mn+ff)}}.$$

§. 26. Ex hac folutione videmus, litteram m nunquam vnitatem excedere poffe, quia alias hae formulae euaderent imaginariae. Cum igitur pofuerimus $m = 1 - \lambda$, patet etiam λ cyphrae maius et pofitiuum effe debere. Pofuimus autem $\lambda = \frac{\Delta\Delta}{n\Gamma}$, quae quantitas vtique nunquam effe poteft negatiua, propterea quod inuenimus

$$\Gamma \partial t^2 = n(\partial v - \partial \theta)^2 + vv\partial\theta^2,$$

quae eft fumma duorum quadratorum, ideoque certe pofitiua. Cum deinde fit $m = 1 - \lambda = \frac{n\Gamma - \Delta\Delta}{n\Gamma}$, videmus femper fore $n\Gamma > \Delta\Delta$, vnde fequitur fore

$$nn(\partial v - \partial\theta)^2 + nvv\partial\theta^2 > [vv\partial\theta - n(\partial v - \partial\theta)]^2,$$

hincque deducitur ifta conditio, qua effe debet $2n\partial v > (vv+n)\partial\theta$; vnde fi ponamus initio fuiffe $v = f$, $\frac{\partial v}{\partial t} = \alpha$ et $\frac{\partial\theta}{\partial t} = \beta$, neceffe eft vt fuerit $2n\alpha > (ff+n)\beta$, quae conditio fi non fuerit obferuata, motus plane fecundum principia mechanica fubfiftere non poterit.

§. 27. Ex his igitur, quae hactenus funt inuenta, pro quouis tempore elapfo t, tam quantitatem v, hincque diftantiam $OT = z = av$, quam angulum $DOT = \theta$ affignare licebit, hincque porro etiam innotefcet motus cylindri gyratorius, cuius celeritas angularis eft $\frac{\partial\Phi}{\partial t} = \frac{\partial v - \partial\theta}{\partial t}$. Eft vero

$$\partial v - \partial\theta = \frac{vv\partial v}{n+vv} - \frac{nv\partial v\sqrt{(1-m)}}{(n+vv)\sqrt{(mn+vv)}};$$

vnde cum effet $\partial t = \frac{nv\partial v\sqrt{(1-m)}}{\Delta\sqrt{(mn+vv)}}$, erit ifta celeritas angularis

$$\frac{\partial\Phi}{\partial t} = \frac{v\Delta\sqrt{(mn+vv)}}{(n+vv)n\sqrt{(1-m)}} - \frac{\Delta}{n+vv},$$

cuius formulae differentiale, fi denuo per ∂t diuidatur, dabit $\frac{\partial\partial\Phi}{\partial t}$, cui aequabatur formula $\frac{2g\Theta}{Mc}$, vnde ergo innotefcet tenfio fili Θ, pariter pro quouis tempore, quo motus durat, ficque omnia

omnia, quae circa hunc motum desiderari possunt, felici successu determinauimus.

§. 28. Quoniam pro littera m duos limites inuenimus, quos transgredi non licet, quorum alter est $m = 0$, ideoque $\lambda = 1$ et $n\Gamma = \Delta\Delta$; alter vero, quo $m = 1$, ideoque $\lambda = 0$, hincque $\Delta = 0$: operae pretium erit hos duos casus extremos seorsim euoluere, quandoquidem reliqui omnes inter hos continentur. Facile autem intelligitur his duobus casibus calculum mirum in modum contrahi debere; vnde eorum solutionem immediate ex aequationibus differentialibus deriuabimus.

Euolutio casus quo $m = 1$ siue $\Delta = 0$.

§. 29. Quia hic est $\Delta = 0$, posterior aequatio, supra integrata, nobis statim praebet $vv\partial\theta - n(\partial v - \partial\theta) = 0$; vnde sequitur $\partial\theta = \frac{n\partial v}{vv+n}$, cuius integrale est $\theta = \sqrt{n} \times \text{A tang.} \frac{v}{\sqrt{n}}$; vbi constantem non adiicimus, quia nihil impedit, quo minus initium ibi statuamus, vbi est $v = 0$, sicque enim quoque sponte fiet $\theta = 0$.

§. 30. Quod iam ad elementum temporis ∂t spectat, id ex posteriore aequatione neutiquam concludere licet, ideoque prior aequatio in subsidium vocari debet, quae, ob $\partial v - \partial\theta = \frac{vv\partial v}{n+vv}$, induet hanc formam:

$$\frac{n v^4 \partial v^2}{(n+vv)^2} + \frac{n^2 vv \partial v^2}{(n+vv)^2} = \Gamma \partial t^2,$$

quae contrahitur in hanc: $\frac{nvv\partial v^2}{n+vv} = \Gamma\partial t^2$, sicque erit $\partial t\sqrt{\Gamma} = \frac{v\partial v\sqrt{n}}{\sqrt{(n+vv)}}$, cuius integrale est $t\sqrt{\Gamma} = \sqrt{n(n+vv)} - n$, siquidem initio, quo $t = 0$, assumimus fore etiam $v = 0$.

§. 31. Cum iam porro ſit $\partial \Phi = \partial v - \partial \theta = \frac{v v \partial v}{n + v v}$, erit $\frac{\partial \Phi}{\partial t \sqrt{\Gamma}} = \frac{v}{\sqrt{n(n + v v)}}$, cuius differentiale praebet

$$\frac{\partial \partial \Phi}{\partial t \sqrt{\Gamma}} = \frac{\partial v \sqrt{n}}{(n + v v)^{\frac{3}{2}}},$$

quod diuiſum per $\partial t \sqrt{\Gamma}$ dat $\frac{\partial \partial \Phi}{\Gamma \partial t^2} = \frac{1}{v(n + v v)}$, vnde colligitur tenſio $\Theta = \frac{M c \Gamma}{2 g} \cdot \frac{1}{v(n + v v)}$, quae ergo initio debuit eſſe infinite magna, dehinc vero continuo valde decreſcit et mox vix ſenſibilis euadit.

§. 32. Examinemus iam etiam ſtatum initialem, vbi fuiſſe aſſumimus $v = 0$ et $\theta = 0$, pro quo igitur fuerant celeritates $\frac{\partial v}{\partial t} = \frac{\sqrt{\Gamma}}{v}$, exiſtente $v = 0$; tum vero $\frac{\partial \theta}{\partial t} = \frac{\sqrt{\Gamma}}{v}$, atque porro $\frac{\partial \Phi}{\partial t} = 0$, ita vt ipſo initio motus gyratorius fuerit nullus. Praeterea vero pro loco centri cylindri C initio erat abſciſſa $A P = x = 0$, applicata vero $P M = y = a$. At vero huius puncti C celeritates reperientur $\frac{\partial x}{a \partial t} = 0$ et $\frac{\partial y}{a \partial t} = a \sqrt{\Gamma}$; vnde patet, centrum grauitatis C hac celeritate ſecundum directionem applicatae ſurſum fuiſſe promotum.

§. 33. Vt iam hinc continuationem motus initio impreſſi inueſtigemus, obſeruemus primo ex aequatione $t \sqrt{\Gamma} = \sqrt{n(n + v v)} - n$ ſequi $v = \frac{\sqrt{\Gamma t t + 2 n t \sqrt{\Gamma}}}{\sqrt{n}}$; vnde patet, ſucceſſu temporis quantitatem v continuo creſcere. Inuenta autem pro quolibet tempore quantitate v, vnde fit diſtantia $O T = z = a v$, angulus $D O T = \theta$ innoteſcet ex hac aequatione: $\theta = \sqrt{n} \times \text{A tang.} \frac{v}{\sqrt{n}}$; vnde diſcimus ſucceſſu temporis etiam angulum θ continuo creſcere, non vero in infinitum, ſed vsque ad certum terminum. Sumto enim $v = \infty$, quod elapſo quoque tempore infinito euenire deberet, prodit iſte angulus $\theta =$

$\theta = 90° \sqrt{n}$, ideoque recto maior euadere poterit. Vnde patet hanc curuam conuergere ad asymptotam ex puncto sub angulo $90° \sqrt{n}$ ad rectam O D inclinatam, vel potius ad rectam huic parallelam, vnde figura huius curuae haud difficulter cognosci poterit.

§. 34. Inquiramus nunc accuratius in lineas curuas, quas tam centrum cylindri C quam punctum contactus T durante motu describent, id quod sine respectu ad tempus habito expedire conueniet, siquidem ad quoduis tempus longitudo lineae v et angulus θ sunt assignati. Hunc in finem statuamus breuitatis gratia radium cylindri $a = 1$ et $\sqrt{n} = \alpha$, vt fiat longitudo fili O T $= v$ (supra $= z$) et angulus $\theta = \alpha$ A tang. $\frac{v}{\alpha}$. Euidens autem est naturam vtriusque curuae quaesitae ex relatione inter v et θ deriuari debere.

§. 35. Incipiamus igitur a curua, quam punctum contactus T durante motu percurret; ac primo quidem eius indolem inuestigemus prope ipsum initium O, quamdiu scilicet longitudo v est valde parua, ita vt assumi possit A tang. $\frac{v}{\alpha} = \frac{v}{\alpha}$ ideoque $\theta = v$. Hunc in finem ex puncto T demittatur applicata T U, vocatisque coordinatis O U $=$ X et U T $=$ Y, erit X $= v$ cos. θ et Y $= v$ sin. θ. Quia igitur circa initium O est $\theta = v$, ideoque sin. $\theta = v$ et cos. $\theta = 1$, fiet X $= v$ et Y $= vv$; consequenter Y $=$ X X; vnde patet hanc curuam in puncto O axem O D tangere eiusque radium curuaturae fore $= \frac{1}{2}$. Tab. II. Fig. 1.

§. 36. Inuestigemus etiam curuae indolem in infinitum porrectae, vbi ob $v = \infty$ erit A tang. $\frac{v}{\alpha} = 90°$, ideoque $\theta = \alpha . 90°$, qui ergo angulus, ob $\alpha > 1$, semper erit angulo recto maior, et quidem pro casu, quo corpus nostrum foret cylindrus

 ex

ex materia homogenea conſtans, ob $c = \frac{1}{2} a$, vti iam ſupra invenimus, foret $n = \frac{3}{2}$, hincque $\alpha = \sqrt{\frac{3}{2}} = 1,2245$, ideoque $\alpha . 90^\circ = 110^\circ . 13'$.

Tab. II. Fig. 2. §. 37. Ex puncto ergo O ſub hoc angulo ducatur recta O E, vt ſit D O E $= \alpha . 90^\circ$, vltra quem terminum angulum θ augeri nequit, et quem cum attigerit, diſtantia v euadet infinita. Quaeſtio ergo iam huc redit, vt determinetur diſtantia puncti T ab iſta recta O E, poſtquam angulus D O T $= \theta$ vsque ad $90^\circ . \alpha$ fuerit auctus. Hunc in finem ex T ad iſtam rectam O E ducatur perpendiculum T S, atque ob angulum T O E $= 90^\circ . \alpha - \theta$ erit iſtud perpendiculum T S $= v$ ſin. $(90^\circ \alpha - \theta)$, cuius ergo valor quaeritur, pro caſu quo $\theta = 90^\circ \alpha$, quo quidem caſu ſin. $(90^\circ \alpha - \theta)$ euaneſceret, ſed quia diſtantia v euadit infinite magna, vtique fieri poteſt, vt haec formula finitum valorem adipiſcatur, quae inueſtigatio cum non ſit vulgaris, eam data opera hic exponamus.

§. 38. Cum ſit $\theta = \alpha$ A tang. $\frac{v}{\alpha}$, ponatur ille arcus cuius tang. eſt $\frac{v}{\alpha} = \omega$, vt ſit $v = \alpha$ tang. ω et $\theta = \alpha \omega$, ſicque habebimus interuallum T S $= \alpha$ tang. ω. ſin. $\alpha (90^\circ - \omega)$, cuius ergo valor requiritur pro caſu $\omega = 90^\circ$, vbi tangens manifeſto fit infinita, ſinus vero euaneſcit. Ad hunc valorem inueſtigandum angulum ω infinite parum infra 90° deprimamus, ſtatuendo $\omega = 90^\circ - o$, fietque ſin. $\alpha (90^\circ - \omega) =$ ſin. $\alpha o = \alpha o$ et tang. $\omega = \frac{\text{ſin.}\, \omega}{\text{coſ.}\, \omega} = \frac{1}{o}$, ex quibus coniunctis deducitur interuallum T S $= \alpha \alpha = n$.

Tab. II. Fig. 3. §. 39. Rectae ergo O G parallela ducatur recta infinita G I, ab ea diſtans interuallo $\alpha \alpha = n$, atque noſtra curua in infinitum extenſa cum iſta recta G I confundetur, quae igitur erit noſtrae

nostrae curuae assymptota, sicque haec curua secundum tractum O T I in infinitum per figuram satis regularem procedet, quoniam nusquam flexum habet contrarium eiusque curuatura satis vniformiter decrescit et tandem in rectam desinit.

§. 40. Inuestigemus nunc etiam curuam, quam punctum C describet, ac primo quidem eius figuram pro initio inuestigemus. Supra autem pro eius loco quocunque dedimus eius coordinatas $OP = x = v \cos.\theta - \sin.\theta$ et $PC = y = v \sin.\theta + \cos.\theta$, maneatque vt ante $\theta = \alpha \,A\,\mathrm{tang}.\frac{v}{\alpha}$; vnde ergo ob v minimum erit $\theta = v$, $\sin.\theta = v$ et $\cos.\theta = 1$, sicque pro initio habebimus $x = 0$ et $y = 1 + vv$. Cum autem accuratius sit $\sin.\theta = v - \frac{1}{6}v^3$ et $\cos.\theta = 1 - \frac{1}{2}vv$, habebimus $x = -\frac{1}{3}v^3$ et $y = 1 + \frac{1}{2}vv$, ideoque $y - 1 = \frac{1}{2}vv$.

§. 41. Hinc igitur discimus figuram huius curuae circa initium contra, ac figura praecedens refert, esse positam, cum abscissa x valorem obtinuerit negatiuum. Scilicet cum ipso initio centrum cylindri C fuerit in Γ, existente perpendiculo $O\Gamma = 1$, vera curua Γ C retro porrigitur, existente abscissa $OP = x = \frac{1}{3}v^3$ et applicata $PC = y = 1 + \frac{1}{2}vv$ ideoque $QC = \frac{1}{2}vv$, quae cum iam sit propria applicata, si ea ponatur $= u$, erit $u = \frac{1}{2}vv$, vnde fiet $8u^3 = 27xx$, siue $u^3 = \frac{27}{8}xx$, vnde patet initium curuae conuenire cum Parabola cubica secunda, siue Neiliana, in vertice Γ cuspidem gerente, cuius parameter est $\frac{27}{8}$ seu $3\frac{3}{8}$. Notum autem est, huius curuae radium osculi in Γ fore $= 0$, ita vt primo initio motus puncti C flexuram infinite magnam fuerit passus, ad quam producendam vtique tensione fili infinite magna opus erat. Tab. II. Fig. 4.

§. 41. Vt nunc etiam indolem portionis infinitesimae cognoscamus, consideremus punctum in loco quocunque C, cui respondeat punc- Tab. II. Fig. 5.

punctum contactus T, ita vt $CT = 1$ et ad $OT = v$ normalis. Iam ducatur recta OC, et sit angulus $COT = \eta$, vt fiat $DOC = \theta + \eta$, eritque $\text{tang.}\,\eta = \frac{1}{v}$ et $OC = \frac{v}{\cos.\,\eta}$. Praeterea vero statuamus vt ante $A\,\text{tang.}\,\frac{v}{\alpha} = \omega$, vt sit $v = \alpha\,\text{tang.}\,\omega$ et $\theta = \alpha\,\omega$, hincque $\text{tang.}\,\eta = \frac{1}{v} = \frac{1}{\alpha\,\text{tang.}\,\omega}$. Quia iam status quaeritur, quando angulus ω abit in rectum, ideoque $\theta = 90°\,\alpha$, ducatur iterum recta OE sub angulo $DOE = 90°\,\alpha$, ad quam ex puncto C demissum perpendiculum CS quaeri debet. Cum igitur sit angulus $COE = 90°\,\alpha - \theta - \eta$, ob $OC = \frac{v}{\cos.\,\eta}$ reperitur $CS = \frac{v\,\sin.(90\alpha - \eta - \theta)}{\cos.\,\eta}$, cuius expressionis postquam loco v, θ et η valores modo assignati fuerint substituti, valor assignari debet pro casu quo fit $\omega = 90°$.

§. 42. Hunc in finem statuamus vt ante $\omega = 90° - o$, eritque $v = \frac{\alpha}{o}$, $\theta = \alpha\,(90° - o)$, $\eta = \frac{o}{\alpha}$ et $\cos.\,\eta = 1$, quibus valoribus introductis erit interuallum quaesitum $CS = \alpha\alpha - 1 = n - 1$; vnde patet, distantiam huius curuae in infinitum extensae a recta OE vnitate minorem esse quam curvae praecedentis, id quod cum rei natura egregie conuenit, cum distantia inter C et T perpetuo maneat vnitati aequalis.

§. 43. Ad hanc ergo curuam describendam, ad rectam
Tab. II. OE sub angulo $90°\,\alpha$ ad axem OD ductam, ducatur perpen-
Fig. 6. diculum $OF = n - 1$, existente interuallo $O\Gamma = 1$, ac per F ducatur recta FI ipsi OE parallela, eaque erit assymptota nostrae curuae quaesitae ΓCI, quippe quae cum ista recta in infinito prorsus congruet. Ceterum ista curua ex praecedente facile construi potuisset, cum pro singulis locis puncti contactus T facillime loca centri C definiri potuissent. Verum quia haec ipsa inuestigatio neutiquam est obuia, haud invtile visum fuit, hanc Analysin fusius exponere.

Euo-

Euolutio alterius casus extremi, quo $m = 0$, siue $\Delta\Delta = n\Gamma$.

§. 44. Cum igitur hoc casu sit $n\Gamma = \Delta\Delta$, erit

$$nn(\partial v - \partial\theta)^2 + nvv\partial\theta^2 = (vv\partial\theta - n(\partial v - \partial\theta))^2,$$

quae aequatio euoluta praebet $(vv + n)\partial\theta = 2n\partial v$, ideoque $\partial\theta = \frac{2n\partial v}{vv+n}$. Quod si iam vt ante assumamus initio fuisse tam $\theta = 0$ quam $v = 0$, habebimus integrando $\theta = 2\sqrt{n} \times \text{A tang.} \frac{v}{\sqrt{n}}$. Hic iam statim vt supra statuamus $\sqrt{n} = \alpha$ et $\text{A tang.} \frac{v}{\sqrt{n}} = \omega$, fiet $v = \alpha \text{ tang. } \omega$ et $\theta = 2\alpha\omega$, ex qua formula iam ambas curvas, quas puncta T et C durante motu describunt, determinare poterimus.

§. 45. Quo autem nostra inuestigatio aliquanto latius pateat, duas has formulas contemplemur: $v = \alpha \text{ tang. } \omega$ et $\theta = \beta\omega$, ita vt hic sit $\beta = 2\alpha$, cum casu praecedente fuisset $\beta = \alpha$. Incipiamus nunc a curua, quam centrum grauitatis C percurrit, pro qua positae erant coordinatae $AP = x = v \text{ cos. } \theta - \text{sin. } \theta$ et $PC = y = v \text{ sin. } \theta + \text{cos. } \theta$, posito scilicet iterum radio cylindri $a = 1$. Ac primo quidem indolem huius curvae circa ipsum motus initium inuestigemus, vbi interuallum v minimum, ideoque etiam angulus ω vt valde paruus erit spectandus, ita vt proxime sit $v = \alpha\omega + \frac{1}{3}\alpha\omega^3$; tum vero Tab. II. Fig. 1.

$$\text{sin. } \theta = \beta\omega - \tfrac{1}{6}\beta^3\omega^3 \text{ et}$$
$$\text{cos. } \theta = 1 - \tfrac{1}{2}\beta^2\omega^2 + \tfrac{1}{24}\beta^4\omega^4$$

vnde pro initio huius curuae erit

$$x = (\alpha - \beta)\omega + (\tfrac{1}{3}\alpha - \tfrac{1}{2}\alpha\beta^2 + \tfrac{1}{6}\beta^3)\omega^3 \text{ et}$$
$$y = 1 + (\alpha\beta - \tfrac{1}{2}\beta^2)\omega^2 + (\tfrac{1}{24}\beta^3 + \tfrac{1}{3}\alpha - \tfrac{1}{6}\alpha\beta^2)\beta\omega^4.$$

§. 46. Hinc igitur pro casu praecedente, quo erat $\beta = \alpha$, debuit esse

 $x =$

$$x = (\tfrac{1}{3}\alpha - \tfrac{1}{3}\alpha^3)\omega^3 = -\tfrac{1}{3}\alpha(\alpha\alpha - 1)\omega^3 \text{ et}$$

$$y = 1 + \tfrac{1}{2}\alpha^2\omega^2 + (\tfrac{1}{3}\alpha^2 - \tfrac{1}{8}\alpha^4)\omega^4,$$

vbi patet, pro abſciſſa x primum terminum ſponte euanuiſſe, ideoque procedi oportuiſſe ad poteſtatem ω^3; pro applicata autem y ſuffeciſſe in poteſtate ω^2 ſubſtitiſſe, ita vt ſtatui poſſit $y - 1 = \frac{1}{2}\alpha^2\omega^2$ et $x = -\frac{1}{3}\alpha(\alpha\alpha - 1)\omega^3$. Hinc ergo accuratius, quam ante fecimus, deducitur fore $\frac{(y-1)^3}{x^2} = \frac{9\alpha^4}{8(\alpha\alpha-1)^2}$; initium ſcilicet huius curuae conueniet cum parabola cubicali ſecunda, cuius autem parameter erit $\frac{9}{8}\frac{\alpha^4}{(\alpha\alpha-1)^2}$. Verum haec correctio non influit in ſuperiorem determinationem, cum ſufficiat noſſe, huius curuae radium oſculi in initio eſſe infinite paruum et ipſam curuam retro vergere.

§. 47. At vero pro praeſenti caſu, quo $\beta = 2\alpha$, prodibit abſciſſa

Tab II. Fig. 7.

$$x = -\alpha\omega - \tfrac{1}{3}\alpha(2\alpha\alpha - 1)\omega^3,$$

ideoque ſatis prope $x = -\alpha\omega$; at vero applicata reperietur

$$y = 1 - \tfrac{2}{3}\alpha\alpha(\alpha\alpha - 1)\omega^4.$$

Quare ſi initio fuerit centrum C in puncto Γ, ſumto in recta axi O D parallela interuallo $\Gamma\gamma = \alpha\omega$, eo repraeſentabitur noſtra abſciſſa minima in plagam contrariam verſa, applicata autem y, ob $\alpha\alpha > 1$, aliquantillo minor erit quam 1, ſicque curvae punctum infra γ in π cadet, interuallo $\gamma\pi$ infinities minore quam $\Gamma\gamma$; vnde haec curua porro per punctum m ſenſim ſenſimque deſcendet, ita vt in initio Γ radius oſculi fuerit infinite magnus et curuatura nulla. Motus ergo primus centri grauitatis retro erat directus cum celeritate finita, vti deinceps videbimus.

§. 48. Inquiramus nunc etiam in naturam huius curvae in infinitum porrectae, atque retenta in calculo littera β primo

primo euidens eſt diſtantiam v euadere infinitam, vbi angulus ω vsque ad $90°$ augetur. Tum autem erit angulus $\theta = \beta . 90°$. Conſtituto ergo angulo $DOE = \beta . 90°$, recta OE nobis poſitionem rectae OT, quando in infinitum fuerit aucta, vel quando punctum C in infinitum proceſſerit, referet; neque vero hinc ſequitur punctum C in ipſam hanc rectam OD incidere, vnde neceſſe eſt eius diſtantiam ab hac recta OE explorare. Tab. II. Fig. 5.

§. 49. Ducta igitur primo recta OC vocetur angulus $COT = \eta$, vt ſit angulus $DOC = \theta + \eta$; at vero ob radium $CT = 1$ et $OT = v$, erit tang. $\eta = \frac{1}{v} = \frac{1}{\alpha \, tang. \, \omega}$, hincque ipſa diſtantia $OC = \frac{v}{cof. \, \eta}$. Nunc ſi ex puncto C ad rectam OE ducamus normalem CS, ob angulum $COS = \beta . 90° - \eta - \theta$ erit

$$CS = \frac{v \, fin. (\beta . 90° - \theta - \eta)}{cof. \, \eta} = \frac{\alpha \, tang. \, \omega}{cof. \, \eta} \, \text{fin.} (\beta . 90° - \beta \omega - \eta),$$

cuius ergo valor quaeritur, quando angulus ω euadit rectus. Euidens autem eſt hoc caſu priorem factorem fieri infinitum, alterum vero euaneſcere, propterea quod etiam angulus η hoc caſu fit nullus.

§. 50. Ad hoc ergo inueſtigandum conſideremus angulum ω adhuc infinite parum a recto deficientem, ac ſtatuamus $\omega = 90° - o$, hincque primo fiet tang. $\omega = \frac{fin. \, \omega}{cof. \, \omega} = \frac{1}{o}$, ideoque tang. $\eta = \frac{o}{\alpha}$, conſequenter $\eta = \frac{o}{\alpha}$, quibus valoribus ſubſtitutis prodibit interuallum

$$CS = \frac{\alpha}{o} \, \text{fin.} \left(\beta o - \frac{o}{\alpha}\right) = \alpha \beta - 1.$$

Hanc ob rem ſi ad rectam OE ducamus normalem $OF = \alpha \beta - 1$, atque per F producamus ipſam rectam OE parallelam FI, in eam punctum C, cum in infinitum protrahetur, incidet, ideoque haec recta FI erit aſſymptota curuae a puncto C deſcriptae.

§. 51. Cum igitur casu, quem hic tractamus, sit $\beta = 2\alpha$,
Tab. II. Fig. 8. erit angulus $DOE = \alpha.\, 180^\circ$ ideoque duobus rectis maior, ob $\alpha > 1$, siue hic angulus erit gibbus. Ad hunc igitur casum constructa est figura 8, vbi angulus gibbus DOE est $\alpha.\, 180^\circ$, ad hanc rectam OE normaliter ducta recta $OF = 2\alpha^2 - 1 = 2n - 1$, assymptota nostrae curuae FI ipsi OE parallela, per hoc punctum F transibit, ad quam igitur nostra curua tractu satis vniformi continuo propius accedet.

§. 52. Multo autem facilius alteram curuam a puncto T descriptam definiemus. Positis enim pro ea coordinatis $OU = X$ et $UT = Y$, ob $DOT = \theta$ et $OT = v$ erit ab-
Fig. 1. scissa $X = v \cos.\theta$ et applicata $Y = v \sin.\theta$. At vero ante iam vidimus coordinatas hasce per angulum ω ita exprimi:

$$X = \alpha\omega + \alpha\left(\tfrac{1}{3} - \tfrac{1}{2}\beta\beta\right)\omega^3 \text{ et}$$

$$Y = \alpha\beta\omega^2 + \alpha\beta\left(\tfrac{1}{3} - \tfrac{1}{6}\beta\beta\right)\omega^4$$

vbi satis erit posuisse $X = \alpha\omega$ et $Y = \alpha\beta\omega\omega$, vnde erit $\frac{Y}{X^2} = \frac{\beta}{\alpha}$, sicque nostra curua congruet cum Parabola rectam OB in suo vertice tangentem, cuiusque axis ad OD est normalis. Nostro igitur casu, quo $\beta = 2\alpha$, eius parameter erit $= \frac{1}{2}$.

§. 53. Pro portione huius curuae in infinitum porrecta ducatur recta OE, sub angulo $DOE = 90^\circ.\,\beta$, ad quam
Fig. 2. ex puncto T demittatur perpendiculum TS, quod erit

$$TS = v \sin.(90^\circ\beta - \theta) = \alpha \tan g.\,\omega \sin.\beta(90^\circ - \omega);$$

quam ob rem posito $\omega = 90^\circ - \rho$, ob $\tan g.\,\omega = \frac{1}{\rho}$ erit $TS = \alpha\beta$. Hinc si rectae OE parallela agatur FI, distans ab illa interuallo $OF = \alpha\beta$, erit haec recta assymptota nostrae
Fig. 3. curuae.

§. 54.

§. 54. Cum igitur pro noſtro caſu ſit $\beta = 2\,\alpha$, recta O E, vt modo ante, cum O B faciet angulum gibbum $= 180^\circ.\,\alpha$, et iam erit interuallum $O\,F = 2\,\alpha\,\alpha = 2\,n$, ſicque hic vnitate maius eſt quam ante. Hinc rectae F I ad interuallum $F\,F' = 1$, vt $O\,F' = 2\,\alpha\,\alpha$, ducatur parallela $F'\,I'$, erit haec aſſymptota curuae a puncto T deſcriptae O T I'. Tab. II. Fig. 8.

§. 55. Quoniam igitur per angulum ω omnia harum curuarum puncta determinauimus, per evndem quoque tempora exprimamus, quibus ſingulae portiones abſoluuntur. Cum igitur pro noſtro caſu, quo $\beta = 2\,\alpha$, ſit $v = \alpha\,\text{tang.}\,\omega$, at $\theta = 2\,\alpha\,\omega$, erit $\partial v = \frac{\alpha\,\partial\omega}{\cos.\,\omega^2}$ et $\partial\theta = 2\,\alpha\,\partial\omega$, vnde ſit

$$\partial v - \partial\theta = \frac{\alpha\,\partial\omega\,(1 - 2\cos.\,\omega^2)}{\cos.\,\omega^2} = -\frac{\alpha\,\partial\omega\,\cos.\,2\,\omega}{\cos.\,\omega^2},$$

hincque ex aequatione principali poſteriore

$$v\,v\,\partial\theta - n\,(\partial v - \partial\theta) = \Delta\,\partial t,$$

ob $n = \alpha\,\alpha$, deducimus hanc: $\frac{\alpha^3\,\partial\omega}{\cos.\,\omega^2} = \Delta\,\partial t$, ſicque nanciſcimur integrando $\Delta\,t = \alpha^3\,\text{tang.}\,\omega = \alpha\,\alpha\,v$; vnde patet, longitudinem fili O T cum tempore vniformiter creſcere. Ceterum hic notetur eſſe $\Delta = \alpha\sqrt{\Gamma}$, id quod obſeruaſſe ideo iuuabit, quod quantitas Γ vi viuae eſt proportionalis, quae corpori noſtro fuit impreſſa, eademque perpetuo conſeruatur.

§. 56. Cum igitur Γ a quantitate motus initio impreſſi pendeat, videmus ipſas curuas deſcriptas non ab hoc motu eiusue quantitate pendere, ſed totum diſcrimen in eo conſiſtere, quod iſtae curuae tardius celeriusue percurrentur. At vero indoles curuarum praecipue a qualitate motus initio impreſſi pendet; quamobrem examinemus, qualis motus corpori initio imprimi debuerit, vt hae ipſae curuae, quas determinauimus, percurrantur.

§. 57. Primo igitur quaeramus motum, qui centro grauitatis corporis C initio imprimi debeat, vt iſtum motum proſequatur. Referat igitur Figura 9. ſitum corporis initialem, pro quo ſupra vidimus eſſe coordinatas $x = -\alpha\,\omega$ et $y = 1 - \frac{2}{3}\alpha^4$, vnde ergo pro celeritatibus colligitur

Tab. II. Fig. 9.

$$\frac{\partial x}{\partial t} = -\frac{\Delta\,cof.\,\omega}{\alpha\alpha} = -\frac{\Delta}{\alpha\alpha} \text{ et } \frac{\partial y}{\partial t} = 0;$$

vnde patet centro grauitatis Γ initio imprimi debuiſſe ſecundum directionem Γa ipſi axi OD parallelam, at in plagam contrariam directam, celeritatem $= + \frac{\Delta}{\alpha\alpha}$. Si haec motus celeritas vocetur $= k$ atque pro data accipiatur, erit $\frac{\Delta}{\alpha\alpha} = k$, ſicque innoteſcit quantitas $\Delta = \alpha\alpha k$, hincque porro $\Gamma = \alpha\alpha k k$.

§. 58. Euidens autem eſt, ſi ſolus hic motus corpori imprimeretur, filum circumuolutum et in O fixum relaxatum iri, quod ne fiat, neceſſe eſt vt corpori noſtro inſuper motus gyratorius in ſenſum $b\,d\,a$ imprimatur, id quod etiam calculus manifeſto declarat. Cum enim pro motu gyratorio ſit $\partial\Phi = \partial v - \partial\theta$, erit generatim $\partial\Phi = \frac{\alpha\,\partial\omega(1 - 2\,cof.\,\omega^2)}{cof.\,\omega^2}$, hincque ipſa celeritas gyratoria $\frac{\partial\Phi}{\partial t} = \frac{\Delta}{\alpha\alpha}(1 - 2\,\text{cof.}\,\omega^2)$; vnde pro ipſo initio, quo $\omega = 0$, erit iſta celeritas angularis $= -\frac{\Delta}{\alpha\alpha}$, in contrariam ſcilicet plagam vergit, eritque $= -k$, ideoque motus iſte ipſo cum motu progreſſiuo conueniet.

§. 59. Quia igitur in genere inuenimus celeritatem angularem $\frac{\partial\Phi}{\partial t} = \frac{\Delta}{\alpha\alpha}(1 - 2\,\text{cof.}\,\omega^2)$, hinc diſcimus, ab initio hanc celeritatem in plagam $b\,d\,a$ directam continuo imminui, atque adeo in nihilum abire, vbi angulus ω euaſerit ſemirectus; poſtea vero, quando iſte angulus vltra hunc terminum 45° increſcit, motus gyratorius generabitur in plagam contrariam verſus, qui continuo augebitur, donec tandem poſt tempus infinitum

tum, quo fit $\omega = 90°$, euadit $\frac{\Delta}{\alpha\alpha}$, ideoque aequalis ipſi motui gyratorio initio impreſſo, at vero illi contrarius.

§. 60. Inſignis autem haec continua motus gyratorii mutatio manifeſto producitur a tenſione fili, quae ergo hic inueſtiganda reſtat, et quam initio deſignauimus littera Θ, atque inuenimus eſſe $\Theta = \frac{Mc\,\partial\partial\Phi}{2g\,\partial t^2}$. Cum igitur ſit

$$\frac{\partial\Phi}{\partial t} = \frac{\Delta}{\alpha\alpha}(1 - 2\,\text{coſ.}\,\omega^2), \text{ erit}$$

$$\frac{\partial\partial\Phi}{\partial t} = \frac{4\Delta}{\alpha\alpha}\partial\omega\,\text{ſin.}\,\omega\,\text{coſ.}\,\omega, \text{ ergo}$$

$$\frac{\partial\partial\Phi}{\partial t^2} = \frac{4\Delta\Delta\,\text{ſin.}\,\omega\,\text{coſ.}\,\omega^3}{\alpha^5} = \frac{4\Gamma\,\text{ſin.}\,\omega\,\text{coſ.}\,\omega^3}{\alpha^3},$$

conſequenter ipſa tenſio fili erit

$$\Theta = \frac{Mc}{2g} \cdot \frac{4\Gamma\,\text{ſin.}\,\omega\,\text{coſ.}\,\omega^3}{\alpha^3}.$$

§. 61. Hinc igitur apparet, ipſo motus initio, quo erat $\omega = 0$, hanc tenſionem prorſus euanuiſſe, hincque paullatim ſucceſſu temporis increſcere, verum non vltra certum terminum; quandoquidem ſumto $\omega = 90°$ tenſio iterum euaneſcit; vnde patet eam alicubi fieri maximam, factoque calculo reperietur hoc euenire quando tang. $\omega = \frac{1}{\sqrt{3}}$; tum enim tenſio maxima ob angulum $\omega = 30°$ erit $\Theta = \frac{Mc}{2g} \cdot \frac{3\Gamma\sqrt{3}}{2\alpha^3}$. Hoc igitur modo omnia phaenomena, quae in his motibus occurrere poſſunt, perfecte ſunt explicata.

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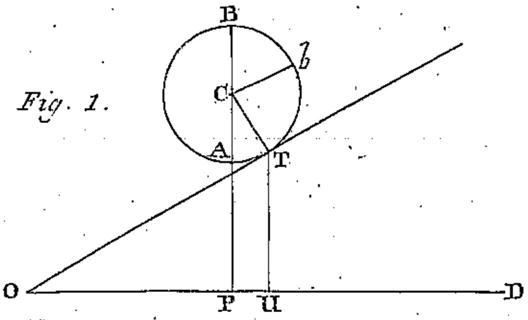

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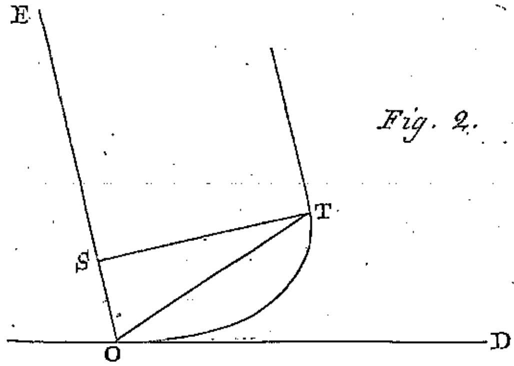

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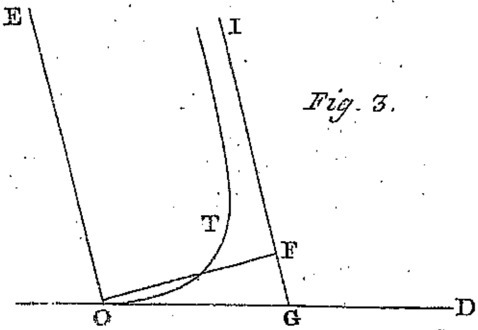

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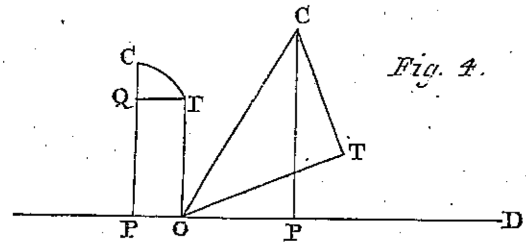

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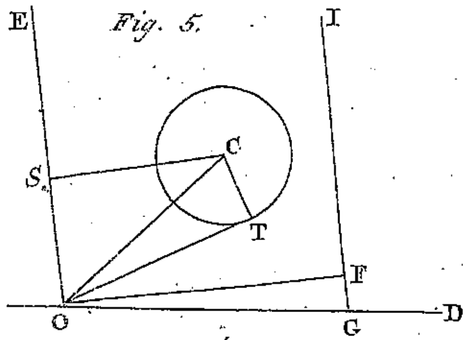
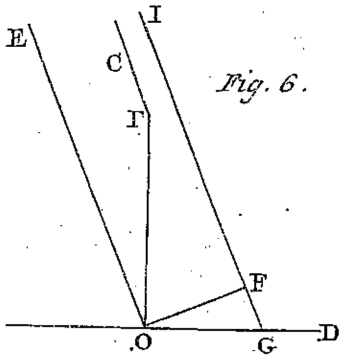


Fig. 5.



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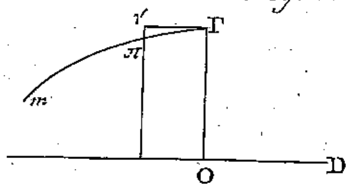

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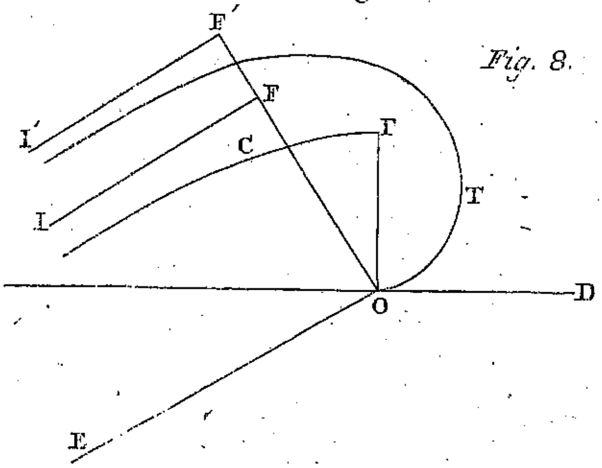

Fig. 8.

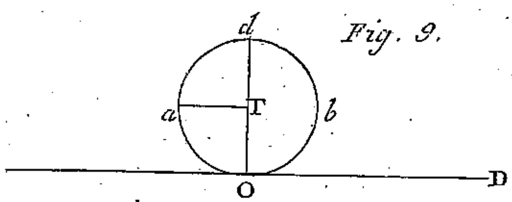

Fig. 9.